NON-TANGENTIAL MAXIMAL FUNCTIONS AND CONICAL SQUARE FUNCTIONS WITH RESPECT TO THE GAUSSIAN MEASURE

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Abstract. We study, in L^1 with respect to the Gaussian measure, non-tangential maximal functions and conical square functions associated with the Ornstein-Uhlenbeck operator by developing a set of techniques which allow us, to some extent, to compensate for the non-doubling character of the gaussian measure. This complements recent results on gaussian Hardy spaces due to Mauceri and Meda.

1. Introduction

Gaussian harmonic analysis, understood as the study of objects associated with the Gaussian measure

$$d\gamma(x) = \pi^{-n/2} \exp(-|x|^2)\, dx$$

on $\mathbb{R}^n$, and the Ornstein-Uhlenbeck operator

$$Lf(x) = -\tfrac{1}{2}\Delta f(x) + x \cdot \nabla f(x)$$

on function spaces such as $L^2(\mathbb{R}^n;\gamma)$, has recently gained new momentum following the development, by Mauceri and Meda [9], of an atomic Hardy space $H^1_{\rm at}(\mathbb{R}^n;\gamma)$, on which various functions of L give rise to bounded operators. Harmonic analysis in $L^p(\gamma)$ has been relatively well established for some time, with results such as the boundedness of Riesz transforms going back to the work of Meyer and Pisier in the 1980's. The $p = 1$ case, however, has always proven to be difficult. Over the last 30 years, some weak type $(1, 1)$ estimates have been obtained, while others have been disproved (see the survey [12]). The proofs of these results have relied on subtle decompositions and estimates of kernels. Until the seminal Mauceri-Meda paper appeared in 2007, a large part of euclidean harmonic analysis, such as end point estimates using Hardy and BMO spaces, seemed to have no gaussian counterpart. Gaussian harmonic analysis in $L^2(\gamma)$ is relatively straightforward given the fact that the Ornstein-Uhlenbeck operator is diagonal with respect to the Hermite polynomials basis. The $L^p(\gamma)$ case, with $1 < p < \infty$, is harder but still manageable through kernel estimates. The end points $p = 1$ and $p = \infty$, however, usually require techniques such as Whitney coverings and Calderón-Zygmund decompositions, for which the non-doubling nature of the gaussian measure, has, so far, not been overcome. Mauceri and Meda's paper [9], though, indicates a possible way. They introduced the notion of *admissible balls*; these are balls $B(x, r)$ with the property that $r \le a \min(1, \frac{1}{|x|})$ for some fixed admissibility parameter $a > 0$. On

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these admissible balls, the gaussian measure turns out to be doubling. The idea is then to follow classical arguments using admissible balls only. This is easier said than done. Indeed, admissible balls need to be very small when their centre is far away from the origin, whereas tools such as Whitney decompositions of open sets require the size of balls to be comparable to their distance to the boundary of the set, hence possibly very large. This may be why, although it contains many breakthrough results, Mauceri and Meda's paper [9] does not yet give a full theory of H^1 and BMO spaces for the gaussian measure. For instance, the boundedness of key operators such as maximal functions, conical square functions (area integrals), and above all Riesz transforms, is still missing. In fact, while this paper was in its final stages, Mauceri, Meda, and Sjögren have posted a result [10] proving that Riesz transforms (more precisely *some* Riesz transforms, see their paper for the details) are bounded on the Mauceri-Meda Hardy space only in dimension one. This suggests that a correct $H^1(\gamma)$ space should be a modification of theirs.

In this paper, we take another step towards a satisfying $H^1(\gamma)$ theory by studying, in $L^1(\gamma)$, non-tangential maximal functions and square functions. These are gaussian analogues of the sublinear operators which, in the euclidean setting, are the cornerstones of the real variable theory of H^1. In the gaussian context, they were first introduced by Fabes and Forzani, who studied a gaussian counterpart of the Lusin area integral. Its L^p-boundedness was shown subsequenly by Forzani, Scotto, and Urbina [6]. Our definition is an averaged version of a non-tangential maximal function from a subsequent paper of Pineda and Urbina [11]. The additional averaging adds some technical difficulties, but experience has shown (see e.g. [7]) that such averaging can be helpful in Hardy space theory and its applications (to boundary value problems for instance).

Here we prove a change of aperture formula for the maximal function in the spirit of one of the key estimates of Coifman, Meyer and Stein [3]. We then show that the non-tangential square function is controlled by the non-tangential maximal function. Such estimates are central in Hardy space theory (see for instance [4, 5]). However, many pieces of the puzzle are still missing, and future work will need to focus on the reverse estimates, along with the closely related issue of molecular decompositions. In this direction we have developed gaussian Whitney covering techniques and studied gaussian tent spaces in [8].

Now let us state the main result of this paper. For test functions $u \in C_c(\mathbb{R}^n)$ and $M_1, M_2 > 0$ we consider the *non-tangential maximal function with parameters* M_1, M_2

$$T^*_{(M_1,M_2)}u(x) := \sup_{(y,t)\in\Gamma_x^{(M_1,M_2)}(\gamma)} \left(\frac{1}{\gamma(B(y,M_1t))}\int_{B(y,M_1t)} |e^{-t^2L}u(z)|^2\,d\gamma(z)\right)^{\frac12},$$

where

$$\Gamma_x^{(M_1,M_2)}(\gamma) := \Big\{(y,t)\in\mathbb{R}^n\times(0,\infty)\colon |y-x| < M_1t < M_2\min\big\{1,\frac{1}{|x|}\big\}\Big\}$$

is the *admissible cone with parameters* M_1, M_2 based at the point $x \in \mathbb{R}^n$. The parameter M_1 is called the aperture of the cone, while $\frac{M_2}{M_1}$ is an admissibility parameter for the balls involved.

The main result of this paper reads as follows.

Theorem 1.1. *For each $u \in L^1(\gamma)$, the square function defined by*

$$Su(x) = \Big(\int_{\Gamma_x^{(1,1)}(\gamma)} \frac{1}{\gamma(B(y,t))} |t\nabla e^{-t^2 L} u(y)|^2 \, d\gamma(y) \, \frac{dt}{t} \Big)^{\frac{1}{2}}$$

is controlled by the non-tangential maximal function in the following sense: there exists an admissibility constant $a \geq 1$, independent of u, such that

$$\|Su\|_{L^1(\gamma)} \lesssim \|T^*_{(1,a)} u\|_{L^1(\gamma)}.$$

2. Covering lemmas

In this section we introduce partitions of $\mathbb{R}^n$ into admissible dyadic cubes and use them to prove two covering lemmas which will be needed later on.

We begin with a brief discussion of admissible balls. Let

$$m(x) := \min \Big\{ 1, \frac{1}{|x|} \Big\}, \quad x \in \mathbb{R}^n.$$

For $a > 0$ we define

$$\mathscr{B}_a := \big\{ B(x,r) \colon x \in \mathbb{R}^n, \ 0 \leq r \leq am(x) \big\}.$$

The balls in $\mathscr{B}_a$ are said to be *admissible at scale a*. It is a fundamental observation of Mauceri and Meda [9] that admissible balls enjoy a doubling property:

Lemma 2.1 (Doubling property)**.** *Let $a, \tau > 0$. There exists a constant $d = d_{\alpha,\tau,n}$, depending only on a, τ, and the dimension n, such that if $B_1 = B(c_1, r_1) \in \mathscr{B}_a$ and $B_2 = B(c_2, r_2)$ have non-empty intersection and $r_2 \leq \tau r_1$, then*

$$\gamma(B_2) \leq d\gamma(B_1).$$

In particular this lemma implies that for all $a > 0$ there exists a constant $d' = d'_a$ such that for all $B(x,r) \in \mathscr{B}_a$ we have

$$\gamma(B(x,2r)) \leq d' \gamma(B(x,r)).$$

The first part of the next lemma, which is taken from [8], says, among other things, that if $B(x,r) \in \mathscr{B}_a$ and $|x-y| < br$, then $B(y,r) \in \mathscr{B}_c$ for some constant $c = c_{a,b}$ which depends only on a and b.

Lemma 2.2. *Let $a, b > 0$ be given.*

(i) *If $r \leq am(x)$ and $|x-y| < br$, then $r \leq c_{a,b} m(y)$, where $c_{a,b} := a(1+ab)$.*
(ii) *If $|x-y| < bm(x)$, then $m(x) \leq (1+b)m(y)$ and $m(y) \leq (2+2b)m(x)$.*

Lemma 2.3. *Let $a, b > 0$ be given. If $B(x,s) \in \mathcal{B}_a$ and $B(y,t) \in \mathcal{B}_b$ have non-empty intersection, then*

$$|x-y| < k \min\{m(x), m(y)\},$$

where $k = k_{a,b} = \max\big\{ 2a \max\{a+b, 1\} + b, \ 2b \max\{a+b, 1\} + a \big\}$.

Proof. If $|y| \leq 1$, then $m(x) \leq 1 = m(y)$.

If $|y| \geq 1$ and $|y| \leq 2(a+b)$, then

$$m(x) \leq 1 \leq 2(a+b) \frac{1}{|y|} = 2(a+b) m(y).$$

Suppose next that $|y| \geq 1$ and $|y| = C(a+b)$ with $C > 2$. Then $|x| \geq |y| - s - t \geq |y| - a - b = (C-1)(a+b)$, so

$$m(x) \leq \frac{1}{|x|} \leq \frac{C}{C-1}\frac{1}{C(a+b)} = \frac{C}{C-1}\frac{1}{|y|} = \frac{C}{C-1}m(y) \leq 2m(y).$$

Hence, in each of these three cases,

$$|x-y| < s+t \leq am(x) + bm(y) \leq (2a\max\{a+b,1\}+b)m(y).$$

By symmetry, the same argument yields $|x-y| < (2b\max\{a+b,1\}+a)m(x)$, and the result follows. □

For $k \geq 0$ let Δ_k be the set of dyadic cubes at scale k, i.e.,

$$\Delta_k = \{2^{-k}(x + [0,1)^n) \colon x \in \mathbb{Z}^n\}.$$

Following [8], in the Gaussian we only use cubes whose diameter depends on another parameter l, which keeps track of the distance from the ball to the origin. More precisely, define the *layers*

$$L_0 = [-1,1)^n, \quad L_l = [-2^l, 2^l)^n \setminus [-2^{l-1}, 2^{l-1})^n \quad (l \geq 1),$$

and define, for $k, l \geq 0$,

$$\Delta^\gamma_{k,l} = \{Q \in \Delta_{l+k} \colon Q \subseteq L_l\}, \quad \Delta^\gamma_k = \bigcup_{l \geq 0} \Delta^\gamma_{k,l}, \quad \Delta^\gamma = \bigcup_{k \geq 0} \Delta^\gamma_k.$$

Note that if $Q \in \Delta^\gamma_k$ with $Q \subseteq L_l$, then its centre x has norm $2^{l-1} \leq |x| \leq 2^l\sqrt{n}$ and $\operatorname{diam}(Q) = 2^{-k-l}\sqrt{n}$.

We denote by $\alpha \circ Q$ the cube with the same centre as Q and α times its side-length; similar notation is used for balls. Cubes in Δ^γ enjoy the following doubling property:

Lemma 2.4. *Let $\alpha > 0$. There exists a constant $C_{\alpha,n}$, depending only on α and the dimension n, such that for every cube $Q \in \Delta^\gamma$, we have*

$$\gamma(\alpha \circ Q) \leq C_{\alpha,n}\gamma(Q).$$

Proof. Without loss of generality we may assume that $\alpha > 1$. Let $Q \in \Delta^\gamma_{k,l}$ with center y and side-length $2s$. Set $B = B(y,s)$ and note that $B \subseteq Q$. Moreover, we have $\alpha \circ Q \subseteq \alpha\sqrt{n} \circ B$. Since, if $|y| > 1$,

$$2s = \frac{\operatorname{diam}(Q)}{\sqrt{n}} = 2^{-k-l} \leq 2^{-l} \leq \frac{\sqrt{n}}{|y|} = \sqrt{n}m(y),$$

and, if $|y| \leq 1$,

$$2s = 2^{-k-l} \leq 1 \leq \sqrt{n}m(y),$$

it follows that $B \in \mathscr{B}_{\sqrt{n}/2}$. Using the doubling property for admissible balls from Lemma 2.1 we now obtain

$$\gamma(\alpha \circ Q) \leq \gamma(\alpha\sqrt{n} \circ B) \leq C_{\alpha,n}\gamma(B) \leq C_{\alpha,n}\gamma(Q).$$

□

Lemma 2.5. *There exists a constant $K \geq 0$, depending only on the dimension n, such that any measurable set $E \subseteq \mathbb{R}^n$ satisfying $\gamma(E) > 0$ admits a covering $(B_k)_{k\geq 1}$ with admissible balls from $\mathscr{B}_1$ such that*

$$\sum_{k\geq 1} \gamma(B_k) \leq K\gamma(E).$$

Proof. By the outer regularity of γ we find an open set $O \supseteq E$ such that $\gamma(O) \leq 2\gamma(E)$. Thus it remains to prove the lemma for a non-empty open set O.

For each $x \in O$ let Q_x be the largest cube in Δ^γ such that $x \in Q_x \subseteq O$. Note that for any two $x, y \in O$ we either have $Q_x = Q_y$ or $Q_x \cap Q_y = \emptyset$. It follows that we can find a sequence $(x_k)_{k\geq 1}$ such that the cubes Q_{x_k} are disjoint and cover O. Let c_k be the centre of Q_{x_k} and let $d_k = \sqrt{n}r_k$, where r_k is the side-length of Q_{x_k}. The balls $B(c_k, \frac{1}{2}d_k)$ cover O. We claim that each of those balls belongs to $\mathscr{B}_{\frac{1}{2}n}$. Indeed, if Q_{x_k} is in layer L_l and $|c_k| \geq 1$, then

$$d_k = \sqrt{n}r_k \leq 2^{-l}\sqrt{n} \leq \frac{n}{|c_k|} = nm(c_k).$$

If $|c_k| < 1$, then $Q_{x_k} \subseteq L_0$ and $r_k \leq 1$, so

$$d_k = \sqrt{n}r_k \leq \sqrt{n} = \sqrt{n}m(c_k) \leq nm(c_k).$$

This proves the claim. Moreover, from $B(c_k, \frac{1}{2}d_k) \subseteq \sqrt{n} \circ Q_{x_k}$ and the doubling property for admissible cubes in Lemma 2.4, we see that

$$\sum_{k\geq 1} \gamma(B(c_k, \tfrac{1}{2}d_k)) \leq \sum_{k\geq 1} \gamma(\sqrt{n} \circ Q_{x_k}) \leq C_n \sum_{k\geq 1} \gamma(Q_{x_k}) = C_n\gamma(O),$$

where C_n is a constant depending only on n.

We claim that there is a number N_n such that each ball $B = B(c_B, r_B)$ in $\mathscr{B}_{\frac{1}{2}n}$ can be covered by at most N_n balls in $\mathscr{B}_1$. Once this has been shown, the lemma now follows since the balls B' used in this covering satisfy $\gamma(B') \leq K_n\gamma(B)$ for some constant K_n depending only on n by the doubling property.

To prove the claim we may assume that $r_B = \frac{1}{2}nm(c_B)$. We distinguish two cases.

Case 1 – If $|c_B|^2 \leq \frac{1}{2}n + \frac{1}{2}$, then B is contained in the set $\{x \in \mathbb{R}^n : |x| \leq \sqrt{\frac{1}{2}n + \frac{1}{2}} + \frac{1}{2}n\}$ and this set can be covered with finitely many balls – the number of which depends only on n – in $\mathscr{B}_1$.

Case 2 – When $|c_B|^2 > \frac{1}{2}n + \frac{1}{2}$ we argue as follows. Clearly, B can be covered with finitely many balls – the number depends only on n – of radius r_B/n and intersecting B. We will check that such balls belong to $\mathscr{B}_1$. Let $B' = B(c', r')$ be such a ball. Using the estimate

$$|c'| \leq |c_B| + r_B + r' \leq |c_B| + r_B + \frac{r_B}{n} = |c_B| + (\tfrac{1}{2}n + \tfrac{1}{2})m(c_B) = |c_B| + \frac{\frac{1}{2}n + \frac{1}{2}}{|c_B|},$$

we obtain

$$r' = \frac{r_B}{n} = \frac{m(c_B)}{2} = \frac{1}{2|c_B|} \leq \frac{1}{|c_B| + (\frac{1}{2}n + \frac{1}{2})/|c_B|} \leq \frac{1}{|c'|},$$

where the second last inequality follows from $(\frac{1}{2}n + \frac{1}{2})/|c_B|^2 \leq 1$. Since also $r' = \frac{r_B}{n} = \frac{1}{2}m(c_B) \leq 1$, it follows that $r' \leq m(c')$. This finishes the proof of the claim. □

Lemma 2.6. *Let $F \subseteq \mathbb{R}^n$ be a non-empty set, let $a, b > 0$ be fixed, and let*

$$O := \{x \in \mathbb{R}^n : 0 < d(x, F) \le am(x)\}.$$

There exists a sequence $(x_k)_{k\ge1}$ in O with the following properties:

(i) $O \subseteq \bigcup_{k\ge1} B(x_k, bd(x_k, F))$;
(ii) $\sum_{k\ge1} \gamma(B(x_k, d(x_k, F))) \lesssim \gamma(O)$ *with a constant depending only on a, b, and the dimension n.*

Proof. We split the proof into four steps.

Step 1 – We begin by noting that if the lemma holds for a certain pair (a, b), then it also holds for all pairs (a, b') with $b' > 0$. This is trivial for $b' \ge b$, and for $0 < b' < b$ this follows from the fact that any ball of radius br may be covered by N balls of radius $b'r$, where N depends only on the ratio b/b' and the dimension n. Thus it suffices to prove the lemma for one specific value of b. We will choose $b = \frac{1}{4}$ because this is the value to which we shall apply the lemma.

Step 2 – Next we shall prove that without loss of generality we may assume that $a \ge a_0$, where $a_0 > 0$ is some fixed number. For this purpose suppose that $0 < a \le a'$, set $a'' := \min\{a, 4\}$, and consider the sets

$$O' := \Big\{z \in \mathbb{R}^n : 0 < d(z, F) \le a'm(z)\Big\},$$
$$O'' := \Big\{z \in \mathbb{R}^n : 0 < d(z, F) \le a''m(z)\Big\}.$$

The claim will be proved once we show that $\gamma(O') \lesssim \gamma(O'')$ with constant depending only on a, a', and n. This, in turn, shows that it suffices to prove an estimate $\gamma(O') \lesssim \gamma(O)$ for any two numbers $0 < a \le a'$ with $a \le 4$.

To prove this inequality we will show that there exists a number M_n, depending only on a, a', and n, and sequence of disjoint cubes $Q_i \in \Delta^\gamma$ contained in O such that

$$O' \setminus O^\circ \subseteq \bigcup_i M_n \circ Q_i.$$

Once this has been shown the claim follows from Lemma 2.4:

$$\gamma(O' \setminus O^\circ) \le \sum_i \gamma(M_n \circ Q_i) \lesssim \sum_i \gamma(Q_i) = \gamma\big(\bigcup_i Q_i\big) \le \gamma(O)$$

and consequently $\gamma(O') \lesssim \gamma(O)$.

Every point $x \in O^\circ$, the interior of O, belongs to some maximal cube $Q_x \in \Delta^\gamma$ with the property that $3 \circ Q_x$ is entirely contained in O°. Since any two such maximal cubes are either equal or disjoint, we may select a sequence (x_i) in O° such that the maximal cubes $Q_{x_i} \in \Delta^\gamma$ are disjoint and cover O°. We will show that these cubes have the desired property for a suitable choice of M_n.

Fix $y \in O' \setminus O^\circ$. Then $d(y, F) = cm(y)$ for some $a \le c \le a'$. Choose $f \in \overline{F}$ with $d(f, y) = cm(y)$ (this is possible since $\overline{F} \cap \{z : d(y, z) \le 2cm(y)\}$ is compact and non-empty). By a continuity argument there exists $0 < \lambda < 1$ such that for $g := (1 - \lambda)f + \lambda y$ we have $d(g, F) = \frac{1}{4}am(g)$. From $d(y, F) = d(y, f)$ and the triangle inequality one easily deduces that also $d(g, F) = d(g, f)$, and therefore we have $d(g, f) = \frac{1}{4}am(g)$. Then $g \in O$ and $|y - g| = (1 - \lambda)|y - f| = (1 - \lambda)cm(y)$. Choose the index i such that $g \in Q_{x_i}$ and let c_i be the centre of Q_{x_i}. Then

$|g - c_i| < \frac{1}{4}a\sqrt{n}m(g)$, since otherwise the side-length of Q_{x_i} would be at least $\frac{1}{2}am(g)$ and then $3 \circ Q_{x_i}$ would contain the point $f \notin O^\circ$. It follows that

$$|c_i - y| < \frac{1}{4}a\sqrt{n}m(g) + (1-\lambda)cm(y) < \frac{1}{4}a\sqrt{n}m(g) + a'm(y).$$

On the other hand, as we will show next, the side-length of $3 \circ Q_{x_i}$ is at least $\frac{1}{7\sqrt{n}}am(g)$.

Suppose the side-length of $3 \circ Q_{x_i}$ were less than $\frac{1}{7\sqrt{n}}am(g)$. Then the side-length of Q_{x_i} is less than $\frac{1}{21\sqrt{n}}am(g)$. We claim that $9 \circ Q_{x_i}$ is still contained in O°. Suppose for the moment we knew this. It would mean that Q_{x_i} is contained in a dyadic cube Q of twice the diameter which satisfies $3 \circ Q \subseteq 9 \circ Q_{x_i'} \subseteq O^\circ$. This contradicts the maximality of Q_{x_i}, since we also have $Q \in \Delta^\gamma$. The latter can be seen as follows. Choose $k, l \geq 0$ such that $Q_{x_i} \in \Delta^\gamma_{k,l}$. The side-length of Q_{x_i} is then 2^{-l-k}. From $\text{diam}(Q_{x_i}) \leq \frac{1}{21}am(g)$ and $g \in Q_{x_i}$ we infer that $2^{-l-k} \leq \frac{1}{21\sqrt{n}}am(g) \leq \frac{1}{21\sqrt{n}}\frac{a}{2^{l-1}}$, so $2^{-k} \leq \frac{2a}{21\sqrt{n}}$, forcing that $k \geq 1$ since we are assuming that $0 < a \leq 4$. But then Q belongs to $\Delta^\gamma_{k-1,l}$ with $k-1 \geq 0$, so $Q \in \Delta^\gamma$.

It remains to show that if the side-length of $3 \circ Q_{x_i}$ were less than $\frac{1}{7\sqrt{n}}am(g)$, then $9 \circ Q_{x_i}$ is contained in O°. If $z \in \mathbb{R}^n$ is such that $|z - g| < \frac{1}{4}am(g)$ and $d(g, F) = \frac{1}{4}am(g)$, then $d(z, F) > 0$ and $d(z, F) < \frac{1}{2}am(g) \leq \frac{1}{2}a(1+\frac{1}{4}a)m(z) \leq am(z)$ (where the second inequality follows from the first part of Lemma 2.2(ii) and the third from the assumption that $0 < a \leq 4$), so that $z \in O^\circ$. Hence the ball $B(g, \frac{1}{4}am(g))$ is contained in O° and therefore it suffices to check that $9 \circ Q_{x_i} \subseteq B(g, \frac{1}{4}am(g))$. But if $z \in 9 \circ Q_{x_i}$, then from $g \in Q_{x_i}$ we infer that $|z - g| < 5\sqrt{n} \cdot \frac{1}{21\sqrt{n}}am(g) < \frac{1}{4}am(g)$, so $z \in B(g, \frac{1}{4}am(g))$ as claimed.

We have now shown that the side-length of $3 \circ Q_{x_i}$ is at least $\frac{1}{7\sqrt{n}}am(g)$. It follows that $y \in M \circ Q_{x_i}$ with

$$M := \frac{42\sqrt{n}}{m(g)}(\frac{1}{4}\sqrt{n}am(g) + a'm(y)) \leq 42\sqrt{n}(\frac{1}{4}\sqrt{n}a + a'(1+a')),$$

where we used the fact that $m(y) \leq (1+a')m(g)$, which follows from Lemma 2.2(ii) and the fact that $|y - g| \leq a'm(y)$. Thus the cubes Q_{x_i} have the desired property for $M_{a,n} := 42\sqrt{n}(\frac{1}{4}\sqrt{n}a + a'(1+a'))$ will do.

Step 3 – With these preliminaries out of the way it remains to prove the lemma for $a \geq 2$ and $b = \frac{1}{4}$. For each $x \in O$ the interval $\big[\frac{1}{4a\sqrt{n}}d(x, F), \frac{1}{2a\sqrt{n}}d(x, F)\big)$ contains a unique number of the form 2^{-j_x} with $j_x \in \mathbb{Z}$; from

$$2^{-j_x} < \frac{1}{2a\sqrt{n}}d(x, F) \leq \frac{1}{2\sqrt{n}}m(x) \leq \frac{1}{2}$$

we see that $j_x \geq 2$. Let Q_x be the unique dyadic cube in Δ_{j_x} containing x. This cube has side-length 2^{-j_x} and diameter $2^{-j_x}\sqrt{n}$. In particular, $\text{diam}(Q_x) < \frac{1}{2a}d(x, F)$. We claim that $Q_x \subseteq O'$, where O' is defined as in step 2 with $a' = \frac{15}{8}a$, i.e.,

$$O' := \Big\{z \in \mathbb{R}^n : 0 < d(z, F) \leq \frac{15}{8}am(z)\Big\}.$$

Indeed, for all $y \in Q_x$ we have $d(y, F) \geq d(x, F) - \text{diam}(Q_x) > (1 - \frac{1}{2a})d(x, F) > 0$ and $d(y, F) \leq d(x, F) + \text{diam}(Q_x) < (1 + \frac{1}{2a})d(x, F) \leq (a + \frac{1}{2})m(x) \leq \frac{3}{2}(a + \frac{1}{2})m(y) \leq \frac{15}{8}am(y)$, where the last inequality uses $a \geq 2$ and the second last follows from the fact that by Lemma 2.2(ii) the inequalities $|x-y| < \operatorname{diam}(Q_x) < \frac{1}{2a} d(x,F) \le \frac{1}{2} m(x)$ imply $m(x) \le \frac{3}{2} m(y)$. This proves the claim.

For any two $x, y \in O$ we either have $Q_x \cap Q_y = \emptyset$ or one of the cubes is (properly or not) contained in the other. As a consequence, every $x \in O$ is contained in a maximal cube of the form $Q_{x'}$ for some (possibly different) $x' \in O$. Clearly, the union V of these maximal cubes satisfies $O \subseteq V \subseteq O'$. Moreover, any two maximal cubes are either the same or else disjoint.

Pick a sequence $(x_i)_{i\ge 1}$ in O such that the cubes Q_{x_i} are maximal and disjoint and their union equals V. Consider the balls $B_i := B(x_i, d_i)$, where $d_i := 2^{-j_{x_i}}\sqrt{n}$ is the diameter of Q_{x_i}. From $Q_{x_i} \subseteq B_i$ we see that

$$O \subseteq \bigcup_{i\ge 1} B_i$$

and (i) follows by noting that $d_i < \frac{1}{2a} d(x_i, F) \le \frac{1}{4} d(x_i, F)$.

We claim that $\gamma(3\circ Q_{x_i}) \lesssim \gamma(Q_{x_i})$ with a constant depending only on n. Taking the claim for granted for the moment, (ii) is obtained as follows. In view of the inequalities $\frac{1}{4a} d(x_i,F) \le d_i < \frac{1}{2a} d(x_i,F) \le \frac{1}{2} m(x_i)$, the doubling property for balls in $\mathscr{B}_{\frac{1}{2}}$, the inclusion $B_i \subseteq 3\circ Q_{x_i}$, and the result proved in Step 2 imply

$$\begin{aligned}\sum_{i\ge 1} \gamma(B(x_i, d(x_i,F))) &\lesssim \sum_{k\ge 1} \gamma(B_i) \le \sum_{i\ge 1} \gamma(3\circ Q_{x_i}) \\ &\lesssim \sum_{i\ge 1} \gamma(Q_{x_i}) \le \gamma(O') \lesssim \gamma(O)\end{aligned}$$

with constants depending only on a and n.

Step 4 – It remains to prove the claim that $\gamma(3\circ Q_{x_i}) \lesssim \gamma(Q_{x_i})$. The point is to show that Q_{x_i} belongs to Δ^γ; once we know this, the claim is a consequence of Lemma 2.4.

We will show that each of the cubes Q_x constructed in Step 3 belong to $\Delta^\gamma_{k,l}$ for suitable $k, l \ge 0$. Fix $x \in O$ and suppose that x belongs to layer L_{l_x}. Suppose first that $l_x = 0$. The side-length of Q_x equals 2^{-j_x} with $j_x \ge 2$. The cubes in $\Delta^\gamma_{0,0}$ have side-length 1. Since x belongs to one of these cubes, we conclude that $Q_x \in \Delta^\gamma_{j_x,0}$.

If $l_x \ge 1$, then $|x| \ge 1$ and therefore $m(x) = 1/|x| \le 2^{-l_x+1}$. Since the side-length of Q_x equals 2^{-j_x} with $2^{-j_x} \le \frac{1}{2} m(x) \le 2^{-l_x}$, it follows that the side-length is at most 2^{-l_x}, say $2^{-l_x-k_x}$ for some integer $k_x \ge 0$. On the other hand, the cubes in Δ^γ_{0,l_x} have side-length 2^{-l_x}. Since x belongs to one of these cubes, we conclude that $Q_x \in \Delta^\gamma_{k_x,l_x}$. This proves the claim. □

3. Change of aperture for maximal functions

In the proof of Theorem 1.1 we need the following change of aperture result for the admissible cone appearing in the definition of non-tangential maximal functions.

Theorem 3.1. *For all $M_1, M_2 > 0$ there exists a constant D, depending only on M_1, M_2, and the dimension n, such that for all $u \in L^1(\gamma)$ and $\sigma > 0$ we have*

$$\gamma\big(\{x \in \mathbb{R}^n : T^*_{(M_1,M_2)} u(x) > \sigma\}\big) \lesssim \gamma\big(\{x \in \mathbb{R}^n :\ T^*_{(1,C_{M_1,M_2})} u(x) > D\sigma\}\big)$$

with $C_{M_1,M_2} = \frac{M_2}{M_1}(1+2M_2)(1+\frac{M_2}{M_1}(1+2M_2))$ and with implied constant independent of u and σ. In particular,

$$\|T^*_{(M_1,M_2)} u\|_{L^1(\gamma)} \lesssim \|T^*_{(1,C_{M_1,M_2})} u\|_{L^1(\gamma)}$$

with implied constant independent of u.

The proof of this theorem depends on a lemma. Both follow known arguments in the euclidean case (see [5]).

Lemma 3.2. *Let* F *be a measurable subset of* $\mathbb{R}^n$*, let* $a > 0$ *and* $C > 0$ *be fixed, and put*

$$\widetilde{F} := \{x \in \mathbb{R}^n : M_a^*(1_F)(x) > C\},$$

where

$$M_a^* f(x) := \sup_{B(x,r)\in\mathcal{B}_a} \frac{1}{\gamma(B(x,r))} \int_{B(x,r)} |f(y)|\, d\gamma(y)$$

Then $\gamma(\widetilde{F}) \lesssim \gamma(F)$*, with the implied constant only depending on* a*,* C*, and the dimension* n.

Proof. We may assume that $\gamma(F) > 0$, since otherwise also $\gamma(\widetilde{F}) = 0$. By Lemma 2.5 there exists a countable cover of F with admissible balls $B_j = B(c_j, m(c_j)) \in \mathscr{B}_1$ which satisfies

$$\sum_j \gamma(B_j) \le K\gamma(F),$$

where K depends only on n. For any $x \in \widetilde{F}$ there is an admissible ball $B(x, r_0) \in \mathcal{B}_a$ centred at x such that

$$\frac{1}{\gamma(B(x,r_0))} \int_{B(x,r_0)} 1_F(y)\, d\gamma(y) > C.$$

In particular, since $1_F \le \sum_j 1_{B_j}$,

$$\begin{aligned}\sum_j \sup_{B(x,r)\in\mathcal{B}_a} &\frac{1}{\gamma(B(x,r))} \int_{B(x,r)\cap B_j} d\gamma(y) \\ &\ge \sum_j \frac{1}{\gamma(B(x,r_0))} \int_{B(x,r_0)} 1_{B_j}(y)\, d\gamma(y) > C.\end{aligned}$$

Integrating over $\widetilde{F}$ we obtain

$$\begin{aligned}\gamma(\widetilde{F}) &\le \frac{1}{C} \sum_j \int_{\widetilde{F}} \sup_{B(x,r)\in\mathcal{B}_a} \frac{1}{\gamma(B(x,r))} \int_{B(x,r)\cap B_j} d\gamma(y)\, d\gamma(x) \\ &= \frac{1}{C} \sum_j \int_{\widetilde{F}} \sup_{B(x,r)\in\mathcal{B}_a} \frac{\gamma(B(x,r)\cap B_j)}{\gamma(B(x,r))}\, d\gamma(x).\end{aligned}$$

Fix j for the moment and suppose that $x \in \widetilde{F}$ is such that the supremum in the integral is non-zero. Then $B_j \cap B(x,r) \ne \emptyset$ for some $0 < r \le am(x)$, and Lemma 2.3 implies that $x \in B_j' := B(c_j, r_j')$, where $r_j' \le bm(c_j)$ for some constant b depending only on a. Therefore,

$$\begin{aligned}\sum_j \int_{\widetilde{F}} \sup_{B(x,r)\in\mathcal{B}_a} &\frac{\gamma(B(x,r)\cap B_j)}{\gamma(B(x,r))}\, d\gamma(x) \\ &\le \sum_j \int_{B_j'} \sup_{B(x,r)\in\mathcal{B}_a} \frac{\gamma(B(x,r)\cap B_j)}{\gamma(B(x,r))}\, d\gamma(x) \\ &\le \sum_j \gamma(B_j').\end{aligned}$$

By the doubling property for admissible balls, this gives

$$\gamma(\widetilde{F}) \lesssim \sum_j \gamma(B'_j) \lesssim \sum_j \gamma(B_j) \lesssim \gamma(F).$$

□

Proof of Theorem 3.1. It suffices to prove the inequality for test functions $u \in C_{\rm c}(\mathbb{R}^n)$.

For the rest of the proof we fix $u \in C_{\rm c}(\mathbb{R}^n)$. We fix a constant $C > 0$ such that

$$\gamma(B(y,(1+4M_1)t)) < \frac{1}{C}\gamma(B(y,t)) \quad \forall B(y,t) \in \mathcal{B}_{c_{M_2/M_1,2M_1}},$$

where $c_{M_2/M_1,2M_1} = (1+2M_2)\frac{M_2}{M_1}$ is the constant arising from Lemma 2.2(i), and define, for $\sigma > 0$,

$$E_\sigma := \{x \in \mathbb{R}^n : T^*_{(1,C_{M_1,M_2})}u(x) > \sigma\},$$
$$\widetilde{E_\sigma} := \{x \in \mathbb{R}^n : M^*_{a_{M_1,M_2}}(1_{E_\sigma})(x) > C\},$$

where $M^*_a f$ is defined as in the lemma and $a_{M_1,M_2} := (1+2M_1)\frac{M_2}{M_1}$. In the estimates that follow, the implicit constants are independent of u and σ.

Fix a point $x \notin \widetilde{E_\sigma}$ and a point $(y,t) \in \Gamma_x^{(2M_1,2M_2)}(\gamma)$. We claim that $B(y,t) \not\subseteq E_\sigma$. To prove this, first note that from $|x-y| \le 2M_1 t$ we have

$$B(y,t) \subseteq B(x,(1+2M_1)t) \subseteq B(y,(1+4M_1)t).$$

Furthermore, $(1+2M_1)t \le (1+2M_1)\frac{M_2}{M_1}m(x)$, and therefore $B(x,(1+2M_1)t) \in \mathscr{B}_{(1+2M_1)\frac{M_2}{M_1}} = \mathscr{B}_{a_{M_1,M_2}}$. If we now assume that the claim is false, we get

$$\begin{aligned}
M^*(1_{E_\sigma})(x) &= \sup_{B(x,r)\in\mathcal{B}_{a_{M_1,M_2}}} \frac{\gamma(B(x,r)\cap E_\sigma)}{\gamma(B(x,r))} \\
&\ge \sup_{B(x,r)\in\mathcal{B}_{a_{M_1,M_2}}} \frac{\gamma(B(x,r)\cap B(y,t))}{\gamma(B(x,r))} \\
&\ge \frac{\gamma(B(x,(1+2M_1)t)\cap B(y,t))}{\gamma(B(x,(1+2M_1)t))} \\
&= \frac{\gamma(B(y,t))}{\gamma(B(x,(1+2M_1)t))} \\
&\ge \frac{\gamma(B(y,t))}{\gamma(B(y,(1+4M_1)t))} \\
&> C,
\end{aligned}$$

where the last inequality follows from the definition of the constant C and the observation that $B(y,t) \in \mathscr{B}_{c_{M_2/M_1,2M_1}}$ by Lemma 2.2(i), using that $B(x,t) \in \mathscr{B}_{M_2/M_1}$ and $|x-y| \le 2M_1 t$. This contradicts the fact that $x \notin \widetilde{E_\sigma}$ and the claim is proved.

So, since $B(y,t) \not\subseteq E_\sigma$, there exists $\tilde{y} \in B(y,t)$ such that $\tilde{y} \notin E_\sigma$, that is,

$$\sup_{(z,s)\in\Gamma_{\tilde{y}}^{(1,C_{M_1,M_2})}(\gamma)} \frac{1}{\gamma(B(z,s))}\int_{B(z,s)} |e^{-s^2L}u(\zeta)|^2\,d\gamma(\zeta) \le \sigma^2. \tag{3.1}$$

In particular, since $t \le c_{M_2/M_1,2M_1} m(y)$, Lemma 2.2 implies that $t \le C_{M_1,M_2} m(\tilde y)$ with $C_{M_1,M_2} = c_{M_2/M_1,2M_1}(1+c_{M_2/M_1,2M_1})$ and $c_{M_2/M_1,2M_1} = \frac{M_2}{M_1}(1+2M_2)$. Thus $(y,t) \in \Gamma_{\tilde y}^{(1,C_{M_1,M_2})}(\gamma)$ and therefore

$$\frac{1}{\gamma(B(y,t))}\int_{B(y,t)} |e^{-t^2L}u(\zeta)|^2\,d\gamma(\zeta) \le \sigma^2, \tag{3.2}$$

and this estimate holds for all $(y,t) \in \Gamma_x^{(2M_1,2M_2)}(\gamma)$ with $x \notin \widetilde{E_\sigma}$.

Next let $(w,t) \in \Gamma_x^{(M_1,M_2)}(\gamma)$ be arbitrary and fixed for the moment. Then $w \in B(x, M_1 t)$. For any $y \in B(w, M_1 t)$ we have $|y-x| \le |y-w| + |w-x| \le 2M_1 t$. Since also $2M_1 t \le 2M_2 m(x)$, it follows that $(y,t) \in \Gamma_x^{(2M_1,2M_2)}(\gamma)$. Also, since $|y-w| \le M_1 t$ implies $B(y,t) \subseteq B(w,(1+M_1)t)$, we have

$$\gamma(B(y,t)) \le \gamma(B(w,(1+M_1)t)) \lesssim \gamma(B(w,M_1 t))$$

by the doubling property for admissible balls; the balls $B(w,M_1t)$ are indeed admissible by Lemma 2.2(i).

We can cover $B(w,M_1t)$ with finitely many balls of the form $B(y_i,t)$ with $y_i \in B(w,M_1t)$; this can be achieved with $N = N(M_1,n)$ balls. We then have, by (3.2),

$$\begin{aligned}
&\frac{1}{\gamma(B(w,M_1t))}\int_{B(w,M_1t)} |e^{-t^2L}u(z)|^2\,d\gamma(z)\\
&\qquad\lesssim \sum_{i=1}^{N} \frac{1}{\gamma(B(y_i,t))}\int_{B(y_i,t)} |e^{-t^2L}u(z)|^2 d\gamma(z) \lesssim \sigma^2.
\end{aligned}$$

Taking the supremum over all $(w,t) \in \Gamma_x^{(M_1,M_2)}(\gamma)$, we have shown that there exists a constant $D > 0$, depending only on M_1, M_2, and the dimension n, such that $T^*_{(M_1,M_2)}u(x) \le D\sigma$ for all $x \notin \widetilde{E_\sigma}$.

We have now shown that $\{T^*_{(M_1,M_2)}u(x) > D\sigma\} \subseteq \widetilde{E_\sigma}$. The first assertion of the theorem follows from this via Lemma 3.2. The second assertion follows from the first by integration:

$$\begin{aligned}
\|T^*_{(M_1,M_2)}u\|_{L^1(\gamma)} &= D\int_0^\infty \gamma(\{x \in \mathbb{R}^n : T^*_{(M_1,M_2)}u(x) > D\sigma\})\,d\sigma\\
&\lesssim \int_0^\infty \gamma(\widetilde{E_\sigma})\,d\sigma \lesssim \int_0^\infty \gamma(E_\sigma)\,d\sigma = \|T^*_{(1,C_{M_1,M_2})}u\|_{L^1(\gamma)}.
\end{aligned}$$

Since the choice of $M_1, M_2 \ge 0$ was arbitrary, this concludes the proof. □

4. Proof of Theorem 1.1

In this section we follow the method pioneered in [5] for proving square function estimates in Hardy spaces. This method has recently been adapted in a variety of contexts (see [1, 2, 7]). Here, we modify the version given in [7] to avoid using the doubling property on non-admissible balls, and to take into account differences between the Laplace and the Ornstein-Uhlenbeck operators. As a typical example of the latter phenomenon, we start by proving a Gaussian version of the parabolic Cacciopoli inequality. Recall that L is the Ornstein-Uhlenbeck operator, defined for $f \in C_c(\mathbb{R}^n)$ by

$$Lf(x) = -\tfrac{1}{2}\Delta f(x) + x\cdot\nabla f(x). \tag{4.1}$$

Note that, for all $f,g \in C_c^\infty(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} Lf \cdot g \, d\gamma = \frac{1}{2}\int_{\mathbb{R}^n} \nabla f \cdot \nabla g \, d\gamma$$

Lemma 4.1. *Let $u : \mathbb{R}^n \times (0,\infty) \to \mathbb{C}$ be a $C^{1,2}$-function such that*

$$\partial_t u + Lu = 0$$

on $I(x_0,t_0,2r) := B(x_0,2cr)\times[t_0-4r^2,t_0+4r^2]$ for some $r\in(0,1)$, $0 < C_0 \le c \le C_1 < \infty$, and $t_0 > 4r^2$. Then

$$\int_{I(x_0,t_0,r)} |\nabla u(x,t)|^2 \, d\gamma(x)\, dt \lesssim \frac{1+r|x_0|}{r^2}\int_{I(x_0,t_0,2r)} |u(x,t)|^2 \, d\gamma(x)\, dt,$$

with implied constant depending only on the dimension n, C_0 and C_1.

Proof. Let $\eta \in C^\infty(\mathbb{R}^n \times (0,\infty))$ be a cutoff function such that $0 \le \eta \le 1$ on $\mathbb{R}^n \times (0,\infty)$, $\eta \equiv 1$ on $I(x_0,t_0,r)$, $\eta \equiv 0$ on the complement of $I(x_0,t_0,2r)$, and

$$\|\nabla\eta\|_\infty \lesssim \frac{1}{r}, \quad \|\partial_t\eta\|_\infty \lesssim \frac{1}{r^2}, \quad \|\Delta\eta\|_\infty \lesssim \frac{1}{r^2}$$

with implied constants depending only on n, C_0, C_1. Then, in view of $\|x\cdot\nabla\eta\|_\infty \lesssim (|x_0|+2r)\cdot\frac{1}{r}$ and recalling that $0<r<1$,

$$\|L\eta\|_\infty \lesssim \frac{1}{r^2} + \frac{1}{r}|x_0| + 1 \lesssim \frac{1+r|x_0|}{r^2}, \tag{4.2}$$

where the implied constants depend only on n, C_0, C_1.

Considering real and imaginary parts separately, we may assume that all functions are real-valued. Integrating the identity

$$(\eta\nabla u)\cdot(\eta\nabla u) = (u\nabla\eta - \nabla(u\eta))\cdot(u\nabla\eta - \nabla(u\eta))$$

and then using that

$$\begin{aligned}\int_{I(x_0,t_0,2r)} \eta^2\nabla(u\eta)\cdot\nabla(u\eta)\, d\gamma\, dt &\le \int_0^\infty\int_{\mathbb{R}^d} \nabla(u\eta)\cdot\nabla(u\eta)\, d\gamma\, dt \\ &= 2\int_0^\infty\int_{\mathbb{R}^n} (u\eta)L(u\eta)\, d\gamma\, dt \\ &= 2\int_{I(x_0,t_0,2r)} u\eta L(u\eta)\, d\gamma\, dt,\end{aligned}$$

we obtain

$$\begin{aligned}\int_{I(x_0,t_0,r)} |\nabla u|^2 \, d\gamma\, dt &\le \int_{I(x_0,t_0,2r)} \eta^2|\eta\nabla u|^2\, d\gamma\, dt \\ &\le \int_{I(x_0,t_0,2r)} \eta^2 |u\nabla\eta|^2\, d\gamma\, dt \\ &\quad + \int_{I(x_0,t_0,2r)} 2u\eta^2\nabla(u\eta)\cdot\nabla\eta\, d\gamma\, dt \\ &\quad + 2\int_{I(x_0,t_0,2r)} (u\eta)L(u\eta)\, d\gamma\, dt.\end{aligned} \tag{4.3}$$

For the first term on the right-hand side we have the estimate

$$\int_{I(x_0,t_0,2r)} \eta^2 |u\nabla\eta|^2 \, d\gamma \, dt \lesssim \frac{1}{r^2} \int_{I(x_0,t_0,2r)} |u|^2 \, d\gamma \, dt.$$

For the second term we have, by (4.2),

$$\begin{aligned}
\Big| \int_{I(x_0,t_0,2r)} 2u\eta^2 \nabla(u\eta)\cdot\nabla\eta \, d\gamma \, dt \Big| &= \frac{1}{2} \Big| \int_{I(x_0,t_0,2r)} \nabla(u\eta)^2 \cdot \nabla\eta^2 \, d\gamma \, dt \Big| \\
&\le \Big| \int_{\mathbb{R}^n} (u\eta)^2 L\eta^2 \, d\gamma \, dt \Big| \\
&\lesssim \frac{1+r|x_0|}{r^2} \int_{I(x_0,t_0,2r)} |u|^2 \, d\gamma \, dt
\end{aligned}$$

where we used the fact that η^2 satisfies the same assumptions as η and (4.2) was applied to η^2. To estimate the third term on the right-hand side of (4.3) we substitute the identity

$$L(u\eta) = \eta L u + u L\eta + \nabla u \cdot \nabla \eta = -\eta \partial_t u + u L \eta + \nabla u \cdot \nabla\eta \tag{4.4}$$

and estimate each of the resulting integrals:

$$\begin{aligned}
\Big| \int_{I(x_0,t_0,2r)} u\eta^2 \partial_t u \, d\gamma \, dt \Big| &= \frac{1}{2} \Big| \int_{I(x_0,t_0,2r)} \eta^2 \partial_t u^2 \, d\gamma \, dt \Big| \\
&= \frac{1}{2} \Big| \int_{I(x_0,t_0,2r)} u^2 \partial_t \eta^2 \, d\gamma \, dt \Big| \\
&= \Big| \int_{I(x_0,t_0,2r)} u^2 \eta \partial_t \eta \, d\gamma \, dt \Big| \\
&\lesssim \frac{1}{r^2} \int_{I(x_0,t_0,2r)} |u|^2 \, d\gamma \, dt, \\
\Big| \int_{I(x_0,t_0,2r)} u^2 \eta L\eta \, d\gamma \, dt \Big| &\lesssim \frac{1+r|x_0|}{r^2} \int_{I(x_0,t_0,2r)} |u|^2 \, d\gamma \, dt, \\
\Big| \int_{I(x_0,t_0,2r)} u\eta \nabla u \cdot \nabla\eta \, d\gamma \, dt \Big| &= \frac{1}{4} \Big| \int_{I(x_0,t_0,2r)} \nabla u^2 \cdot \nabla \eta^2 \, d\gamma \, dt \Big| \\
&= \frac{1}{2} \Big| \int_{I(x_0,t_0,2r)} u^2 L\eta^2 \, d\gamma \, dt \Big| \\
&\lesssim \frac{1+r|x_0|}{r^2} \int_{I(x_0,t_0,2r)} |u|^2 \, d\gamma \, dt.
\end{aligned}$$

□

We can now prove the main result of this paper. Recall that

$$\begin{aligned}
Su(x) &= \Big(\int_{\Gamma_x^{(1,1)}(\gamma)} \frac{1}{\gamma(B(y,t))} |t\nabla e^{-t^2 L} u(y)|^2 \, dy \, \frac{dt}{t} \Big)^{\frac{1}{2}} \\
&= \Big(\int_{\mathbb{R}^n \times (0,\infty)} \frac{1_{B(x,t)}(y)}{\gamma(B(y,t))} 1_{(0,m(y))}(t) |t\nabla e^{-t^2 L} u(y)|^2 \, dy \, \frac{dt}{t} \Big)^{\frac{1}{2}}.
\end{aligned}$$

It will be convenient to define, for $\varepsilon > 0$,

$$S^\varepsilon u(x) := \Big(\int_{\mathbb{R}^n \times (0,\infty)} \frac{1_{B(x,t)}(y)}{\gamma(B(y,t))} 1_{(\varepsilon,m(y))}(t) |t\nabla e^{-t^2 L} u(y)|^2 \, dy \, \frac{dt}{t} \Big)^{\frac{1}{2}}.$$

Proof of Theorem 1.1. As in the proof of Lemma 4.1 it suffices to consider real-valued $u \in C_c(\mathbb{R}^n)$.

Let $F \subseteq \mathbb{R}^n$ be an arbitrary closed set and define

$$F^* := \big\{x \in \mathbb{R}^n \colon \gamma(F \cap B(x,r)) \geq \tfrac{1}{2}\gamma(B(x,r)) \ \forall r \in (0, c_{2,2}m(x)] \ \big\},$$

where $c_{2,2}$ has been defined in Lemma 2.2. For $0 < \varepsilon < 1$ and $1 < \alpha < 2$ put

$$R^\varepsilon_\alpha(F^*) := \{(y,t) \in \mathbb{R}^n \times (0,\infty) \colon d(y,F^*) < \alpha t \text{ and } t \in (\alpha^{-1}\varepsilon, \alpha m(y))\}$$

and let $\partial R^\varepsilon_\alpha(F^*)$ be its topological boundary. As in [5, page 162] and [13, page 206] we may regularise this set and thus assume it admits a surface measure $d\sigma^\varepsilon_\alpha(y,t)$. Applying first Green's formula in $\mathbb{R}^n$ to the section of $R^\varepsilon_\alpha(F^*)$ at level t and using (4.1), and subsequently the fundamental theorem of calculus in the t-variable, we obtain the estimate

$$\begin{aligned}
\int_{F^*} |S^\varepsilon u(x)|^2 \, d\gamma(x) &\leq \int_{R^\varepsilon_\alpha(F^*)} |t\nabla e^{-t^2L}u(y)|^2 \, dy \, \frac{dt}{t} \\
&\lesssim \int_{R^\varepsilon_\alpha(F^*)} tLe^{-t^2L}u(y) \cdot e^{-t^2L}u(y) \, d\gamma(y) \, dt \\
&\qquad + \int_{\partial R^\varepsilon_\alpha(F^*)} |t\nabla e^{-t^2L}u \cdot \nu^{/\!/}(y,t)||e^{-t^2L}u(y)|e^{-|y|^2} \, d\sigma^\varepsilon_\alpha(y,t) \\
&\lesssim \int_{R^\varepsilon_\alpha(F^*)} -\partial_t|e^{-t^2L}u(y)|^2 \, d\gamma(y) \, dt \\
&\qquad + \int_{\partial R^\varepsilon_\alpha(F^*)} |t\nabla e^{-t^2L}u(y)||e^{-t^2L}u(y)|e^{-|y|^2} \, d\sigma^\varepsilon_\alpha(y,t) \\
&\lesssim \int_{\partial R^\varepsilon_\alpha(F^*)} |e^{-t^2L}u(y)\nu^\perp(y,t)|^2 e^{-|y|^2} \, d\sigma^\varepsilon_\alpha(y,t) \\
&\qquad + \int_{\partial R^\varepsilon_\alpha(F^*)} |t\nabla e^{-t^2L}u(y)||e^{-t^2L}u(y)|e^{-|y|^2} \, d\sigma^\varepsilon_\alpha(y,t).
\end{aligned}$$

In the above computation, $\nu^{/\!/}$ denotes the projection of the normal vector ν to R^ε_α onto $\mathbb{R}^n$ and $\nu^\perp$ the projection of ν in the t direction. Of course, all implied constants in the above inequalities are independent of F, ε, α, and u.

Next we note that $\partial R^\varepsilon_\alpha(F^*) \subseteq B^\varepsilon$, where

$$B^\varepsilon := \tilde{B}^\varepsilon_1 \cup \tilde{B}^\varepsilon_2 \cup \tilde{B}^\varepsilon_3$$

with

$$\begin{aligned}
\tilde{B}^\varepsilon_1 &:= \{(y,t) \in \mathbb{R}^n \times (0,\infty) \colon t \in [\tfrac{1}{2}\varepsilon, \min\{\varepsilon, m(y)\}] \text{ and } d(y,F^*) \leq 2t\}, \\
\tilde{B}^\varepsilon_2 &:= \{(y,t) \in \mathbb{R}^n \times (0,\infty) \colon t \in [\varepsilon, m(y)] \text{ and } t \leq d(y,F^*) \leq 2t\}, \\
\tilde{B}^\varepsilon_3 &:= \{(y,t) \in \mathbb{R}^n \times (0,\infty) \colon t \in [m(y), 2m(y)] \text{ and } d(y,F^*) \leq 2t\}.
\end{aligned}$$

Now notice that, on $\partial R^\varepsilon_\alpha(F^*)$, we have either $t = \frac{\varepsilon}{\alpha}$, $t = \alpha m(y)$, or $t = \frac{d(y,F^*)}{\alpha}$. Integrating over $\alpha \in (1,2)$ with respect to $\frac{d\alpha}{\alpha}$ and changing variables using that $\frac{d\alpha}{\alpha} \sim \frac{dt}{t}$, we get

$$\int_{F^*} |S^\varepsilon u|^2 \, d\gamma \lesssim \int_{B^\varepsilon} |e^{-t^2L}u(y)|^2 dy \, \frac{dt}{t}$$

$$
\begin{aligned}
&+\Big(\int_{B^\varepsilon}|e^{-t^2L}u(y)|^2dy\,\frac{dt}{t}\Big)^{\frac12}\Big(\int_{B^\varepsilon}|t\nabla e^{-t^2L}u(y)|^2dy\,\frac{dt}{t}\Big)^{\frac12}\\
&\lesssim \int_{B^\varepsilon}|e^{-t^2L}u(y)|^2dy\,\frac{dt}{t}+\int_{B^\varepsilon}|t\nabla e^{-t^2L}u(y)|^2dy\,\frac{dt}{t}.
\end{aligned}
$$

Here, and in the estimates to follow, the implied constants are independent of F, ε, and u.

We have to estimate the following six integrals:

$$
\begin{aligned}
I_1 &:= \int_{\tilde B_1^\varepsilon}|e^{-t^2L}u(y)|^2dy\,\frac{dt}{t}, & I_2 &:= \int_{\tilde B_1^\varepsilon}|t\nabla e^{-t^2L}u(y)|^2dy\,\frac{dt}{t},\\
I_3 &:= \int_{\tilde B_2^\varepsilon}|e^{-t^2L}u(y)|^2dy\,\frac{dt}{t}, & I_4 &:= \int_{\tilde B_2^\varepsilon}|t\nabla e^{-t^2L}u(y)|^2dy\,\frac{dt}{t},\\
I_5 &:= \int_{\tilde B_3^\varepsilon}|e^{-t^2L}u(y)|^2dy\,\frac{dt}{t}, & I_6 &:= \int_{\tilde B_3^\varepsilon}|t\nabla e^{-t^2L}u(y)|^2dy\,\frac{dt}{t}.
\end{aligned}
$$

We start with I_1 and remark that, for $(y,t)\in\tilde B_1^\varepsilon$, there exists $x\in F^*$ such that $|x-y|\le 2t$. Since $t\le\min\{\varepsilon,m(y)\}\le m(y)$, by Lemma 2.2(i) we have $t\le c_{1,2}m(x)\le c_{2,2}m(x)$ (the last estimate looks redundant, but the reader may check that in the estimation of I_5 below we shall only get an estimate with $c_{2,2}$). Therefore, by the definition of F^*,

$$\gamma(F\cap B(x,t))\ge\tfrac12\gamma(B(x,t)).$$

This implies, via the doubling property for the admissible ball $B(x,t)\in\mathscr{B}_{c_{1,2}}$,

$$\gamma(F\cap B(y,3t))\ge\gamma(F\cap B(x,t))\ge\tfrac12\gamma(B(x,t))\gtrsim\gamma(B(x,3t))\ge\gamma(B(y,t)),$$

and therefore

$$
\begin{aligned}
I_1 &\lesssim \int_{\tilde B_1^\varepsilon}\int_{F\cap B(y,3t)}|e^{-t^2L}u(y)|^2\frac{d\gamma(z)}{\gamma(B(y,t))}\,d\gamma(y)\,\frac{dt}{t}\\
&\le \int_{\mathbb{R}^n}\int_{\frac12\varepsilon}^{\frac12\varepsilon\vee\min\{\varepsilon,m(y)\}}\int_F 1_{B(y,3t)}(z)|e^{-t^2L}u(y)|^2\frac{d\gamma(z)}{\gamma(B(y,t))}\,\frac{dt}{t}\,d\gamma(y) \qquad (4.5)\\
&\le \int_F\int_{\frac12\varepsilon}^{\frac12\varepsilon\vee\min\{\varepsilon,c_{1,3}m(z)\}}\int_{B(z,3t)}|e^{-t^2L}u(y)|^2\frac{d\gamma(y)}{\gamma(B(y,t))}\,\frac{dt}{t}\,d\gamma(z),
\end{aligned}
$$

where in the last inequality we used that $t\le m(y)$ and $|y-z|<3t$ imply $t\le c_{1,3}m(z)$ by Lemma 2.2(i).

Fix $(z,t)\in F\times(\frac12\varepsilon,\frac12\varepsilon\vee\min\{\varepsilon,c_{1,3}m(z)\})$ and pick any $z'\in\mathbb{R}^n$ such that $|z-z'|<t$. For all $y\in B(z,3t)$ we have $B(z,3t)\subseteq B(z',4t)\subseteq B(y,8t)$ and therefore, by the doubling property for $B(y,t)$ (noting that from $t<c_{1,3}m(z)$ and $|z-y|<3t$ it follows that $t<c_{c_{1,3},3}m(y)$, so $B(y,t)$ is admissible of class $\mathscr{B}_{c_{c_{1,3},3}}$),

$$
\begin{aligned}
\int_{B(z,3t)}|e^{-t^2L}u(y)|^2\,\frac{d\gamma(y)}{\gamma(B(y,t))} &\lesssim \frac{1}{\gamma(B(z',4t))}\int_{B(z',4t)}|e^{-t^2L}u(y)|^2\,d\gamma(y)\\
&\le |T^*_{(4,4c_{1,3})}u(z)|^2,
\end{aligned}
$$

where the last inequality follows from $(z',t)\in\Gamma_z^{(1,c_{1,3})}(\gamma)\subseteq\Gamma_z^{(4,4c_{1,3})}(\gamma)$. Combining this with the previous inequality it follows that

$$I_1\lesssim\int_F\int_{\frac12\varepsilon}^{\varepsilon}|T^*_{(4,4c_{1,3})}u(z)|^2\,\frac{dt\,d\gamma(z)}{t}\lesssim\int_F|T^*_{(4,4c_{1,3})}u(z)|^2\,d\gamma(z).$$

We proceed similarly for I_2, using Lemma 4.1 to handle the gradient. With $\tau(z) := c_{1,3}m(z)$ we have, proceeding as in (4.5),

$$
\begin{aligned}
I_2 &\lesssim \int_F \int_{\frac12\varepsilon}^{\frac12\varepsilon\vee\min\{\varepsilon,\tau(z)\}} \int_{B(z,3t)} |t\nabla e^{-t^2L}u(y)|^2 \frac{d\gamma(y)}{\gamma(B(y,t))}\,\frac{dt}{t}\,d\gamma(z)\\
&\overset{\text{(i)}}{\lesssim} \int_{F\cap\{\tau(z)\geq\frac12\varepsilon\}} \int_{\frac12\varepsilon}^{\varepsilon} \frac{1}{\gamma(B(z,3\varepsilon))}\int_{B(z,3\varepsilon)} |t\nabla e^{-t^2L}u(y)|^2\,d\gamma(y)\,\frac{dt}{t}\,d\gamma(z)\\
&\overset{\text{(ii)}}{\lesssim} \int_{F\cap\{\tau(z)\geq\frac12\varepsilon\}} \sum_{l=2}^{7}\int_{\frac{l\varepsilon^2}{8}}^{\frac{(l+1)\varepsilon^2}{8}} \frac{1}{\gamma(B(z,3\varepsilon))}\int_{B(z,3\varepsilon)} |\nabla e^{-sL}u(y)|^2\,d\gamma(y)\,ds\,d\gamma(z).
\end{aligned}
$$

In (i) we used the inclusions $B(z,3t)\subseteq B(z,3\varepsilon)\subseteq B(z,6t)\subseteq B(y,9t)$ together with the doubling property for $B(y,t)$, and in (ii) we substituted $t^2 = s$.

For each $l\in\{2,\dots,7\}$ we apply Lemma 4.1 with $t_0^l = \frac12(\frac{l\varepsilon^2}{8}+\frac{(l+1)\varepsilon^2}{8}) = \frac{(2l+1)\varepsilon^2}{16}$, $c^l = 12$ and $(r^l)^2 = \frac{\varepsilon^2}{16}$. Together with the doubling property for $B(z,\varepsilon)$ (noting that $B(z,\varepsilon)\in\mathscr{B}_{2c_{1,3}}$ in view of $\varepsilon\leq 2t\leq 2c_{1,3}m(z)$), this gives

$$
\begin{aligned}
I_2 \lesssim \int_{F\cap\{\tau(z)\geq\frac12\varepsilon\}} &\sum_{l=2}^{7}\int_{\frac{(2l-3)\varepsilon^2}{16}}^{\frac{(2l+5)\varepsilon^2}{16}} \frac{1+r^l|z|}{(r^l)^2}\\
&\times\frac{1}{\gamma(B(z,6\varepsilon))}\int_{B(z,6\varepsilon)} |e^{-sL}u(y)|^2\,d\gamma(y)\,ds\,d\gamma(z).
\end{aligned}
$$

Fix $(z,s)\in(F\cap\{\tau(z)\geq\frac12\varepsilon\})\times(\frac{1}{16}\varepsilon^2,\frac{19}{16}\varepsilon^2)$ and pick any $z'\in\mathbb{R}^n$ such that $|z-z'|<\sqrt{s}$. Then from $B(z,6\varepsilon)\subseteq B(z,24\sqrt{s})\subseteq B(z',25\sqrt{s})\subseteq B(z,26\sqrt{s})\subseteq B(z,52\varepsilon)$ and the doubling property for the balls $B(z,\varepsilon)\in\mathscr{B}_{2c_{1,3}}$ (note that $\varepsilon\leq 2\tau(z) = 2c_{1,3}m(z)$),

$$
\begin{aligned}
&\frac{1}{\gamma(B(z,6\varepsilon))}\int_{B(z,6\varepsilon)} |e^{-sL}u(y)|^2\,d\gamma(y)\\
&\qquad\lesssim \frac{1}{\gamma(B(z',25\sqrt{s}))}\int_{B(z',25\sqrt{s})} |e^{-sL}u(y)|^2\,d\gamma(y) \leq |T^*_{(25,100c_{1,3})}u(z)|^2,
\end{aligned}
$$

where the last step follows from $(z',\sqrt{s})\in\Gamma_z^{(1,4c_{1,3})}(\gamma)\subseteq\Gamma_z^{(25,100c_{1,3})}(\gamma)$ (indeed, this follows from $|z-z'|<\sqrt{s}<2\varepsilon\leq 4c_{1,3}m(z)$). Combining this with the previous estimate we obtain

$$
\begin{aligned}
I_2 &\lesssim \int_F \sum_{l=2}^{7}\int_{\frac{(2l-3)\varepsilon^2}{16}}^{\frac{(2l+5)\varepsilon^2}{16}} \frac{1+r^l|z|}{(r^l)^2}|T^*_{(25,100c_{1,3})}u(z)|^2\,ds\,d\gamma(z)\\
&\lesssim \int_F (1+\varepsilon|z|)|T^*_{(25,100c_{1,3})}u(z)|^2\,d\gamma(z),
\end{aligned}
$$

where the last step follows from the fact that $r^l = \frac14\varepsilon$.

We proceed with an estimate for I_3. Let

$$G := \{y\in\mathbb{R}^n : 0 < d(y,F^*)\leq 2m(y)\}.$$

Using Lemma 2.6, we cover G with a sequence of balls $B(x_k,r_k)$ with $x_k\in G$ and $r_k = \frac14 d(x_k,F^*)$ for all k, and

$$\sum_{k\geq1}\gamma(B(x_k,d(x_k,F^*)))\lesssim\gamma(G)\leq\gamma(\complement F^*). \tag{4.6}$$

with implied constant independent of u and F. Note that $B(x_k, r_k) \in \mathscr{B}_{\frac12}$ for all k.

If $(y,t) \in \tilde{B}_2^{\varepsilon}$, then $y \in G$ and therefore $y \in B(x_k, r_k)$ for some k, and $\frac12 d(y,F^*) \le t \le d(y,F^*)$. It follows that

$$
\begin{aligned}
I_3 &\le \sum_k \int_{B(x_k,r_k)} \int_{\frac12 d(y,F^*)}^{d(y,F^*)} |e^{-t^2L}u(y)|^2 \, \frac{dt}{t} \, d\gamma(y) \\
&\le \sum_k \int_{B(x_k,r_k)} \int_{\frac14 d(x_k,F^*)}^{\frac54 d(x_k,F^*)} |e^{-t^2L}u(y)|^2 \, \frac{dt}{t} \, d\gamma(y) \qquad (4.7)\\
&\le \sum_k \int_{\frac14 d(x_k,F^*)}^{\frac54 d(x_k,F^*)} \int_{B(x_k,t)} |e^{-t^2L}u(y)|^2 \, d\gamma(y) \, \frac{dt}{t}.
\end{aligned}
$$

In the second inequality we used that $y \in B(x_k, r_k)$ implies $|x_k - y| < r_k = \frac14 d(x_k, F^*)$, and the third inequality follows from Fubini's theorem and the inequality $r_k = \frac14 d(x_k,F^*) \le \frac12 d(y,F^*) \le t$.

Fix an index k and a number $t \in (\frac14 d(x_k,F^*), \frac54 d(x_k,F^*))$. Since F^* is contained in the closure of F we may pick $z_k \in F$ such that $|x_k - z_k| < 2d(x_k,F^*)$. By the choice of t this implies $|x_k - z_k| < 8t$. Since by assumption we have $t \le \frac54 d(x_k,F^*) \le \frac52 m(x_k)$ (the second inequality being a consequence of $x_k \in G$), and since $|x_k - z_k| < 8t$, from Lemma 2.2 we conclude that $t \le dm(z_k)$ with $d := c_{\frac52,8}$. We conclude that $(x_k,t) \in \Gamma_{z_k}^{(8,8d)}(\gamma)$ (since by definition this means that $|x_k - z_k| \le 8t \le 8dm(z_k)$) and consequently, using the doubling property for the admissible ball $B(x_k,t) \in \mathscr{B}_{\frac52}$,

$$
\begin{aligned}
&\frac{1}{\gamma(B(x_k,t))} \int_{B(x_k,t)} |e^{-t^2L}u(y)|^2 \, d\gamma(y) \\
&\quad \lesssim \frac{1}{\gamma(B(x_k,8t))} \int_{B(x_k,8t)} |e^{-t^2L}u(y)|^2 \, d\gamma(y) \le |T^*_{(8,8d)}u(z_k)|^2.
\end{aligned}
$$

Combining this with the previous inequalities we obtain

$$
\begin{aligned}
I_3 &\lesssim \Big(\sup_{z\in F} |T^*_{(8,8d)}u(z)|^2 \Big) \sum_k \int_{\frac14 d(x_k,F^*)}^{\frac54 d(x_k,F^*)} \gamma(B(x_k,t)) \, \frac{dt}{t} \\
&\lesssim \Big(\sup_{z\in F} |T^*_{(8,8d)}u(z)|^2 \Big) \sum_k \gamma(B(x_k, \tfrac54 d(x_k,F^*))) \\
&\lesssim \Big(\sup_{z\in F} |T^*_{(8,8d)}u(z)|^2 \Big) \gamma(\complement F^*),
\end{aligned}
$$

where the last step used (4.6) and the doubling property (recall that $d(x_k,F^*) \le 2m(x_k)$, so the balls $B(x_k, d(x_k,F^*))$ belong to $\mathscr{B}_2$).

For estimating I_4, we let G and $B(x_k,r_k)$ be as in the previous estimate. Proceeding as in the first two lines of (4.7) and applying the Fubini theorem, we get

$$
\begin{aligned}
I_4 &\lesssim \sum_k \int_{\frac14 d(x_k,F^*)}^{\frac54 d(x_k,F^*)} \int_{B(x_k,r_k)} |t\nabla e^{-t^2L}u(y)|^2 \, d\gamma(y) \, \frac{dt}{t} \\
&= \frac12 \sum_k \sum_{l=2}^{49} \int_{\frac{2l}{64} d^2(x_k,F^*)}^{\frac{2l+2}{64} d^2(x_k,F^*)} \int_{B(x_k,r_k)} |\nabla e^{-sL}u(y)|^2 \, d\gamma(y) \, ds.
\end{aligned}
$$

By Lemma 4.1, applied with $t_0 = \frac{2l+1}{64} d^2(x_k, F^*)$, $c = 2$ and $r = \frac{1}{8} d(x_k, F^*)$, this gives the estimate

$$
\begin{aligned}
I_4 &\lesssim \sum_k \sum_{l=2}^{49} \int_{\frac{2l-3}{64} d^2(x_k,F^*)}^{\frac{2l+5}{64} d^2(x_k,F^*)} \frac{1 + d(x_k, F^*)|x_k|}{d^2(x_k, F^*)} \int_{B(x_k, \frac{1}{2} d(x_k,F^*))} |e^{-sL} u(y)|^2 \, d\gamma(y) \, ds \\
&\le \sum_k \sum_{l=2}^{49} \int_{\frac{2l-3}{64} d^2(x_k,F^*)}^{\frac{2l+5}{64} d^2(x_k,F^*)} \frac{3}{d^2(x_k, F^*)} \int_{B(x_k, 4\sqrt{s})} |e^{-sL} u(y)|^2 \, d\gamma(y) \, ds,
\end{aligned}
$$

where we used that $d(x_k, F^*) \le 2m(x_k) \le \frac{2}{|x_k|}$ and that $s \ge \frac{1}{64} d^2(x_k, F^*)$ implies $\frac{1}{2} d(x_k, F^*) \le 4\sqrt{s}$.

Fix k and pick an element $z_k \in F$ such that $|x_k - z_k| < 2d(x_k, F^*)$. Then for all s in the range of integration we have $|x_k - z_k| < 16\sqrt{s}$. Since $\sqrt{s} \le \frac{3}{2} d(x_k, F^*) \le 3m(x_k)$, from Lemma 2.2 we conclude that $\sqrt{s} \le dm(z_k)$ with $d := c_{3,16}$. We conclude that $(x_k, 4\sqrt{s}) \in \Gamma_{z_k}^{(4,16d)}(\gamma)$. This gives

$$
\begin{aligned}
I_4 &\lesssim \Big(\sup_{z \in F} |T^*_{(4,16d)} u(z)|^2 \Big) \sum_k \frac{1}{d^2(x_k, F^*)} \int_{\frac{1}{64} d^2(x_k,F^*)}^{\frac{103}{64} d^2(x_k,F^*)} \gamma(B(x_k, 4\sqrt{s}) \, ds \\
&\lesssim \Big(\sup_{z \in F} |T^*_{(4,16d)} u(z)|^2 \Big) \sum_k \gamma(B(x_k, \tfrac{1}{2}\sqrt{103} d(x_k, F^*))) \\
&\lesssim \Big(\sup_{z \in F} |T^*_{(4,16d)} u(z)|^2 \Big) \sum_k \gamma(B(x_k, d(x_k, F^*))) \\
&\lesssim \Big(\sup_{z \in F} |T^*_{(4,16d)} u(y)|^2 \Big) \gamma(\complement F^*),
\end{aligned}
$$

where the second last step used the doubling property for admissible balls (recalling that $B(x_k, d(x_k, F^*)) \in \mathscr{B}_2$) and the last one used (4.6).

To estimate I_5, we proceed as we did for I_1:

$$
\begin{aligned}
I_5 &\lesssim \int_{\tilde{B}_3^\varepsilon} \int_{F \cap B(y,3t)} |e^{-t^2 L} u(y)|^2 \frac{d\gamma(z)}{\gamma(B(y,t))} \, d\gamma(y) \, \frac{dt}{t} \\
&\le \int_{\mathbb{R}^n} \int_{m(y)}^{2m(y)} \int_F 1_{B(y,3t)}(z) |e^{-t^2 L} u(y)|^2 \frac{d\gamma(z)}{\gamma(B(y,t))} \, \frac{dt}{t} \, d\gamma(y) \\
&\overset{\text{(i)}}{\le} \int_F \int_{(1+3c_{2,3})^{-1} m(z)}^{c_{2,3} m(z)} \int_{B(z,3t)} |e^{-t^2 L} u(y)|^2 \frac{d\gamma(y)}{\gamma(B(y,t))} \, \frac{dt}{t} \, d\gamma(z) \\
&\lesssim \int_F |T^*_{(4,4c_{2,3})} u(z)|^2 \, d\gamma(z),
\end{aligned}
$$

where in step (i) we used that $m(y) \le t \le 2m(y)$ and $|y - z| < 3t$ imply $t \le c_{2,3} m(z)$ by Lemma 2.2(i), so $|y - z| < 3c_{2,3} m(z)$, and by an application of Lemma 2.2(ii) the latter implies $m(z) \le (1 + 3c_{2,3}) m(y) \le (1 + 3c_{2,3}) t$.

Finally we turn to I_6, which is treated as I_2. With $c = c_{2,3}$ and $d = (1+3c_{2,3})^{-1}$ as in the previous estimate, and using Lemma 4.1 as in the estimate for I_2, we get

$$\begin{aligned} I_6 &\lesssim \int_F \int_{dm(z)}^{cm(z)} \frac{1}{\gamma(B(z,3t))} \int_{B(z,3t)} |t\nabla e^{-t^2L}u(y)|^2 \, d\gamma(y) \, \frac{dt}{t} \, d\gamma(z) \\ &= \frac{1}{2} \int_F \int_{d^2m(z)^2}^{c^2m(z)^2} \frac{1}{\gamma(B(z,3t))} \int_{B(z,3t)} |\nabla e^{-sL}u(y)|^2 \, d\gamma(y) \, ds \, d\gamma(z) \\ &\lesssim \int_F (1+m(z)|z|)|T^*_{(M_1,M_2)}u(z)|^2 \, d\gamma(z) \\ &\lesssim \int_F |T^*_{(M_1,M_2)}u(z)|^2 \, d\gamma(z), \end{aligned}$$

for certain M_1, M_2 independent of u, F, and ε.

Combining all these estimates, we obtain six couples $(M_1^{(j)}, M_2^{(j)})$ $(j = 1, ..., 6)$, and, passing to the limit $\varepsilon \downarrow 0$, the following estimate, valid for arbitrary closed subsets $F \subseteq \mathbb{R}^n$:

$$\begin{aligned} &\int_{F^*} |Su(x)|^2 \, d\gamma(x) \\ &\lesssim \sum_{j=1}^{6} \Big(\big(\sup_{z\in F} |T^*_{M_1^{(j)},M_2^{(j)}}u(z)|^2\big)\gamma(\complement F^*) + \int_F |T^*_{M_1^{(j)},M_2^{(j)}}u(z)|^2 \, d\gamma(z) \Big), \end{aligned} \tag{4.8}$$

with constants independent of F and u.

To finish the proof, we consider the distribution functions

$$\begin{aligned} \gamma_{Su}(\sigma) &:= \gamma\big(\{x \in \mathbb{R}^n \colon Su(x) > \sigma\}\big), \\ \gamma_{T^*_{(M_1^{(j)},M_2^{(j)})}}(\sigma) &:= \gamma\big(\{x \in \mathbb{R}^n \colon T^*_{(M_1^{(j)},M_2^{(j)})}u(x) > \sigma\}\big), \quad j = 1, \dots, 6. \end{aligned}$$

We apply (4.8) to the set

$$F := \big\{z \in \mathbb{R}^n \colon T^*_{(M_1^{(j)},M_2^{(j)})}u(z) \le \sigma, \ j = 1, \dots, 6\big\},$$

and remark that $\complement F^* \subseteq \{z \in \mathbb{R}^n \colon M^*_{c_{2,2}}(1_{\complement F})(z) > \frac{1}{2}\} = \widetilde{\complement F}$ using the notation of Lemma 3.2 (with $a = c_{2,2}$ and $C = \frac{1}{2}$). Indeed, if $x \in \mathbb{R}^n$ and $r \in (0, c_{2,2}m(x)]$ are such that $\gamma(B(x,r) \cap F) < \frac{1}{2}\gamma(B(x,r))$, then

$$\sup_{B(x,r)\in\mathcal{B}_{c_{2,2}}} \frac{\gamma(B(x,r) \cap \complement F)}{\gamma(B(x,r))} > \frac{1}{2}.$$

Lemma 3.2 gives us $\gamma(\complement F^*) \le \gamma(\widetilde{\complement F}) \lesssim \gamma(\complement F)$. Using this in combination with the definition of F,

$$\frac{1}{\sigma^2}\big(\sup_{z\in F} |T^*_{M_1^{(j)},M_2^{(j)}}u(z)|^2\big)\gamma(\complement F^*) \le \gamma(\complement F^*) \lesssim \gamma(\complement F) \le \sum_{k=1}^{6} \gamma\big(\{T_{(M_1^{(k)},M_2^{(k)})} > \sigma\}\big).$$

Hence, from (4.8) we infer

$$\begin{aligned}\gamma_{Su}(\sigma) &\le \gamma(F^* \cap \{Su > \sigma\}) + \gamma(\complement F^*)\\ &\lesssim \frac{1}{\sigma^2}\int_{F^*} |Su(x)|^2\,d\gamma(x) + \gamma(\complement F)\\ &\lesssim \sum_{j=1}^{6}\Big[\gamma_{T^*_{(M_1^{(j)},M_2^{(j)})}}(\sigma) + \frac{1}{\sigma^2}\int_F |T^*_{(M_1^{(j)},M_2^{(j)})}u(z)|^2\,d\gamma(z)\Big]\\ &\lesssim \sum_{j=1}^{6}\Big[\gamma_{T^*_{(M_1^{(j)},M_2^{(j)})}}(\sigma) + \frac{1}{\sigma^2}\int_0^\sigma t\gamma_{T^*_{(M_1^{(j)},M_2^{(j)})}}(t)\,dt\Big].\end{aligned}$$

Integrating over σ and noting that

$$\begin{aligned}\int_0^\infty \frac{1}{\sigma^2}\int_0^\sigma t\gamma_{T^*_{(M_1^{(j)},M_2^{(j)})}}(t)\,dt\,d\sigma &= \int_0^\infty t\gamma_{T^*_{(M_1^{(j)},M_2^{(j)})}}(t)\int_t^\infty \frac{1}{\sigma^2}\,d\sigma\,dt\\ &= \int_0^\infty \gamma_{T^*_{(M_1^{(j)},M_2^{(j)})}}(t)\,dt = \big\|T^*_{(M_1^{(j)},M_2^{(j)})}\big\|_{L^1(\gamma)},\end{aligned}$$

we get, by Theorem 3.1,

$$\begin{aligned}\|Su\|_{L^1(\gamma)} &\lesssim \sum_{j=1}^{6}\big\|T^*_{(M_1^{(j)},M_2^{(j)})}u\big\|_{L^1(\gamma)}\\ &\lesssim \sum_{j=1}^{6}\big\|T^*_{(1,C_{M_1^{(j)},M_2^{(j)}})}u\big\|_{L^1(\gamma)} \le 6\big\|T^*_{(1,C)}u\big\|_{L^1(\gamma)},\end{aligned}$$

where $C = \max_{j=1,\dots,6} C_{M_1^{(j)},M_2^{(j)}}$. □

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