MALLIAVIN CALCULUS AND DECOUPLING INEQUALITIES IN BANACH SPACES

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ABSTRACT. We develop a theory of Malliavin calculus for Banach space-valued random variables. Using radonifying operators instead of symmetric tensor products we extend the Wiener-Itô isometry to Banach spaces. In the white noise case we obtain two sided L^p-estimates for multiple stochastic integrals in arbitrary Banach spaces. It is shown that the Malliavin derivative is bounded on vector-valued Wiener-Itô chaoses. Our main tools are decoupling inequalities for vector-valued random variables. In the opposite direction we use Meyer's inequalities to give a new proof of a decoupling result for Gaussian chaoses in UMD Banach spaces.

1. INTRODUCTION

The theory of Malliavin calculus [12, 28] has been developed in the seventies by Malliavin [17], who used it to give a probabilistic proof of Hörmander's "sums of squares"-theorem. The Malliavin calculus generalises in a natural way to Hilbert space-valued random variables. We refer to [6] for a recent account of this infinite dimensional setting with applications to stochastic (partial) differential equations.

In recent years many Hilbert space results in stochastic (and harmonic) analysis have been transferred to a Banach space setting [11, 13]. Of particular relevance for this work is the theory of stochastic integration in Banach spaces developed by van Neerven, Veraar and Weis [24, 26]. Motivated by these developments we construct in this paper a theory of Malliavin calculus for random variables taking values in a Banach space.

Vector-valued Malliavin calculus has been consider by several authors [18, 19, 20, 33]. The main focus in this work is on the interplay between Malliavin calculus and decoupling inequalities. On the one hand, decoupling inequalities are our main tools in the proof of Theorems 3.2, 4.2 and 5.3. In the opposite direction, we apply the theory developed in this paper to give a new proof of a known decoupling result in Theorem 6.12.

In a follow-up paper with van Neerven [16] the vector-valued Malliavin calculus is used to construct a Skorokhod integral in UMD spaces which extends the stochastic integral from [24]. This is used to obtain a Clark-Ocone representation formula in UMD spaces.

It has been proved by Pisier [31] that the fundamental Meyer inequalities remain valid if the Banach space is a UMD space, provided that the norm of the derivative is taken in the appropriate space. These spaces turn out to be spaces of so-called γ-radonifying operators, which have been used to transfer classical Hilbert space results to a more general Banach space setting in various recent works.

Firstly, in the work of Kalton and Weis on H^∞-functional calculus [13] γ-radonifying operators appear as generalisations of classical square functions from harmonic analysis. Secondly, γ-radonifying operators are used in [24, 26] to obtain two-sided estimates for moments of vector-valued stochastic integrals, and provide a generalisation of the classical Itô-isometry.

Let us describe some of the main results in this paper. For details we refer to later sections. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let H be a Hilbert space, and let E be a Banach space. We consider an isometry $W : H \to L^2(\Omega)$ onto a closed subspace consisting of Gaussian random variables, and assume that $\mathcal{F}$ is the σ-field generated by $\{W(h) : h \in H\}$. The classical Wiener-Itô decomposition says that $L^2(\Omega, \mathcal{F}, \mathbb{P})$ admits an orthogonal decomposition into Gaussian chaoses $L^2(\Omega, \mathcal{F}, \mathbb{P}) = \bigoplus_{m \geq 0} \mathcal{H}_m$. Moreover, there exist canonical isometries Φ_m from the symmetric Hilbert space tensor powers $H^{\circledS m}$ onto $\mathcal{H}_m$.

We show in Theorem 3.2 that this result admits a natural Banach space-valued generalisation. For this purpose we consider the space of symmetric γ-radonifying operators $\gamma^{\circledS m}(H, E)$ (cf. Section 3), which turns out to be the natural vector-valued analogue of the symmetric Hilbert space tensor powers. We prove that Φ_m extends to an L^p-isomorphism between $\gamma^{\circledS m}(H, E)$ and the vector-valued Gaussian chaos $\mathcal{H}_m(E)$ for $1 \leq p < \infty$,

$$\|(\Phi_m \otimes I)T\|_{L^p(\Omega;E)} \eqsim_{m,p} \|T\|_{\gamma^{\circledS m}(H,E)}, \qquad T \in \gamma^{\circledS m}(H, E).$$

Here and in the rest of this paper, we use the Vinogradov notation $A \lesssim B$ (and similarly $A \gtrsim B$ and $A \eqsim B$) to indicate that there exists a constant $C \geq 0$ not depending on A and B, such that $A \leq CB$. If the constant depends on additional parameters, we include them as subscripts.

In Section 4 we consider the particular case where $H = L^2(M, \mu)$ for some σ-finite measure space (M, μ). Theorem 4.2 shows that the Wiener-Itô isomorphism between $\gamma^{\circledS m}(L^2(M, \mu), E)$ and $\mathcal{H}_m(E)$ is given by a multiple stochastic integral I_m for Banach space-valued functions. This result gives two-sided bounds for L^p-norms of multiple stochastic integrals, for $1 \leq p < \infty$,

$$\|I_m F\|_{L^p(\Omega;E)} \eqsim_{m,p} \|F\|_{\gamma^{\circledS m}(L^2(M),E)}, \qquad F \in \gamma^{\circledS m}(L^2(M, \mu), E),$$

thereby generalising the (single) Banach space-valued stochastic integral of [26].

The proofs of both results rely on (different) decoupling inequalities. The idea to use decoupling in the study of multiple stochastic integrals is not new. In fact, applications to multiple stochastic integration appear already in the pioneering work on decoupling by McConnell and Taqqu [21, 22], Kwapień [14] and others. The decoupling results that we will use, as well as a some preliminaries on γ-radonifying operators, can be found in Section 2.

In Section 5 we consider the Banach space-valued Malliavin derivative D, which for $1 \leq p < \infty$ acts as a closed operator

$$D : \mathbb{D}^{1,p}(E) \subset L^p(\Omega; E) \to L^p(\Omega; \gamma(H, E)).$$

The main result in this section (Theorem 5.3) asserts that the restriction of the Malliavin derivative to each chaos is an L^p-isomorphism for $1 \leq p < \infty$,

$$\|DF\|_{L^p(\Omega;\gamma(H,E))} \eqsim_{p,m} \|F\|_{L^p(\Omega;E)}, \qquad F \in \mathcal{H}_m(E),$$

a fact which is by no means obvious for general Banach spaces. The use of decoupling in this context appears to be new. In UMD spaces this result is an easy consequence of Meyer's inequalities.

These inequalities are considered in more detail in Section 6. We discuss several of its consequences and obtain a version of Meyer's multiplier theorem in UMD spaces. We return to decoupling in Theorem 6.12 where we give a new proof of

a known decoupling result for Gaussian chaoses in UMD spaces based on Meyer's inequalities.

The Malliavin calculus provides powerful tools for the study of stochastic differential equations both in finite (see, e.g., [28]) and infinite dimensions [3, 10]. The theory developed in this paper makes these tools available for the study of stochastic evolution equations in (UMD) Banach spaces [25], which model stochastic partial differential equations.

2. Preliminaries

2.1. **Decoupling.** Decoupling inequalities go back to the work of McConnell and Taqqu [21, 22], Kwapień [14], Arcones and Giné [1], and de la Peña and Montgomery-Smith [8] among others. We refer to the monographs [15, 7] for extensive information on this topic.

First we introduce some notation which will be used throughout the paper. For $j \geq 1$ and a finite sequence $\boldsymbol{i} = (i_1, \ldots, i_n)$ with values in $\{1, 2, \ldots\}$ we set

$$j(\boldsymbol{i}) = \#\{i_k : 1 \leq k \leq n, i_k = j\}, \quad |\boldsymbol{i}| = n, \quad |\boldsymbol{i}|_\infty := \max_{1 \leq k \leq n} i_k, \quad \boldsymbol{i}! = \prod_{j=1}^\infty j(\boldsymbol{i})!.$$

Note that the latter product contains only finitely many factors.

Let $(\gamma_n)_{n \geq 1}$ be a Gaussian sequence, i.e. a sequence of independent standard Gaussian random variables on a (sufficiently rich) probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let $(\gamma_n^{(k)})_{n \geq 1}$ be independent copies for each $k \geq 1$. The Hermite polynomials H_m satisfy $H_0(x) = 1$, $H_1(x) = x$ and the recurrence relation

$$(m+1)H_{m+1}(x) = xH_m(x) - H_{m-1}(x), \quad m \geq 1.$$

We set

$$\Psi_{\boldsymbol{i}} = (\boldsymbol{i}!)^{1/2} \prod_{j \geq 1} H_{j(\boldsymbol{i})}(\gamma_j).$$

The next theorem states two well-known decoupling results which were obtained in [22, 14, 1]. A general result containing both parts of the next theorem is due to Giné [7, Theorem 4.2.7].

Theorem 2.1. *Let E be a Banach space, let $m, n \geq 1$, and suppose that we are in one of the following two situations:*

  (1) *(symmetric case) Let $(x_{\boldsymbol{i}})_{|\boldsymbol{i}|=m} \subset E$ satisfy $x_{\boldsymbol{i}} = x_{\boldsymbol{i}'}$ whenever $\boldsymbol{i}'$ is a permutation of $\boldsymbol{i}$, and set*

$$F := \sum_{|\boldsymbol{i}|=m, |\boldsymbol{i}|_\infty \leq n} (\boldsymbol{i}!/m!)^{1/2} \Psi_{\boldsymbol{i}} x_{\boldsymbol{i}}.$$

  (2) *(tetrahedral case) Let $(x_{\boldsymbol{i}})_{|\boldsymbol{i}|=m} \subset E$ satisfy $x_{\boldsymbol{i}} = 0$ whenever $j(\boldsymbol{i}) > 1$ for some $j \geq 1$, and set*

$$F := \sum_{|\boldsymbol{i}|=m, |\boldsymbol{i}|_\infty \leq n} \gamma_{i_1} \cdot \ldots \cdot \gamma_{i_m} x_{\boldsymbol{i}}.$$

In both cases we put

$$\widetilde{F} := \sum_{|\boldsymbol{i}|=m, |\boldsymbol{i}|_\infty \leq n} \gamma_{i_1}^{(1)} \cdot \ldots \cdot \gamma_{i_m}^{(m)} x_{\boldsymbol{i}}.$$

Then there exists a constant $C_m \geq 1$ depending only on m, such that for all $t > 0$ we have

$$\frac{1}{C_m}\mathbb{P}(\|\widetilde{F}\|_E > C_m t) \leq \mathbb{P}(\|F\|_E > t) \leq C_m \mathbb{P}(\|\widetilde{F}\|_E > \frac{t}{C_m}).$$

Consequently, for $1 \leq p < \infty$ we have

$$\|F\|_{L^p(\Omega;E)} \eqsim_{p,m} \|\widetilde{F}\|_{L^p(\Omega;E)}.$$

Remark 2.2. The requirement that $|\boldsymbol{i}|_\infty \leq n$ is chosen for convenience, to ensure that we are dealing with finite sums exclusively. Note however that the constants in all of our estimates do not depend on n.

2.2. **Spaces of γ-radonifying operators.** In this section we will review some well-known results about γ-radonifying operators. For more information we refer to [4, 13]. Let H be a real separable Hilbert space with orthonormal basis $(u_n)_{n\geq 1}$, and let E be a real Banach space. Let $(\gamma_n)_{n\geq 1}$ be a Gaussian sequence.

An operator $T \in \mathcal{L}(H, E)$ is said to be γ-radonifying if the sum $\sum_{n=1}^\infty \gamma_n T u_n$ converges in $L^2(\Omega; E)$. The convergence and the L^2-norm of this sum do not depend on the choice of the orthonormal basis and the Gaussian sequence. The space $\gamma(H, E)$ consisting of all γ-radonifying operators in $\mathcal{L}(H, E)$ is a Banach space endowed with the norm

$$\|T\|_{\gamma(H,E)} := \left(\mathbb{E}\Big\|\sum_{n=1}^\infty \gamma_n T u_n\Big\|_E^2\right)^{1/2}.$$

Obviously all rank-1 operators

$$h \otimes x : h' \mapsto [h', h] \cdot x, \quad h, h' \in H, x \in E,$$

are contained in $\gamma(H, E)$ and one easily sees that they span a dense subspace of $\gamma(H, E)$.

It is well known (see, e.g., [13, Proposition 4.3]) that the space $\gamma(H, E)$ enjoys the following ideal property: let $\widetilde{H}$ be a Hilbert space and let $\widetilde{E}$ be a Banach space. For $S \in \mathcal{L}(\widetilde{H}, H)$, $T \in \gamma(H, E)$ and $R \in \mathcal{L}(E, \widetilde{E})$ we have

$$(2.1) \qquad \|R \circ T \circ S\|_{\gamma(\widetilde{H}, \widetilde{E})} \leq \|R\|_{\mathcal{L}(E, \widetilde{E})} \|T\|_{\gamma(H, E)} \|S\|_{\mathcal{L}(\widetilde{H}, H)}.$$

For later use we state the following lemma which has been proved in [13, Proposition 2.6].

Lemma 2.3. *Let (S, Σ, μ) be a σ-finite measure space and let $1 \leq p < \infty$. The mapping $F : L^p(S; \gamma(H, E)) \to \mathcal{L}(H, L^p(S; E))$ defined by $(F(X)h)(s) := X(s)h$ for $s \in S$ and $h \in H$, defines an isomorphism*

$$L^p(S; \gamma(H, E)) \eqsim \gamma(H, L^p(S; E)).$$

An important role in this work will be played by spaces of the form $\gamma^m(H, E)$, which we define inductively by

$$\gamma^1(H, E) := \gamma(H, E), \quad \gamma^{m+1}(H, E) := \gamma(H, \gamma^m(H, E)), \quad m \geq 1.$$

To improve readability we will write $T(h, h')$ instead of $(Th)(h')$ if $T \in \gamma^2(H, E)$. Furthermore we will write $(h \otimes h') \otimes x$ to denote the operator $h \otimes (h' \otimes x) \in \gamma^2(H, E)$. Similar remarks apply when $m > 2$. For future use we record that for operators of the form

$$(2.2) \qquad T = \sum_{|\boldsymbol{i}|=m, |\boldsymbol{i}|_\infty \leq n} (u_{i_1} \otimes \cdots \otimes u_{i_m}) \otimes x_{\boldsymbol{i}}, \quad x_{\boldsymbol{i}} \in E,$$

the norm in $\gamma^m(H, E)$ is given by

$$(2.3) \qquad \|T\|^2_{\gamma^m(H,E)} = \mathbb{E}\Big\| \sum_{|\boldsymbol{i}|=m, |\boldsymbol{i}|_\infty \leq n} \gamma^{(1)}_{i_1} \cdot \ldots \cdot \gamma^{(m)}_{i_m} x_{\boldsymbol{i}} \Big\|^2_E,$$

where we use the multi-index notation from Section 2.1.

If K is a Hilbert space then $\gamma^m(H, K)$ is canonically isometric to the Hilbert space tensor product $H^{\widehat{\otimes} m} \widehat{\otimes} K$. It has been shown in [13] (see also [27]) that $\gamma^m(H, E)$ is isomorphic to $\gamma(H^{\widehat{\otimes} m}, E)$ for all $m \geq 1$ if and only if the Banach space E has Pisier's property (α) [29].

It is well known (see [13, Section 5]) that the pairing

$$(2.4) \qquad [T, S]_\gamma := \operatorname{tr}(T^* S), \quad T \in \gamma(H, E), S \in \gamma(H, E^*),$$

defines a duality between $\gamma(H, E)$ and $\gamma(H, E^*)$, which allows us to identify $\gamma(H, E^*)$ with a weak*-dense subspace of the dual space $\gamma(H, E)^*$. It has been proved by Pisier [32] that the Banach spaces $\gamma(H, E)^*$ and $\gamma(H, E^*)$ are isomorphic if E is K-convex. The notion of K-convexity and its relevance for vector-valued Malliavin calculus will be discussed in Remark 3.4. It is not difficult to check that

$$[T, S]_\gamma = \sum_{j=1}^\infty \langle Tu_j, Su_j \rangle \quad \text{and} \quad [T, S]_\gamma \leq \|T\|_{\gamma(H,E)} \|S\|_{\gamma(H,E^*)}.$$

Let us now consider the important special case that $H = L^2(M, \mu)$ for some σ-finite measure space (M, μ). A strongly measurable function $\phi : M^m \to E$ is said to be weakly-L^2 if $\langle \phi, x^* \rangle \in L^2(M^m)$ for all $x^* \in E^*$. We say that such a function represents an operator $T_\phi \in \gamma^m(L^2(M), E)$ if for all $f_1, \ldots, f_m \in L^2(M)$ and for all $x^* \in E^*$ we have

$$\langle T_\phi(f_1, \ldots, f_m), x^* \rangle = \int_{M^m} f_1(t_1) \cdot \ldots \cdot f_m(t_m) \langle \phi(t_1, \ldots, t_m), x^* \rangle \, d\mu^{\otimes m}(t_1, \ldots, t_m).$$

We will not always notationally distinguish between a function ϕ and the operator $T_\phi \in \gamma^m(L^2(M), E)$ that it represents. The subspace of operators which can be represented by a function is dense in $\gamma^m(L^2(M), E)$.

2.3. **UMD Banach spaces.** A Banach space E is said to be a *UMD space* (*U*n-conditionality of *M*artingale *D*ifferences) if there exists a constant C_E such that for all L^2-integrable E-valued martingale difference sequences $(d_j)_{j=1}^n$ and all sequences $(r_j)_{j=1}^n \subset \{-1, 1\}^n$ we have

$$\Big(\mathbb{E}\Big\| \sum_{j=1}^n r_j d_j \Big\|^2\Big)^{1/2} \leq C_E \Big(\mathbb{E}\Big\| \sum_{j=1}^n d_j \Big\|^2\Big)^{1/2}.$$

The exponent 2 may be replaced by any $p \in (1, \infty)$ without changing the class of spaces under consideration (at the cost of changing the constant C_E). Examples of UMD spaces include Hilbert spaces and $L^p(S)$ spaces for $1 < p < \infty$ and σ-finite measure spaces (S, Σ, μ).

The class of UMD spaces provides a natural setting for Banach space-valued stochastic and harmonic analysis. For more information on these spaces we refer to Burkholder's review article [5].

3. Wiener-Itô chaos in Banach spaces

In this section we will prove a Banach space analogue of the classical Wiener-Itô isometry. First we fix some notations.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a (sufficiently rich) probability space, let H be a real separable Hilbert space and let $W : H \to L^2(\Omega)$ be an isonormal Gaussian process on H, i.e. W maps H isometrically onto a closed subspace consisting of Gaussian random

variables. We assume that $\mathcal{F}$ is the σ-field generated by $\{W(h) : h \in H\}$. We fix an orthonormal basis $(u_j)_{j \geq 1}$ of H, and consider the Gaussian sequence defined by $\gamma_j := W(u_j)$ for $j \geq 1$. For $m \geq 0$ we consider the m-th Wiener-Itô chaos,

$$\mathcal{H}_m := \overline{\mathrm{lin}}\{H_m(W(h)) : \|h\| = 1\},$$

where the closure is taken in $L^2(\Omega)$. Furthermore, let $H^{\otimes m}$ be the m-fold symmetric tensor power which is defined to be the range of the orthogonal projection $P_{\circledS} \in \mathcal{L}(H^{\widehat{\otimes} m})$ given by

$$P_{\circledS}(h_1 \otimes \cdots \otimes h_m) = \frac{1}{m!} \sum_{\pi \in S_m} h_{\pi(1)} \otimes \cdots \otimes h_{\pi(m)}, \qquad h_1, \ldots, h_m \in H,$$

where S_m is the group of permutations of $\{1, \ldots, m\}$.

A classical result of Wiener states that the following orthogonal decomposition holds:

$$L^2(\Omega, \mathcal{F}, \mathbb{P}) = \bigoplus_{m \geq 0} \mathcal{H}_m.$$

Moreover, the mapping Φ_m defined by

$$(3.1) \qquad \Phi_m : P_{\circledS}(u_{i_1} \otimes \cdots \otimes u_{i_m}) \mapsto (\boldsymbol{i}!/m!)^{1/2} \Psi_{\boldsymbol{i}},$$

extends to an isometry from $H^{\otimes m}$ onto $\mathcal{H}_m$. Recall that $\Psi_{\boldsymbol{i}}$ is the generalised Hermite polynomial defined in Section 2.1, to which we refer for notations.

Let us consider the vector-valued Gaussian chaos

$$\mathcal{H}_m(E) := \overline{\mathrm{lin}}\{f \otimes x : f \in \mathcal{H}_m, x \in E\},$$

where the closure is taken in $L^2(\Omega; E)$. The following well-known result is a consequence of the decoupling result in Theorem 2.1(1) and the Kahane-Khintchine inequalities. Extensive information on this topic can be found in the monographs [15, 7].

Proposition 3.1. *Let E be a Banach space, let $m \geq 1$, and let $1 \leq p, q < \infty$. For all $F \in \mathcal{H}_m(E)$ we have*

$$\|F\|_{L^p(\Omega; E)} \eqsim_{m,p,q} \|F\|_{L^q(\Omega; E)}.$$

Our next goal is the construction of the spaces $\gamma^{\otimes m}(H, E)$, which will be the Banach space substitutes for the symmetric Hilbert space tensor powers. We refer to Section 2.2 for the definition of the space $\gamma^m(H, E)$. For $T \in \gamma^m(H, E)$ we define its symmetrisation $P_{\circledS} T \in \gamma^m(H, E)$ by

$$(P_{\circledS} T)(h_1, \ldots, h_m) := \frac{1}{m!} \sum_{\pi \in S_m} T(h_{\pi(1)}, \ldots, h_{\pi(m)}), \quad h_1, \ldots, h_m \in H,$$

and we will say that $T \in \gamma^m(H, E)$ is symmetric if $P_{\circledS} T = T$. The mapping $P_{\circledS}$ is easily seen to be a projection in $\mathcal{L}(\gamma^m(H, E))$ and we define $\gamma^{\otimes m}(H, E)$ to be its range.

We remark that if K is a Hilbert space, then $\gamma^{\otimes m}(H, K)$ is isometrically isomorphic to the space $H^{\otimes m} \widehat{\otimes} K$, where $\widehat{\otimes}$ denotes the Hilbert space tensor product.

Now we are ready to state the main result of this section, which is a Banach space-valued extension of the canonical isometry (3.1).

Theorem 3.2. *Let E be a Banach space, let $1 \leq p < \infty$, and let $m \geq 1$. The mapping*

$$(\Phi_m \otimes I) : P_{\circledS}(h_{i_1} \otimes \cdots \otimes h_{i_m}) \otimes x \mapsto (\boldsymbol{i}!/m!)^{1/2} \Psi_{\boldsymbol{i}} \otimes x,$$

extends to a bounded operator $(\Phi_m \otimes I) : \gamma^{\circledS m}(H, E) \to L^p(\Omega; E)$, *which maps* $\gamma^{\circledS m}(H, E)$ *onto* $\mathcal{H}_m(E)$. *Moreover, we have equivalence of norms*

$$\|(\Phi_m \otimes I)T\|_{L^p(\Omega;E)} \eqsim_{m,p} \|T\|_{\gamma^m(H,E)}, \quad T \in \gamma^{\circledS m}(H, E).$$

Proof. Let T be a symmetric operator of the form (2.2) and observe that

$$T = \sum_{|\boldsymbol{i}|=m, |\boldsymbol{i}|_\infty \le n} P_{\circledS}(u_{i_1} \otimes \cdots \otimes u_{i_m}) \otimes x_{\boldsymbol{i}}.$$

Using (2.3), the decoupling result from Theorem 2.1(1) and the Kahane-Khintchine inequalities we obtain

$$\mathbb{E}\|(\Phi_m \otimes I)T\|_E^p = \mathbb{E}\Big\| \sum_{|\boldsymbol{i}|=m, |\boldsymbol{i}|_\infty \le n} (\boldsymbol{i}!/m!)^{1/2}\Psi_{\boldsymbol{i}} x_{\boldsymbol{i}} \Big\|_E^p$$
$$\eqsim_{m,p} \mathbb{E}\Big\| \sum_{|\boldsymbol{i}|=m, |\boldsymbol{i}|_\infty \le n} \gamma_{i_1}^{(1)} \cdot \ldots \cdot \gamma_{i_m}^{(m)} x_{\boldsymbol{i}} \Big\|_E^p \eqsim_{m,p} \|T\|_{\gamma^m(H,E)}^p.$$

In view of Proposition 3.1 it is clear that $\Phi_m \otimes I$ maps $\gamma^{\circledS m}(H, E)$ into $\mathcal{H}_m(E)$. To show that its range is $\mathcal{H}_m(E)$, we observe that $\Phi_m \otimes I(h^{\otimes m} \otimes x) = H_m(W(h)) \cdot x$ for all $h \in H$ with $\|h\| = 1$ and all $x \in E$. Now the result follows from the norm estimate above and the identity

$$\mathcal{H}_m(E) = \overline{\mathrm{lin}}\{H_m(W(h)) \cdot x : \|h\| = 1, x \in E\},$$

where the closure is taken in $L^p(\Omega; E)$. $\qquad\square$

Remark 3.3. In the special case that $E = \mathbb{R}$ and $p = 2$ we recover the classical Wiener-Itô isometry from $H^{\circledS m}$ onto $\mathcal{H}_m$. More generally, if E is a Hilbert space and $p = 2$, our result reduces to the well-known Hilbert space-valued Wiener-Itô isometry from $H^{\circledS m} \otimes E$ onto $\mathcal{H}_m(E)$.

Remark 3.4. Let $m \ge 1$ and let J_m be the orthogonal projection onto $\mathcal{H}_m$. It is well known that for all $1 < p < \infty$ the restriction of J_m to $L^p(\Omega) \cap L^2(\Omega)$ extends to a bounded projection on $L^p(\Omega)$. A Banach space E is said to be K-convex if $J_1 \otimes I$ extends to a bounded operator on $L^2(\Omega; E)$. Actually, this notion is usually defined using Rademacher instead of Gaussian random variables, but this does not affect the class of Banach spaces under consideration [9]. It has been shown by Pisier [30] that in this case the operators $J_m \otimes I$ (which will be denoted by J_m below) are bounded for all $m \ge 1$ and all $1 < p < \infty$. Every UMD space is K-convex. These facts will be used in Sections 5 and 6.

4. Multiple Wiener-Itô integrals in Banach spaces

As in the previous section we consider a real separable Hilbert space H and an isometry $W : H \to L^2(\Omega)$ onto a closed linear subspace consisting of Gaussian random variables.

In addition we assume in this section that $H = L^2(M, \mathcal{B}, \mu)$ for some σ-finite non-atomic measure space M. We let $\mathcal{B}_0 := \{B \in \mathcal{B} : \mu(B) < \infty\}$. For $A \in \mathcal{B}_0$ we write with some abuse of notation $W(A) := W(\mathbf{1}_A)$. In this way W defines an $L^2(\Omega)$-valued measure on $\mathcal{B}_0$ which is called the white noise based on μ.

Our next goal is to construct multiple stochastic integrals for Banach space-valued functions. Our construction generalises the well known multiple stochastic integral for Hilbert space-valued functions, and in another direction, the (single) stochastic integral for Banach space-valued functions which has been constructed in [26].

For fixed $m \geq 1$ we define $\mathcal{E}_m(E)$ to be the linear space of tetrahedral simple functions $F : M^m \to E$ of the form

$$(4.1) \qquad F = \sum_{|\boldsymbol{i}|=m, |\boldsymbol{i}|_\infty \leq n} \mathbf{1}_{A_{i_1} \times \cdots \times A_{i_m}} \cdot x_{\boldsymbol{i}},$$

where the A_j's are pairwise disjoint sets in $\mathcal{B}_0$, $n \geq 1$, and the coefficients $x_{\boldsymbol{i}} \in E$ vanish whenever $j(\boldsymbol{i}) > 1$ for some $j \geq 1$. It is easy to see that such a function F represents an operator $T_F \in \gamma^m(L^2(M), E)$ in the sense described in Section 2.2, and by taking an orthonormal basis $(u_j)_{j \geq 1}$ of $L^2(M)$ with $u_j = \mu(A_j)^{-1/2} \mathbf{1}_{A_j}$ for $j = 1, \ldots, n$, one can check that

$$(4.2)$$
$$\|T_F\|^2_{\gamma^m(L^2(M),E)} = \mathbb{E} \Big\| \sum_{|\boldsymbol{i}|=m, |\boldsymbol{i}|_\infty \leq n} \gamma^{(1)}_{i_1} \cdot \ldots \cdot \gamma^{(m)}_{i_m} \cdot \mu(A_1)^{1/2} \cdot \ldots \cdot \mu(A_n)^{1/2} \cdot x_{\boldsymbol{i}} \Big\|^2_E.$$

We recall that $(\gamma^{(k)}_j)_{j \geq 1}$ are independent Gaussian sequences for $k \geq 1$.

Lemma 4.1. *The collection of operators represented by functions in $\mathcal{E}_m(E)$ is dense in $\gamma^m(L^2(M), E)$ for all $m \geq 1$.*

Proof. This follows by reasoning as in the proof of the corresponding scalar-valued result [28, p.10], taking into account that the measure space M is non-atomic. $\square$

Suppose that $T_F \in \gamma^m(L^2(M), E)$ is represented by a strongly measurable weakly-L^2 function F. Then T_F belongs to $\gamma^{\circledS m}(L^2(M), E)$ if and only if F agrees almost everywhere with its symmetrisation $\widetilde{F}$ defined by

$$\widetilde{F}(t_1, \ldots, t_m) := \frac{1}{m!} \sum_{\pi \in S_m} F(t_{\pi(1)}, \ldots, t_{\pi(m)}).$$

For $F \in \mathcal{E}_m(E)$ of the form (4.1) we define the multiple Wiener-Itô integral $I_m(F) \in L^2(\Omega; E)$ by

$$(4.3) \qquad I_m(F) = \sum_{|\boldsymbol{i}|=m, |\boldsymbol{i}|_\infty \leq n} W(A_{i_1}) \cdot \ldots \cdot W(A_{i_m}) \cdot x_{\boldsymbol{i}}.$$

One easily checks that this definition does not depend on the representation of F as an element of $\mathcal{E}_m(E)$. Moreover, I_m is linear and $I_m(F) = I_m(\widetilde{F})$. The next theorem may be considered as a generalisation of the classical Wiener-Itô-isometry for multiple stochastic integrals to the Banach space setting.

Theorem 4.2. *Let $m \geq 1$ and $1 \leq p < \infty$. The operator $I_m : \mathcal{E}_m(E) \to L^p(\Omega; E)$ extends uniquely to a bounded operator*

$$I_m : \gamma^m(L^2(M), E) \to L^p(\Omega; E),$$

which maps $\gamma^m(L^2(M), E)$ onto $\mathcal{H}_m(E)$. Moreover, for all $F \in \gamma^m(L^2(M), E)$ we have:

(i) $I_m F = I_m \widetilde{F}$;

(ii) $\|I_m F\|_{L^p(\Omega;E)} \eqsim_{m,p} \|\widetilde{F}\|_{\gamma^m(L^2(M),E)} \leq \|F\|_{\gamma^m(L^2(M),E)}$.

Proof. First we show that for all $F \in \mathcal{E}_m(E)$ the following equivalence of norms holds:

$$\|I_m F\|_{L^p(\Omega;E)} \eqsim_{m,p} \|\widetilde{F}\|_{\gamma^m(L^2(M),E)}.$$

For that purpose we take $F \in \mathcal{E}_m(E)$ of the form (4.1). Since $I_m(F) = I_m(\widetilde{F})$ we may assume that F is symmetric, hence $x_{(i_{\pi(1)}, \ldots, i_{\pi(m)})} = x_{(i_1, \ldots, i_m)}$ for all permutations $\pi \in S_m$. Let $(u_j)_{j \geq 1}$ be an orthonormal basis of $L^2(M)$ with $u_j = \mu(A_j)^{-1/2} \mathbf{1}_{A_j}$ for $j = 1, \ldots, n$, and let $(\gamma_j)_{j \geq 1}$ be the Gaussian sequence $\gamma_j =$

$W(u_j)$ for $j \geq 1$. Using the decoupling inequalities from Theorem 2.1(2), (4.2), and the Kahane-Khintchine inequalities we obtain

$$
\begin{aligned}
\|I_m F\|^p_{L^p(\Omega,E)} &= \mathbb{E} \Big\| \sum_{|\boldsymbol{i}|=m, |\boldsymbol{i}|_\infty \leq n} W(A_{i_1}) \cdot \ldots \cdot W(A_{i_m}) \cdot x_{\boldsymbol{i}} \Big\|^p_E \\
&= \mathbb{E} \Big\| \sum_{|\boldsymbol{i}|=m, |\boldsymbol{i}|_\infty \leq n} \gamma_{i_1} \cdot \ldots \cdot \gamma_{i_m} \cdot \mu(A_1)^{1/2} \cdot \ldots \cdot \mu(A_n)^{1/2} \cdot x_{\boldsymbol{i}} \Big\|^p_E \\
&\eqsim_{m,p} \mathbb{E} \Big\| \sum_{|\boldsymbol{i}|=m, |\boldsymbol{i}|_\infty \leq n} \gamma_{i_1}^{(1)} \cdot \ldots \cdot \gamma_{i_m}^{(m)} \cdot \mu(A_1)^{1/2} \cdot \ldots \cdot \mu(A_n)^{1/2} \cdot x_{\boldsymbol{i}} \Big\|^p_E \\
&\eqsim_{m,p} \|F\|^p_{\gamma^m(L^2(M),E)}.
\end{aligned}
$$

Now the first claim follows from Lemma 4.1. To prove that $I_m T \in \mathcal{H}_m(E)$ for all $T \in \gamma^m(L^2(M), E)$ we first let $T = T_F$ for some tetrahedral function F of the form (4.1). It follows from (4.3) and the fact that

$$
W(A_{j_1}) \cdot \ldots \cdot W(A_{j_m}) \in \mathcal{H}_m
$$

whenever all j_k's are different, that $I_m T \in \mathcal{H}_m(E)$. Since I_m is continuous the same holds for general $T \in \gamma^m(L^2(M), E)$ by Lemma 4.1. To show that the mapping $I_m : \gamma^m(L^2(M), E) \to \mathcal{H}_m(E)$ is surjective we proceed as in Theorem 3.2. The other statements are clear in view of Lemma 4.1. □

5. The Malliavin derivative

In this section we consider a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a real separable Hilbert space H, and an isonormal Gaussian process $W : H \to L^2(\Omega)$. As before we assume that $\mathcal{F}$ is the σ-algebra generated by W.

Let us introduce some notation. For $n \geq 1$ we denote by $C_{pol}^\infty(\mathbb{R}^n)$ the vector space of all C^∞-functions $f : \mathbb{R}^n \to \mathbb{R}$ such that f and its partial derivatives of all orders have polynomial growth, i.e. for every multi-index α there exist positive constants C_α, p_α such that

$$
|\partial_\alpha f(x)| \leq C_\alpha (1 + |x|)^{p_\alpha}.
$$

Let $\mathcal{S}$ be the collection of all random variables $f : \Omega \to \mathbb{R}$ of the form

$$
(5.1) \qquad\qquad f = \varphi(W(h_1), \ldots, W(h_n))
$$

for some $\varphi \in C_{pol}^\infty(\mathbb{R}^n)$, $h_1, \ldots, h_n \in H$ and $n \geq 1$.

For a real Banach space E we consider the dense subspace $\mathcal{S}(E)$ of $L^p(\Omega; E)$, $1 \leq p < \infty$, consisting of all functions $F : \Omega \to E$ of the form

$$
F = \sum_{i=1}^n f_i \cdot x_i,
$$

where $f_i \in \mathcal{S}$ and $x_i \in E$, $i = 1, \ldots n$. Occasionally it will be convenient to work with the space $\mathcal{P}(E)$, which is defined similarly, except that the functions φ are required to be polynomials.

For a function $F = f \cdot x \in \mathcal{S}(E)$ with f of the form (5.1) we define its Malliavin derivative DF by

$$
(5.2) \qquad\qquad DF = \sum_{j=1}^n \partial_j \varphi(W(h_1), \ldots, W(h_n)) h_j \otimes x.
$$

This definition extends to $\mathcal{S}(E)$ by linearity. For $F \in \mathcal{S}(E)$ the Malliavin derivative DF is a random variable which takes values in the algebraic tensor product $H \otimes E$, which we endow with the norm $\|\cdot\|_{\gamma(H,E)}$ (cf. Section 2.2).

The following result is the simplest case of the integration by parts formula. We omit the proof, which is the same as in the scalar-valued case [28, Lemma 1.2.1].

Lemma 5.1. *If $F \in \mathcal{S}(E)$, then $\mathbb{E}(DF(h)) = \mathbb{E}(W(h)F)$ for all $h \in H$.*

A straightforward computation shows that the following product rule holds:

$$D\langle F, G\rangle = \langle DF, G\rangle + \langle F, DG\rangle, \quad F \in \mathcal{S}(E), G \in \mathcal{S}(E^*).$$

Here $\langle \cdot, \cdot \rangle$ denotes the duality between E and E^*. Combining this with Lemma 5.1 we obtain the following integration by parts formula:

$$(5.3) \quad \mathbb{E}\langle DF(h), G\rangle = \mathbb{E}(W(h)\langle F, G\rangle) - \mathbb{E}\langle F, DG(h)\rangle, \quad F \in \mathcal{S}(E), G \in \mathcal{S}(E^*).$$

This identity is the main ingredient in the proof of the following result which can be found in [16, Proposition 2.3].

Proposition 5.2. *The Malliavin derivative D is closable as an operator from $L^p(\Omega; E)$ into $L^p(\Omega; \gamma(H, E))$ for all $1 \le p < \infty$.*

With a slight abuse of notation we will denote the closure of D again by D. Its domain in $L^p(\Omega; E)$ will be denoted by $\mathbb{D}^{1,p}(\Omega; E)$, which is a Banach space endowed with the norm

$$\|F\|_{\mathbb{D}^{1,p}(\Omega;E)} := \left(\|F\|_{L^p(\Omega;E)}^p + \|DF\|_{L^p(\Omega;\gamma(H,E))}^p\right)^{1/p}.$$

Furthermore we will write $\mathbb{D}^{1,p}(\Omega) := \mathbb{D}^{1,p}(\Omega; \mathbb{R})$.

Derivatives of higher order are defined inductively. For $n \ge 1$ we define

$$\mathbb{D}^{n+1,p}(\Omega; E) := \{F \in \mathbb{D}^{n,p}(\Omega; E) : D^n F \in \mathbb{D}^{1,p}(\Omega; \gamma^n(H, E))\},$$
$$D^{n+1}F := D(D^n F), \quad F \in \mathbb{D}^{n+1,p}(\Omega; E).$$

It follows from Proposition 5.2 that D^n is a closed and densely defined operator from $\mathbb{D}^{n-1,p}(\Omega; E)$ into $L^p(\Omega; \gamma^n(H, E))$. Its domain is denoted by $\mathbb{D}^{n,p}(\Omega; E)$ which is a Banach space endowed with the norm

$$\|F\|_{n,p} := \|F\|_{\mathbb{D}^{n,p}(\Omega;E)} := \left(\|F\|_{L^p(\Omega;E)}^p + \sum_{k=1}^n \|D^k F\|_{L^p(\Omega;\gamma^k(H,E))}^p\right)^{1/p}.$$

The main result in this section describes the behaviour of the Malliavin derivative on the E-valued Wiener-Itô chaoses. It extends [28, Proposition 1.2.2] to Banach spaces (and to $1 \le p < \infty$, but this is well-known in the scalar case).

Theorem 5.3. *Let E be a Banach space, let $1 \le p < \infty$ and let $m \ge 1$. Then we have $\mathcal{H}_m(E) \subset \mathbb{D}^{1,p}(\Omega; E)$ and $D(\mathcal{H}_m(E)) \subset \mathcal{H}_{m-1}(\gamma(H, E))$. Moreover, the following equivalence of norms holds:*

$$\|DF\|_{L^p(\Omega;\gamma(H,E))} \eqsim_{p,m} \|F\|_{L^p(\Omega;E)}, \quad F \in \mathcal{H}_m(E).$$

Proof. Let $(u_j)_{j\ge1}$ be an orthonormal basis of H and put $\gamma_j := W(u_j)$. Let $(\gamma_j^{(k)})_{j\ge1}$ and $(\widetilde{\gamma}_j)_{j\ge1}$ be independent copies of $(\gamma_j)_{j\ge1}$. For $\boldsymbol{i} = (i_1, \ldots, i_m)$ and $k \ge 1$ we will write $(\boldsymbol{i}, k) = (i_1, \ldots, i_m, k)$.

First we take $F \in \mathcal{H}_m(E)$ of the form

$$F = \sum_{|\boldsymbol{i}|=m, |\boldsymbol{i}|_\infty \le n} \frac{\boldsymbol{i}!}{m!^{1/2}} \prod_{j=1}^n H_{j(\boldsymbol{i})}(\gamma_j) \, x_{\boldsymbol{i}}.$$

Clearly we may assume without loss of generality that the coefficients $x_{\boldsymbol{i}}$ are symmetric, i.e. $x_{\boldsymbol{i}} = x_{\boldsymbol{i}'}$ whenever $\boldsymbol{i}'$ is a permutation of $\boldsymbol{i}$.

It follows from Theorem 2.1(1) that

$$
\begin{aligned}
\mathbb{E}\|F\|_E^p &= \mathbb{E}\left\| \sum_{|\boldsymbol{i}|=m,|\boldsymbol{i}|_\infty \leq n} \frac{\boldsymbol{i}!}{m!^{1/2}} \prod_{j=1}^n H_{j(\boldsymbol{i})}(\gamma_j) x_{\boldsymbol{i}} \right\|_E^p \\
&\eqsim_{m,p} \mathbb{E}\left\| \sum_{|\boldsymbol{i}|=m} \gamma_{i_1}^{(1)} \cdot \ldots \cdot \gamma_{i_m}^{(m)} x_{\boldsymbol{i}} \right\|_E^p.
\end{aligned}
$$
(5.4)

On the other hand, by a change of variables to modify the range of summation from $\{|\boldsymbol{i}|=m\}$ to $\{|\boldsymbol{i}|=m-1\}$, and rearranging terms, we obtain with the convention that $H_{-1}=0$,

$$
\begin{aligned}
DF &= \sum_{|\boldsymbol{i}|=m,|\boldsymbol{i}|_\infty \leq n} \frac{\boldsymbol{i}!}{m!^{1/2}} \sum_{k=1}^n \prod_{j\neq k} H_{j(\boldsymbol{i})}(\gamma_j) H_{k(\boldsymbol{i})-1}(\gamma_k) \cdot u_k \otimes x_{\boldsymbol{i}}, \\
&= \sum_{k=1}^n u_k \otimes \left(m^{1/2} \sum_{|\boldsymbol{i}|=m-1,|\boldsymbol{i}|_\infty \leq n} \frac{\boldsymbol{i}!}{(m-1)!} \prod_{j=1}^n H_{j(\boldsymbol{i})}(\gamma_j) x_{(\boldsymbol{i},k)} \right) \\
&= \sum_{k=1}^n u_k \otimes \left(\frac{m^{1/2}}{(m-1)!^{1/2}} \sum_{|\boldsymbol{i}|=m-1,|\boldsymbol{i}|_\infty \leq n} \frac{\boldsymbol{i}!^{1/2}}{(m-1)!^{1/2}} \Psi_{\boldsymbol{i}} x_{(\boldsymbol{i},k)} \right).
\end{aligned}
$$

Using the Kahane-Khintchine inequalities and Theorem 2.1(1) once more, we find

$$
\begin{aligned}
\mathbb{E}\|DF\|_{\gamma(H,E)}^p &\eqsim_p \mathbb{E}\widetilde{\mathbb{E}}\left\| \sum_{k=1}^n \widetilde{\gamma}_k DF(u_k) \right\|_E^p \\
&= \mathbb{E}\widetilde{\mathbb{E}}\left\| \frac{m^{1/2}}{(m-1)!^{1/2}} \sum_{k=1}^n \widetilde{\gamma}_k \sum_{|\boldsymbol{i}|=m-1,|\boldsymbol{i}|_\infty \leq n} \frac{\boldsymbol{i}!^{1/2}}{(m-1)!^{1/2}} \Psi_{\boldsymbol{i}} x_{(\boldsymbol{i},k)} \right\|_E^p \\
&= \widetilde{\mathbb{E}}\mathbb{E}\left\| \frac{m^{1/2}}{(m-1)!^{1/2}} \sum_{|\boldsymbol{i}|=m-1,|\boldsymbol{i}|_\infty \leq n} \frac{\boldsymbol{i}!^{1/2}}{(m-1)!^{1/2}} \Psi_{\boldsymbol{i}}\left(\sum_{k=1}^n \widetilde{\gamma}_k x_{(\boldsymbol{i},k)} \right) \right\|_E^p \\
&\eqsim_{m,p} \widetilde{\mathbb{E}}\mathbb{E}\left\| \sum_{|\boldsymbol{i}|=m-1,|\boldsymbol{i}|_\infty \leq n} \sum_{k=1}^n \gamma_{i_1}^{(1)} \cdot \ldots \cdot \gamma_{i_{m-1}}^{(m-1)} \widetilde{\gamma}_k x_{(\boldsymbol{i},k)} \right\|_E^p.
\end{aligned}
$$
(5.5)

Comparing (5.4) and (5.5) yields the norm estimate. The theorem follows by the closedness of D and the fact that functions F of the form considered above are dense in $\mathcal{H}_m(E)$. □

Remark 5.4. In the special case that E is a UMD Banach space the result above is known. Indeed, it follows from Meyer's inequalities (Theorem 6.3) that

$$
\|DF\|_{L^p(\Omega;\gamma(H,E))} \eqsim_{p,E} m^{1/2}\|F\|_{L^p(\Omega;E)}, \quad F \in \mathcal{H}_m(E).
$$

This formula gives an explicit dependence on m, but in contrast with Theorem 5.3 the constants depend on (the Hilbert transform constants of) E. We return to this observation in Section 6.

In the remainder of this section we consider the case where $H = L^2(M,\mathcal{B},\mu)$ as in Section 4. In this setting the Malliavin derivative of a random variable $F \in \mathbb{D}^{1,p}(\Omega;E)$ is an element of $L^p(\Omega;\gamma(L^2(M),E))$ and according to Lemma 2.3, the latter space can be identified with $\gamma(L^2(M),L^p(\Omega;E))$. If DF can be represented by a function from M to $L^p(\Omega;E)$ in the sense of Section 2.2, we will follow the classical notation and denote this function by $t \mapsto D_t F$.

The next result is the Banach space-valued extension of [28, Proposition 1.2.7].

Proposition 5.5. *Let E be a Banach space, let $1 \leq p < \infty$, and let $m \geq 1$. Let $\phi : M^m \to E$ be a strongly measurable weakly-L^2 function representing an operator in $\gamma^{\circledS m}(L^2(M), E)$. Then $I_m\phi \in \mathbb{D}^{1,p}(\Omega; E)$ and for a.e. $t \in M$ we have*

$$D_t I_m(\phi) = m I_{m-1}\phi(\cdot, t).$$

Proof. Using Lemma 4.1 we take a sequence of symmetric functions $(\phi_n)_{n \geq 1} \subset \mathcal{E}_m(E)$ converging to ϕ in $\gamma^{\circledS m}(L^2(M), E)$. It follows from the corresponding scalar valued result that for a.e. $t \in M$,

(5.6) $D_t I_m(\phi_n) = m I_{m-1}(\phi_n(\cdot, t)).$

Theorem 4.2 implies that

(5.7) $I_m(\phi_n) \to I_m(\phi) \quad \text{in } L^p(\Omega; E).$

On the other hand, by definition of the γ^m-norms, we have $\phi_n(\cdot, t) \to \phi(\cdot, t)$ in $\gamma(L^2(M), \gamma^{m-1}(L^2(M), E))$. Hence by another application of Theorem 4.2, the ideal property of γ-norms (2.1), and the fact that

$$\gamma(L^2(M), L^p(\Omega; E)) \approx L^p(\Omega; \gamma(L^2(M), E))$$

according to Lemma 2.3, we infer that

(5.8) $I_{m-1}(\phi_n(\cdot, t)) \to I_{m-1}(\phi(\cdot, t)) \quad \text{in } L^p(\Omega; \gamma(L^2(M), E)).$

Now the result follows by combining the closedness of D with (5.6), (5.7) and (5.8). $\square$

6. Meyer's inequalities and its consequences

Let $(P(t))_{t \geq 0} \subset \mathcal{L}(L^2(\Omega))$ be the Ornstein-Uhlenbeck semigroup defined by

(6.1) $P(t) := \sum_{m \geq 0} e^{-mt} J_m.$

As is well known, this semigroup extends to a C_0-semigroup of positive contractions on $L^p(\Omega)$ for all $1 \leq p < \infty$. We refer the reader to [28] for proofs of these and other elementary properties.

Let E be an arbitrary Banach space. By positivity of P, $(P(t) \otimes I)_{t \geq 0}$ extends to a C_0-semigroup of contractions on the Lebesgue-Bochner spaces $L^p(\Omega; E)$ for $1 \leq p < \infty$ which will be denoted by $(P_E(t))_{t \geq 0}$. The domain in $L^p(\Omega; E)$ of its infinitesimal generator L_E is denoted $\mathsf{D}_p(L_E)$. The subordinated semigroup $(Q_E(t))_{t \geq 0}$ is defined by

(6.2) $Q_E(t)f := \int_0^\infty P_E(s)f \, d\nu_t(s),$

where the probability measure ν_t is given by

(6.3) $d\nu_t(s) = \dfrac{t}{2\sqrt{\pi s^3}} e^{-t^2/4s} \, ds, \quad t > 0.$

The generator of $(Q_E(t))_{t \geq 0}$ will be denoted by C_E. As is well known we have

$$C_E = -(-L_E)^{1/2}.$$

Often, when there is no danger of confusion, we will omit the subscripts E.

The next lemma is a vector-valued analogue of the representation of L as a generator associated with a Dirichlet form. We omit the proof, which follows from the scalar-valued analogue in a straightforward way. Recall that the notation $[\cdot, \cdot]_\gamma$ has been introduced in (2.4).

Lemma 6.1. *Let E be a Banach space and let $1 \leq p < \infty$. For all $F \in \mathcal{P}(E)$ and $G \in \mathbb{D}^{1,p}(\Omega; E^*)$ we have*

$$\mathbb{E}\langle L_E F, G\rangle = \mathbb{E}[DF, DG]_\gamma.$$

In the following Lemma we collect some useful commutation relations, which follow easily from the corresponding scalar-valued results.

Lemma 6.2. *Let E be a Banach space and let $1 \leq p < \infty$.*

(i) *For $F \in \mathbb{D}^{1,p}(\Omega; E)$ we have $P_E(t)F \in \mathbb{D}^{1,p}(\Omega; E)$ and*

$$DP_E(t)f = e^{-t}P_{\gamma(H,E)}DF.$$

(ii) *For $F \in \mathbb{D}^{1,p}(\Omega; E)$ we have $Q_E(t)F \in \mathbb{D}^{1,p}(\Omega; E)$ and*

$$DQ_E(t)f = Q^{(1)}_{\gamma(H,E)}DF,$$

where $Q^{(1)}_{\gamma(H,E)}$ is the semigroup generated by $-(I - L_{\gamma(H,E)})^{1/2}$.

(iii) *For $F \in \mathcal{P}(E)$ we have $L_E F \in \mathbb{D}^{1,p}(\Omega; E)$ and $DL_E F = -(I - L_{\gamma(H,E)})DF$.*

(iv) *For $F \in \mathcal{P}(E)$ we have $C_E F \in \mathbb{D}^{1,p}(\Omega; E)$ and $DC_E F = -(I - L_{\gamma(H,E)})^{1/2}DF$.*

Pisier proved in [31] that Meyer's inequalities extend to UMD spaces. Formulated in the language of γ-norms his result reads as follows.

Theorem 6.3 (Meyer's inequalities)**.** *Let E be a UMD space and let $1 < p < \infty$. Then $\mathsf{D}_p(C_E) = \mathbb{D}^{1,p}(\Omega; E)$ and for all $f \in \mathbb{D}^{1,p}(\Omega; E)$ the following two-sided estimate holds:*

$$(6.4) \qquad \|C_E f\|_{L^p(\Omega;E)} \eqsim_{p,E} \|Df\|_{L^p(\Omega;\gamma(H,E))}.$$

In Theorem 6.8 we shall state an extension of this result.

The following lemma is the crucial ingredient in the proof of Meyer's multiplier Theorem. The proof in the scalar case in [28, Lemma 1.4.1] does not extend to the vector-valued setting, since it depends heavily on the Hilbert space structure of $L^2(\Omega)$. We give a simple proof in the case that E is a UMD space, which is based on Meyer's inequalities. Recall that J_m denotes the chaos projection considered in Remark 3.4.

Lemma 6.4. *Let $1 < p < \infty$ and let E be a UMD space. For each $N \geq 1$ and $t > 0$ we have*

$$\|P(t)(I - J_0 - J_1 - \ldots - J_{N-1})\|_{\mathcal{L}(L^p(\Omega;E))} \lesssim_{E,p,N} e^{-Nt}.$$

Proof. For $F \in \mathcal{P}(E)$ we set

$$RF = D\sum_{m=1}^\infty m^{-1/2}J_m F, \qquad S\Big(D\sum_{m=0}^\infty J_m F\Big) := \sum_{m=1}^\infty m^{1/2}J_m F.$$

Note that the sums consists of finitely many terms since $F \in \mathcal{P}(E)$. Both operators are well-defined and bounded by Theorem 6.3. Using the fact that

$$S^N R^N F = \sum_{m=N}^\infty J_m F,$$

we obtain by Lemma 6.2 and Theorem 6.3,

$$\|P(t)(I - J_0 - J_1 - \ldots - J_{N-1})F\|_{L^p(\Omega;E)}$$

$$= \Big\|\sum_{m=N}^\infty e^{-mt}J_m F\Big\|_{L^p(\Omega;E)} = \|S^N R^N P(t)F\|_{L^p(\Omega;E)}$$

$$= \|S^N e^{-Nt}P(t)R^N F\|_{L^p(\Omega;E)} \leq e^{-Nt}\|S\|^N\|R\|^N\|F\|_{L^p(\Omega;E)}.$$

$\square$

Using this lemma, the remainder of the proof of Meyer's multiplier Theorem [23] in the scalar case as given in [28, Theorem 1.4.2] extends verbatim to the vector-valued setting. It is even possible to allow operator-valued multipliers.

Theorem 6.5 (Meyer's Multiplier Theorem). *Let $1 < p < \infty$, let E be a UMD space, and let $(a_k)_{k=0}^\infty \subset \mathcal{L}(L^p(\Omega; E))$ be a sequence of bounded linear operators such that $\sum_{k=0}^\infty \|a_k\|_{\mathcal{L}(L^p(\Omega;E))} N^{-k} < \infty$ for some $N \geq 1$. If $(\phi(n))_{n \geq 0} \subset \mathcal{L}(L^p(\Omega; E))$ is a sequence of operators satisfying $\phi(n) := \sum_{k=0}^\infty a_k n^{-k}$ for $n \geq N$, then the operator T_ϕ defined by*

$$T_\phi F := \sum_{n=0}^\infty \phi(n) J_n F, \quad F \in \mathcal{P}(E)$$

extends to a bounded operator on $L^p(\Omega; E)$.

As a first application of the multiplier theorem we determine the spectrum of L. We start with a simple but useful lemma.

Lemma 6.6. *Let E be a K-convex Banach space, let $1 < p < \infty$, and let $F \in L^p(\Omega; E)$ such that $J_m F = 0$ for all $m \geq 0$. Then $F = 0$ in $L^p(\Omega; E)$.*

Proof. For $G \in \mathcal{P}(E^*)$ we have

$$\mathbb{E}\langle F, G \rangle = \mathbb{E}\langle F, \sum_{m \geq 0} J_m G \rangle = \mathbb{E}\langle \sum_{m \geq 0} J_m F, G \rangle = 0.$$

This implies the result, since $\mathcal{P}(E^*)$ is dense in $L^q(\Omega; E^*)$, hence weak*-dense in $L^p(\Omega; E)^*$.
$\square$

Proposition 6.7. *Let $1 < p < \infty$ and let E be a UMD space. Then*

$$\sigma(-L) = \{0, 1, 2, \ldots\}.$$

Moreover, every integer $m \geq 0$ is an eigenvalue of $-L$ and $\ker(m + L) = \mathcal{H}_m(E)$.

Proof. To prove that $\{0, 1, 2, \ldots\} \subset \sigma(-L)$ we take an integer $m \geq 0$ and a non-zero $F \in \mathcal{H}_m(E)$. Since $P(t)F = e^{-mt}F$ it follows that $F \in \mathsf{D}_p(L)$ and $(m + L)F = 0$, hence $m \in \sigma(-L)$ and $\ker(m + L) \supset \mathcal{H}_m(E)$.

To show the converse inclusion for the spectrum, take $\lambda \in \mathbb{C} \setminus \{0, 1, 2, \ldots\}$. To prove that $\lambda + L$ is injective, take $F \in \ker(\lambda + L)$. Since J_m is bounded for $m \geq 0$ by Remark 3.4 (UMD spaces are K-convex), it follows that $J_m L F = L J_m F = -m J_m F$. This implies that $(\lambda - m) J_m F = J_m (\lambda + L) F = 0$, hence $J_m F = 0$ for all $m \geq 0$, so that $F = 0$ by Lemma 6.6.

To prove surjectivity, we conclude from the Multiplier Theorem 6.5 that

$$R_\lambda := \sum_{m=0}^\infty \frac{1}{\lambda - m} J_m$$

extends to a bounded operator on $L^p(\Omega; E)$. Using the fact that L is closed, we infer that $(\lambda + L)R_\lambda = I$, hence $\lambda + L$ is surjective.

It remains to show that $\ker(m+L) \subset \mathcal{H}_m(E)$ for all $m \geq 0$. Take $F \in \ker(m+L)$. Since

$$(m - k) J_k F = (m + L) J_k F = J_k (m + L) F = 0$$

for all integers $k \geq 0$, we have $J_k F = 0$ for all $k \neq m$. This implies that $J_k(F - J_m F) = 0$, hence $F = J_m F \in \mathcal{H}_m(E)$ by Lemma 6.6.
$\square$

Next we give the general form of Meyer's inequalities in the language of γ-radonifying norms. This result is stated in a slightly different setting in [18, Theorem 1.17], but the proof given there contains a gap. More precisely, the last formula for the function ψ defined in [18, p.300] should be replaced by $\psi(t) = \frac{1}{2}e^{-t/2}(I_0(\frac{t}{2}) + I_1(\frac{t}{2}))$. This function however is not contained in $L^1(0, \infty)$; but this is needed to conclude the proof.

The proof given below uses Lemma 6.4, which is based on the first order Meyer inequalities from Theorem 6.3. This allows us to adapt the argument in the scalar case from [28, Theorem 1.5.1].

Theorem 6.8 (Meyer's inequalities, general case). *Let E be a UMD space, let $1 < p < \infty$ and let $n \geq 1$. Then $\mathsf{D}_p(C^n) = \mathbb{D}^{n,p}(\Omega; E)$, and for all $F \in \mathbb{D}^{n,p}(\Omega; E)$ we have*

$$
(6.5) \qquad
\begin{aligned}
\|D^n F\|_{L^p(\Omega;\gamma^n(H,E))} &\lesssim_{p,E,n} \|C^n F\|_{L^p(\Omega;E)} \\
&\lesssim_{p,E,n} \|F\|_{L^p(\Omega;E)} + \|D^n F\|_{L^p(\Omega;\gamma^n(H,E))}.
\end{aligned}
$$

Proof. The proof proceeds by induction. The case $n = 1$ has been treated in Theorem 6.3. Suppose that (6.5) holds for some $n \geq 1$. Using Lemma 6.2 and the fact that the operator $C^n(I-L)^{-n/2} = (-L)^{n/2}(I-L)^{-n/2}$ is bounded on $L^p(\Omega; E)$ we obtain by the induction hypothesis

$$
\begin{aligned}
\mathbb{E}\|D^{n+1} F\|^p_{\gamma^{n+1}(H,E)} &\lesssim_{p,E,n} \mathbb{E}\|C^n DF\|^p_{\gamma(H,E)} \lesssim_{p,E,n} \mathbb{E}\|(I-L)^{n/2} DF\|^p_{\gamma(H,E)} \\
&= \mathbb{E}\|DC^n F\|^p_{\gamma(H,E)} \eqsim_{p,E} \mathbb{E}\|C^{n+1} F\|^p_E.
\end{aligned}
$$

To prove the second inequality, we note that according to Remark 3.4,

$$
\|C^n(J_0 + \ldots + J_{n-1})F\|_p \lesssim_{p,E,n} \|F\|_p, \quad F \in L^p(\Omega; E).
$$

Therefore it suffices to show by induction that

$$
\|C^n F\|_{L^p(\Omega;E)} \lesssim_{p,E,n} \|D^n F\|_{L^p(\Omega;\gamma^n(H,E))}
$$

for all $F \in \mathcal{P}(E)$ with $J_0 F = \ldots = J_{n-1} F = 0$.

Let us assume that this statement holds for some $n \geq 1$ and take $F \in \mathcal{P}(E)$ satisfying $J_0 F = \ldots = J_n F = 0$. It follows from Lemma 6.4 that $(P(t))_{t \geq 0}$ restricts to a C_0-semigroup $(P_n(t))_{t \geq 0}$ on

$$
X_{n,p}(E) := \overline{\bigoplus_{m \geq n} \mathcal{H}_m(E)}^{L^p(\Omega;E)},
$$

satisfying the growth bound $\|P_n(t)\|_{\mathcal{L}(X_{n,p}(E))} \lesssim_{E,p,n} e^{-nt}$ for some constant K depending on n. Consequently (see e.g. [2, Proposition 3.8.2]), we have

$$
\|(\alpha - L)^{1/2} F\|_p \eqsim_{p,E} \|(\beta - L)^{1/2} F\|_p, \quad F \in X_{n,p}(E),
$$

for all $\alpha, \beta > -n$, and in particular is $(I-L)^{1/2} C^{-1}$ bounded on $X_{n,p}(E)$. Using Lemma 6.2 and the fact that $C^n DF \in X_{n,p}(\gamma(H,E))$, it follows that

$$
\begin{aligned}
\mathbb{E}\|C^{n+1} F\|^p_E &\eqsim_{p,E} \mathbb{E}\|DC^n F\|^p_{\gamma(H,E)} = \mathbb{E}\|(I-L)^{n/2} DF\|^p_{\gamma(H,E)} \\
&\lesssim_{p,E,n} \|(I-L)^{n/2} C^{-n}\|^p_{\mathcal{L}(X_{n,p}(\gamma(H,E)))} \mathbb{E}\|C^n DF\|^p_{\gamma(H,E)} \\
&\eqsim_{p,E,n} \mathbb{E}\|D^{n+1} F\|^p_{\gamma^{n+1}(H,E)}.
\end{aligned}
$$

$\square$

As an application of Meyer's inequalities we will show that $\gamma(H, E)$-valued Malliavin differentiable random variables are contained in the domain of the divergence operator δ. First we give the precise definition of δ.

Let E be a Banach space, fix an exponent $1 < p < \infty$ and let $\frac{1}{p} + \frac{1}{q} = 1$. For the moment let D denote the Malliavin derivative on $L^q(\Omega; E^*)$, which is a densely defined closed operator with domain $\mathbb{D}^{1,q}(\Omega; E^*)$ and taking values in $L^q(\Omega; \gamma(H, E^*))$. We let the domain $\mathsf{D}_p(\delta)$ consist of all $u \in L^p(\Omega; \gamma(H, E))$ for which there exists an $F_u \in L^p(\Omega; E)$ such that

$$\mathbb{E}[u, DG]_\gamma = \mathbb{E}\langle F_u, G\rangle \quad \text{for all } G \in \mathbb{D}^{1,q}(\Omega; E^*).$$

The function F_u, if it exists, is uniquely determined. We set

$$\delta(u) := F_u, \quad X \in \mathsf{D}_p(\delta).$$

In other words, δ is the part of the adjoint operator D^* in $L^p(\Omega; \gamma(H, E))$ which maps into $L^p(\Omega; E)$. Here we identify $L^p(\Omega; \gamma(H, E))$ and $L^p(\Omega; E)$ in a natural way with subspaces of $(L^q(\Omega; \gamma(H, E^*)))^*$ and $(L^q(\Omega; E^*))^*$ respectively.

The divergence operator δ is easily seen to be closed and densely defined. For more information we refer to [16].

In the next result we generalise some classical identities to the vector-valued setting.

Proposition 6.9. *Let E be a Banach space and let $1 < p < \infty$.*

(i) *For $u \in \mathbb{D}^{2,p}(\Omega; H) \otimes E$ we have*

$$D\delta(u) = u + \delta D(u).$$

If E is a UMD space, this identity holds for all $u \in \mathbb{D}^{2,p}(\Omega; \gamma(H, E))$.

(ii) *Let $1 < q, r < \infty$ be such that $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. Let $F \in \mathbb{D}^{1,p}(\Omega)$ and let $u \in L^q(\Omega; \gamma(H, E))$ be contained in $\mathsf{D}_q(\delta)$. Then $Fu \in \mathsf{D}_r(\delta)$ and*

$$\delta(Fu) = F\delta(u) - u(DF).$$

Proof. (i): For $v \in \mathbb{D}^{2,p}(\Omega; H)$ and $x \in E$ we obtain using the corresponding scalar result,

$$D\delta(v \otimes x) = (D\delta(v)) \otimes x = (v + \delta D(v)) \otimes x$$
$$= v \otimes x + \delta D(v \otimes x)$$

This proves the identity for all $u \in \mathbb{D}^{2,p}(\Omega; H) \otimes E$. The final statement is a consequence of Meyer's inequalities (Theorem 6.3).

(ii): For $G \in \mathcal{S}$ (see (5.1)) and $x^* \in E^*$ we obtain using the corresponding scalar result,

$$\begin{aligned}
\mathbb{E}[Fu, DG \otimes x^*]_\gamma &= \mathbb{E}[DG, Fx^*(u)]_H \\
&= \mathbb{E}\big(G\delta(Fx^*(u))\big) \\
&= \mathbb{E}\big(G\big(F\delta(x^*(u)) - [x^*(u), DF]_H\big)\big) \\
&= \mathbb{E}\langle F\delta(u), G \otimes x^*\rangle - \langle u(DF), G \otimes x^*\rangle.
\end{aligned}$$

It follows that $Fu \in \mathsf{D}_r(\delta)$ and the desired identity holds. $\square$

The proof of the following result is a variation of the proof of the scalar-valued result in [28, Proposition 1.5.4].

Proposition 6.10. *Let $1 < p < \infty$ and let E be a UMD space. The operator δ is bounded from $\mathbb{D}^{1,p}(\Omega; \gamma(H, E))$ into $L^p(\Omega; E)$.*

Proof. Let $u \in \mathbb{D}^{1,p}(\Omega; \gamma(H, E))$ and $G \in \mathcal{P}(E^*)$. Using Theorem 5.3 we find that $\|DJ_1G\|_p \eqsim_p \|J_1G\|_p$, and therefore

$$\begin{aligned}
(6.6) \qquad \mathbb{E}[u, D(J_0 + J_1)G]_\gamma &\leq \|u\|_{L^p(\Omega;\gamma(H,E))} \|D(J_0 + J_1)G\|_{L^q(\Omega;\gamma(H,E^*))} \\
&\lesssim_{p,E} \|u\|_{L^p(\Omega;\gamma(H,E))} \|G\|_{L^q(\Omega;E^*)}.
\end{aligned}$$

Now we assume that $J_0 G = J_1 G = 0$. By the Multiplier Theorem 6.5 the operator

$$T := \sum_{m=2}^{\infty} \frac{m}{m-1} J_m$$

is bounded on $L^p(\Omega; \gamma(H, E))$. By Lemma 6.4 the operator L^{-1} is well defined on $X_{1,p}(E)$, where we use the notation from the proof of Theorem 6.8. This justifies the use of L^{-1} in the following computation. Using Lemma 6.1 and Theorem 6.8 we obtain

(6.7)
$$\begin{aligned}
\mathbb{E}[u, DG]_\gamma = \mathbb{E}[u, LL^{-1}DG]_\gamma &= \mathbb{E}[Du, DL^{-1}DG]_\gamma \\
&\leq \|Du\|_{L^p(\Omega; \gamma^2(H,E))} \|DL^{-1}DG\|_{L^q(\Omega; \gamma^2(H,E^*))} \\
&= \|Du\|_{L^p(\Omega; \gamma^2(H,E))} \|D^2 L^{-1} TG\|_{L^q(\Omega; \gamma^2(H,E^*))} \\
&\lesssim_{p,E} \|Du\|_{L^p(\Omega; \gamma^2(H,E))} \|G\|_{L^q(\Omega; E^*)}.
\end{aligned}$$

Combining (6.6) and (6.7) we conclude that for all $G \in \mathcal{P}(E^*)$ we have

$$\mathbb{E}[u, DG]_\gamma \lesssim_{p,E} \|u\|_{\mathbb{D}^{1,p}(\Omega; E)} \|G\|_{L^q(\Omega; E^*)}.$$

It follows that there exists an $F_u \in (L^q(\Omega; E^*))^*$ such that $\mathbb{E}[u, DG]_\gamma = \mathbb{E}\langle F_u, G\rangle$. Since E is a UMD space, we conclude that $F_u \in L^p(\Omega; E)$ and we obtain the desired result. $\qquad\square$

For $1 \leq p < \infty$ we define the vector space of exact E-valued processes as

$$L_e^p(\Omega; \gamma(H, E)) = \{DF : F \in \mathbb{D}^{1,p}(\Omega; E)\}.$$

The next result is concerned with the representation of random variables as divergences of exact processes. This representation may be regarded as a Clark-Ocone type formula. In [16] we provide a related result stating, loosely speaking, that Malliavin differentiable random variables can be represented as E-valued stochastic Itô integrals in the sense of [24].

Proposition 6.11. *Let E be a UMD space, let $1 < p < \infty$ and let $F \in L^p(\Omega; E)$. Then $U := DL^{-1}(F - \mathbb{E}(F))$ is the unique element in $L_e^p(\Omega; \gamma(H, E))$ satisfying*

$$F = \mathbb{E}(F) + \delta(U).$$

Proof. By an easy computation we see that

(6.8)
$$F = \mathbb{E}(F) + \delta D(L^{-1}(F - \mathbb{E}(F)))$$

for all $F \in \mathcal{P}(E)$. It follows from Lemma 6.4 (or Proposition 6.7) that L^{-1} is well-defined and bounded on $\{G \in L^p(\Omega; E) : \mathbb{E}(G) = 0\}$. Meyer's inequalities imply that D is bounded from $\mathsf{D}_p(L)$ into $\mathbb{D}^{1,p}(\Omega; \gamma(H, E))$, and by Proposition 6.10 we have that δ is bounded from $\mathbb{D}^{1,p}(\Omega; \gamma(H, E))$ into $L^p(\Omega; E)$. Using these facts and an approximation argument with elements from $\mathcal{P}(E)$ we conclude that the right hand side of (6.8) is well-defined for all $F \in L^p(\Omega; E)$, and the identity remains valid.

To prove uniqueness, suppose that $F = \mathbb{E}(F) + \delta(DF')$ for some $F' \in \mathbb{D}^{1,p}(\Omega; E)$ with $DF' \in \mathsf{D}_p(\delta)$, and put $G := F' - L^{-1}(F - \mathbb{E}(F))$. Then $\delta DG = 0$, hence $\langle G, LP\rangle = 0$ for all polynomials $P \in \mathcal{P}(E^*)$. In particular, for all $m \geq 1$ and all $P \in \mathcal{P}(E^*) \cap \mathcal{H}_m(E^*)$ one has $\langle G, mP\rangle = 0$, and since $\mathcal{P}(E^*) \cap \mathcal{H}_m(E^*)$ is dense in $\mathcal{H}_m(E^*)$, we have $\langle J_m G, \widetilde{F}\rangle = \langle G, J_m \widetilde{F}\rangle = 0$ for all $\widetilde{F} \in L^q(\Omega; E^*)$. It follows that $J_m G = 0$ for all $m \geq 1$, which implies $J_m(G - J_0 G) = 0$ for all $m \geq 0$. We conclude that $G = J_0 G$ by Lemma 6.6, hence $F' = L^{-1}(F - \mathbb{E}F) + x$ for some $x \in E$. We conclude that $DF' = DL^{-1}(F - \mathbb{E}F)$, which is the desired identity. $\qquad\square$

We conclude the paper with an application of the vector-valued Malliavin calculus developed in this work. We give a new proof of Theorem 2.1(1) under the additional assumption that E is a UMD space, which is based on Meyer's inequalities. This approach seems to be new even in the scalar-valued case.

Theorem 6.12. *Let E be a UMD space, let $1 < p < \infty$, and define F and $\widetilde{F}$ as in Theorem 2.1(1). Then we have*

$$\|F\|_p \eqsim_{p,m,E} \|\widetilde{F}\|_p.$$

Proof. We argue as in the proof of Theorem 5.3. By (5.4) we have

$$\mathbb{E}\|F\|_E^p = \mathbb{E}\Big\| \sum_{|\boldsymbol{i}|=m,|\boldsymbol{i}|_\infty \leq n} \frac{\boldsymbol{i}!^{1/2}}{m!^{1/2}} \Psi_{\boldsymbol{i}} x_{\boldsymbol{i}} \Big\|_E^p$$

and according to (5.5),

$$\mathbb{E}\|DF\|_{\gamma(H,E)}^p \eqsim_p \widetilde{\mathbb{E}}\mathbb{E}\Big\| \frac{m^{1/2}}{(m-1)!^{1/2}} \sum_{|\boldsymbol{i}|=m-1,|\boldsymbol{i}|_\infty \leq n} \frac{\boldsymbol{i}!^{1/2}}{(m-1)!^{1/2}} \Psi_{\boldsymbol{i}} \Big(\sum_{k=1}^n \widetilde{\gamma}_k x_{(\boldsymbol{i},k)} \Big) \Big\|_E^p.$$

Noting that $CF = m^{1/2}F$, Meyer's inequalities imply that

$$\mathbb{E}\Big\| \sum_{|\boldsymbol{i}|=m,|\boldsymbol{i}|_\infty \leq n} \frac{\boldsymbol{i}!^{1/2}}{m!^{1/2}} \Psi_{\boldsymbol{i}} x_{\boldsymbol{i}} \Big\|_E^p$$

$$\eqsim_{p,m,E} \widetilde{\mathbb{E}}\mathbb{E}\Big\| \sum_{|\boldsymbol{i}|=m-1,|\boldsymbol{i}|_\infty \leq n} \frac{\boldsymbol{i}!^{1/2}}{(m-1)!^{1/2}} \Psi_{\boldsymbol{i}} \Big(\sum_{k=1}^n \widetilde{\gamma}_k x_{(\boldsymbol{i},k)} \Big) \Big\|_E^p.$$

The desired result is obtained by repeating this procedure $m-1$ times. $\qquad\square$

References

[1] M. A. Arcones and E. Giné, *On decoupling, series expansions, and tail behavior of chaos processes*, J. Theoret. Probab. **6** (1993), no. 1, 101–122.

[2] W. Arendt, C. J. K. Batty, M. Hieber, and F. Neubrander, *Vector-valued Laplace transforms and Cauchy problems*, Monographs in Mathematics, vol. 96, Birkhäuser Verlag, Basel, 2001.

[3] Y. Bakhtin and J. C. Mattingly, *Malliavin calculus for infinite-dimensional systems with additive noise*, J. Funct. Anal. **249** (2007), no. 2, 307–353.

[4] V. I. Bogachev, *Gaussian measures*, Mathematical Surveys and Monographs, vol. 62, American Mathematical Society, Providence, RI, 1998.

[5] D. L. Burkholder, *Martingales and singular integrals in Banach spaces*, Handbook of the geometry of Banach spaces, Vol. I, North-Holland, Amsterdam, 2001, pp. 233–269.

[6] R. A. Carmona and M. R. Tehranchi, *Interest rate models: an infinite dimensional stochastic analysis perspective*, Springer Finance, Springer-Verlag, Berlin, 2006.

[7] V. H. de la Peña and E. Giné, *Decoupling*, Probability and its Applications (New York), Springer-Verlag, New York, 1999.

[8] V. H. de la Peña and S. J. Montgomery-Smith, *Decoupling inequalities for the tail probabilities of multivariate U-statistics*, Ann. Probab. **23** (1995), no. 2, 806–816.

[9] T. Figiel and N. Tomczak-Jaegermann, *Projections onto Hilbertian subspaces of Banach spaces*, Israel J. Math. **33** (1979), no. 2, 155–171.

[10] M. Hairer, J. C. Mattingly, and É. Pardoux, *Malliavin calculus for highly degenerate 2D stochastic Navier-Stokes equations*, C. R. Math. Acad. Sci. Paris **339** (2004), no. 11, 793–796.

[11] T. P. Hytönen, *Littlewood-Paley-Stein theory for semigroups in UMD spaces*, Rev. Mat. Iberoam. **23** (2007), no. 3, 973–1009.

[12] N. Ikeda and S. Watanabe, *Stochastic differential equations and diffusion processes*, second ed., North-Holland Mathematical Library, vol. 24, North-Holland Publishing Co., Amsterdam, 1989.

[13] N. J. Kalton and L. Weis, *The H^∞-functional calculus and square function estimates*, in preparation.

[14] S. Kwapień, *Decoupling inequalities for polynomial chaos*, Ann. Probab. **15** (1987), no. 3, 1062–1071.

[15] S. Kwapień and W. A. Woyczyński, *Random series and stochastic integrals: single and multiple*, Probability and its Applications, Birkhäuser Boston Inc., Boston, MA, 1992.

[16] J. Maas and J. M. A. M. van Neerven, *A Clark-Ocone formula in UMD Banach spaces*, Electron. Commun. Probab. **13** (2008), 151–164.

[17] P. Malliavin, *Stochastic calculus of variation and hypoelliptic operators*, Proceedings of the International Symposium on Stochastic Differential Equations (Res. Inst. Math. Sci., Kyoto Univ., Kyoto, 1976) (New York), Wiley, 1978, pp. 195–263.

[18] P. Malliavin and D. Nualart, *Quasi-sure analysis and Stratonovich anticipative stochastic differential equations*, Probab. Theory Related Fields **96** (1993), no. 1, 45–55.

[19] E. Mayer-Wolf and M. Zakai, *The divergence of Banach space valued random variables on Wiener space*, Probab. Theory Related Fields **132** (2005), no. 2, 291–320.

[20] _____, *Erratum: "The divergence of Banach space valued random variables on Wiener space" [Probab. Theory Related Fields* **132** *(2005), no. 2, 291–320; MR2199294]*, Probab. Theory Related Fields **140** (2008), no. 3-4, 631–633.

[21] T. R. McConnell and M. S. Taqqu, *Decoupling inequalities for multilinear forms in independent symmetric random variables*, Ann. Probab. **14** (1986), no. 3, 943–954.

[22] _____, *Decoupling of Banach-valued multilinear forms in independent symmetric Banach-valued random variables*, Probab. Theory Related Fields **75** (1987), no. 4, 499–507.

[23] P.-A. Meyer, *Transformations de Riesz pour les lois gaussiennes*, Seminar on probability, XVIII, Lecture Notes in Math., vol. 1059, Springer, Berlin, 1984, pp. 179–193.

[24] J. M. A. M. van Neerven, M. C. Veraar, and L. Weis, *Stochastic integration in UMD Banach spaces*, Ann. Probab. **35** (2007), no. 4, 1438–1478.

[25] _____, *Stochastic evolution equations in UMD Banach spaces*, J. Funct. Anal. **255** (2008), no. 4, 940–993.

[26] J. M. A. M. van Neerven and L. Weis, *Stochastic integration of functions with values in a Banach space*, Studia Math. **166** (2005), no. 2, 131–170.

[27] _____, *Stochastic integration of operator-valued functions with respect to Banach space-valued Brownian motion*, Potential Anal. **29** (2008), no. 1, 65–88.

[28] D. Nualart, *The Malliavin calculus and related topics*, second ed., Probability and its Applications (New York), Springer-Verlag, Berlin, 2006.

[29] G. Pisier, *Some results on Banach spaces without local unconditional structure*, Compositio Math. **37** (1978), no. 1, 3–19.

[30] _____, *Holomorphic semigroups and the geometry of Banach spaces*, Ann. of Math. (2) **115** (1982), no. 2, 375–392.

[31] _____, *Riesz transforms: a simpler analytic proof of P.-A. Meyer's inequality*, Séminaire de Probabilités, XXII, Lecture Notes in Math., vol. 1321, Springer, Berlin, 1988, pp. 485–501.

[32] _____, *The volume of convex bodies and Banach space geometry*, Cambridge Tracts in Mathematics, vol. 94, Cambridge University Press, Cambridge, 1989.

[33] I. Shigekawa, *Sobolev spaces of Banach-valued functions associated with a Markov process*, Probab. Theory Related Fields **99** (1994), no. 3, 425–441.

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