BOUNDEDNESS OF RIESZ TRANSFORMS FOR ELLIPTIC OPERATORS ON ABSTRACT WIENER SPACES

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Dedicated to Professor Alan M[c]Intosh on the occasion of his 65th birthday

ABSTRACT. Let (E, H, μ) be an abstract Wiener space and let $D_V := VD$, where D denotes the Malliavin derivative and V is a closed and densely defined operator from H into another Hilbert space $\underline{H}$. Given a bounded operator B on $\underline{H}$, coercive on the range $\overline{\mathsf{R}(V)}$, we consider the operators $A := V^*BV$ in H and $\underline{A} := VV^*B$ in $\underline{H}$, as well as the realisations of the operators $L := D_V^* B D_V$ and $\underline{L} := D_V D_V^* B$ in $L^p(E, \mu)$ and $L^p(E, \mu; \underline{H})$ respectively, where $1 < p < \infty$. Our main result asserts that the following four assertions are equivalent:

(1) $\mathsf{D}(\sqrt{L}) = \mathsf{D}(D_V)$ with $\|\sqrt{L}f\|_p \eqsim \|D_V f\|_p$ for $f \in \mathsf{D}(\sqrt{L})$;
(2) $\underline{L}$ admits a bounded H^∞-functional calculus on $\overline{\mathsf{R}(D_V)}$;
(3) $\mathsf{D}(\sqrt{A}) = \mathsf{D}(V)$ with $\|\sqrt{A}h\| \eqsim \|Vh\|$ for $h \in \mathsf{D}(\sqrt{A})$;
(4) $\underline{A}$ admits a bounded H^∞-functional calculus on $\overline{\mathsf{R}(V)}$.

Moreover, if these conditions are satisfied, then $\mathsf{D}(L) = \mathsf{D}(D_V^2) \cap \mathsf{D}(D_A)$.

The equivalence (1)–(4) is a nonsymmetric generalisation of the classical Meyer inequalities of Malliavin calculus (where $\underline{H} = H$, $V = I$, $B = \frac{1}{2}I$). A one-sided version of (1)–(4), giving L^p-boundedness of the Riesz transform $D_V/\sqrt{L}$ in terms of a square function estimate, is also obtained.

As an application let $-A$ generate an analytic C_0-contraction semigroup on a Hilbert space H and let $-L$ be the L^p-realisation of the generator of its second quantisation. Our results imply that two-sided bounds for the Riesz transform of L are equivalent with the Kato square root property for A.

1. Introduction

Let (E, H, μ) be an abstract Wiener space, i.e., E is a real Banach space and μ is a centred Gaussian Radon measure on E with reproducing kernel Hilbert space H. In this paper we prove square function estimates and boundedness of Riesz transforms for abstract second order elliptic operators L in divergence form acting on $L^p(E, \mu)$, $1 < p < \infty$. Our main result (Theorem 2.1) gives necessary and sufficient conditions for the domain equality

$$\mathsf{D}(\sqrt{L}) = \mathsf{D}(D_V) \tag{1.1}$$

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in $L^p(E,\mu)$ with equivalence of norms

$$\|\sqrt{L}f\|_p \eqsim \|D_V f\|_p \tag{1.2}$$

for a class of divergence form elliptic operators of the form

$$L = D_V^* B D_V.$$

Here $D_V := VD$, where D is the Malliavin derivative in the direction of H, $V : \mathsf{D}(V) \subseteq H \to \underline{H}$ is a closed and densely defined operator, and B is a bounded operator on $\underline{H}$ which is coercive on $\overline{\mathsf{R}(V)}$. Our main result asserts that (1.1) and (1.2) hold if and only if the sectorial operator $\underline{A} := VV^*B$ on $\underline{H}$ admits a bounded H^∞-functional calculus. In particular, if (1.1) and (1.2) hold for one $1 < p < \infty$, then they hold for all $1 < p < \infty$. By well-known examples, cf. [38, Theorem 4 and its proof], sectorial operators on $\underline{H}$ of the form TB with $T : \mathsf{D}(T) \subseteq \underline{H} \to \underline{H}$ positive and self-adjoint and B coercive on $\underline{H}$ need not always have a bounded H^∞-calculus. In our setting, such examples can be translated into examples of operators L for which (1.2) fails (e.g., take $H = \underline{H}$ and $V = \sqrt{T}$).

Returning to (1.2), we shall prove the more precise result that the inclusions

$$\mathsf{D}(\sqrt{L}) \hookrightarrow \mathsf{D}(D_V),$$

respectively

$$\mathsf{D}(\sqrt{L}) \hookleftarrow \mathsf{D}(D_V),$$

hold in $L^p(E,\mu)$ if and only if the operator $\underline{A}$ satisfies a lower, respectively upper square function estimate in $\overline{\mathsf{R}(V)}$.

The simplest example to which our results apply is the classical Ornstein-Uhlenbeck operator of Malliavin calculus. This example is obtained by taking $\underline{H} = H$, $V = I$, and $B = \frac{1}{2}I$. With these choices, D_V reduces to the Malliavin derivative D in the direction of H and $L = \frac{1}{2}D^*D$ is the classical Ornstein-Uhlenbeck operator. The equivalences (1.1) and (1.2) then reduce to the classical Meyer inequalities [40]. Various proofs of these inequalities have been given; see, e.g., [21, 47]. For further references on this subject we refer to Nualart [45].

A second and non-trivial application concerns the computation of the L^p-domains of second quantised operators. Let (E,H,μ) be an abstract Wiener space and suppose that $S = (S(z))_{z\in\overline{\Sigma}}$ is an analytic C_0-contraction semigroup defined on the closed sector $\overline{\Sigma}$. By this we mean that S is a C_0-semigroup of contractions on $\overline{\Sigma}$ which is analytic on the interior of $\overline{\Sigma}$. Let $-A$ denote the generator of S. For $1 < p < \infty$, by second quantisation (see Section 3 for the details) we obtain an analytic C_0-contraction semigroup $(\Gamma(S(t)))_{z\in\overline{\Sigma}}$, with generator $-L$, on $L^p(E,\mu)$. As we will show, the operators A and L are always of the form $A = V^*BV$ and $L = D_V^* B D_V$ for suitable choices of V and B, and Theorem 2.1 implies that

$$\mathsf{D}(\sqrt{L}) = \mathsf{D}(D_V) \ \text{ with } \ \|\sqrt{L}f\|_p \eqsim \|D_V f\|_p, \quad 1 < p < \infty, \tag{1.3}$$

if and only if A admits a bounded H^∞-calculus on the homogeneous form domain associated with A. As before, one-sided versions of this result can be formulated in terms of square function estimates. By restricting (1.3) to the first Wiener-Itô chaos of $L^p(E,\mu)$ (see section 3) and using that the L^p-norms are pairwise equivalent on every chaos, we see that a necessary condition for (1.3) is given by

$$\mathsf{D}(\sqrt{A}) = \mathsf{D}(V) \ \text{ with } \ \|\sqrt{A}h\| \eqsim \|Vh\|.$$

Since $\mathsf{D}(V)$ equals the domain of the form associated with A, this is nothing but Kato's square root property for A. Thus our main result asserts that this necessary condition is also sufficient.

Second quantised operators arise naturally as generators of transition semigroups associated with solutions of linear stochastic evolution equations with additive noise; see for instance [11, 12]. In a forthcoming paper we shall apply our results to obtain Meyer inequalities for non-symmetric analytic Ornstein-Uhlenbeck operators in infinite dimensions. These extend previous results of Shigekawa [49] and Chojnowska-Michalik and Goldys [12] for the symmetric case, and of Metafune, Prüss, Rhandi, and Schnaubelt [39] for the finite dimensional case, and they improve results of [36] where a slightly more general class of non-symmetric Ornstein-Uhlenbeck operators was considered.

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2. Statement of the main results

The domain, kernel, and range of a (possibly unbounded) linear operator T are denoted by $\mathsf{D}(T)$, $\mathsf{N}(T)$, and $\mathsf{R}(T)$, respectively. When considering an operator T acting consistently on a scale of (vector-valued) L^p-spaces, $\mathsf{D}_p(T)$, $\mathsf{R}_p(T)$, and $\mathsf{N}_p(T)$ denote the domain, range, and kernel of the L^p-realisation of T.

We introduce the setting studied in this paper in the form of a list of assumptions which will be in force throughout the paper, with the exception of the intermediate Sections 3, 6, and 7.

Assumption (A1). *(E, H, μ) is an abstract Wiener space.*

More precisely, we assume that E is real Banach space, H is a real Hilbert space with inner product $[\cdot,\cdot]$, and μ is a centred Gaussian Radon measure on E with reproducing kernel Hilbert space H. Recall that this implies that H is continuously embedded in E; we shall write $i : H \hookrightarrow E$ for the inclusion mapping. The covariance operator of μ equals $i \circ i^*$ (here and in what follows, we identify H^* and H via the Riesz representation theorem).

For $h \in H$ we may define a linear function $\phi_h : iH \to \mathbb{R}$ by $\phi_h(ig) := [h, g]$. Although $\mu(iH) = 0$ if H is infinite dimensional [7, Theorem 2.4.7], there exists a μ-measurable linear extension $\phi_h : E \to \mathbb{R}$ which is uniquely defined μ-almost everywhere [7, Theorem 2.10.11]. Note that for $x^* \in E^*$ we have $\phi_{i^*x^*}(x) = \langle x, x^*\rangle$ μ-almost everywhere. The identity

$$\int_E \langle x, x^*\rangle^2 \, d\mu(x) = \|i^*x^*\|^2, \quad x^* \in E^*,$$

shows that $h \mapsto \phi_h$, as a mapping from H into $L^2(E, \mu)$, is an isometric embedding.

Assumption (A2). *V is a closed and densely defined linear operator from H into another real Hilbert space $\underline{H}$.*

When H_0 is a linear subspace of H and $k \geqslant 0$ is an integer, we let $\mathscr{F}C^k_{\mathrm{b}}(E; H_0)$ denote the vector space of all (μ-almost everywhere defined) functions $f : E \to \mathbb{R}$ of the form

$$f(x) := \varphi(\phi_{h_1}(x), \dots, \phi_{h_n}(x))$$

with $n \geqslant 1$, $\varphi \in C_{\rm b}^k(\mathbb{R}^n)$, and $h_1, \dots, h_n \in H_0$. Here $C_{\rm b}^k(\mathbb{R}^n)$ is the space consisting of all bounded continuous functions having bounded continuous derivatives up to order k. In case $H_0 = H$ we simply write $\mathscr{F}C_{\rm b}^k(E)$. For $f \in \mathscr{F}C_{\rm b}^1(E; \mathsf{D}(V))$ as above the gradient $D_V f$ 'in the direction of V' is defined by

$$D_V f(x) := V(Df(x)) = \sum_{j=1}^{n} \partial_j \varphi(\phi_{h_1}(x), \dots, \phi_{h_n}(x)) \otimes V h_j,$$

where D denotes the Malliavin derivative and $\partial_j \varphi$ denotes the j-th partial derivative of φ.

We shall write

$$L^p := L^p(E, \mu), \quad \underline{L}^p := L^p(E, \mu; \underline{H})$$

for brevity. As in [19, Theorem 3.5], the proof of which can be repeated almost *verbatim*, the operator D_V is closable as an operator from L^p into $\underline{L}^p$ for all $1 \leqslant p < \infty$. From now on, D_V denotes its closure; domain and range of this closure will be denoted by $\mathsf{D}_p(D_V)$ and $\mathsf{R}_p(D_V)$ respectively.

Assumption (A3). *B is a bounded operator on $\underline{H}$ which is coercive on $\overline{\mathsf{R}(V)}$.*

More precisely, B is a bounded operator on $\underline{H}$ which satisfies the coercivity condition

$$[BVh, Vh] \geqslant k\|Vh\|^2, \qquad h \in \mathsf{D}(V),$$

where $k > 0$ is a constant independent of $h \in \mathsf{D}(V)$. Clearly B satisfies (A3) if and only if B^* satisfies (A3).

At this point we pause to observe that the assumptions (A1), (A2), (A3) continue to hold after complexifying. In what follows we shall be mostly dealing with the complexified operators, which we do not distinguish notationally from their real counterparts as this would only overburden the notations. The complexified version of (A3) reads

$$\mathrm{Re}\, [BVh, Vh] \geqslant k\|Vh\|^2, \qquad h \in \mathsf{D}(V).$$

If (A1), (A2), (A3) hold, the operator

$$L := D_V^* B D_V$$

is well defined, and $-L$ generates an analytic C_0-contraction semigroup $(P(t))_{t \geqslant 0}$ on L^p for all $1 < p < \infty$, which coincides with the second quantisation of the analytic C_0-contraction semigroup on H generated by $-A$, where

$$A := V^* B V$$

(Theorem 4.4). In the converse direction we show that every second quantised analytic C_0-contraction semigroup arises in this way (Theorem 3.2).

It is not hard to see (Proposition 5.1) that the operator $\underline{A} := VV^*B$ is sectorial on $\underline{H}$ and that $-\underline{A}$ generates a bounded analytic C_0-semigroup on this space. Associated with this operator is the operator $\underline{L} = D_V D_V^* B$ on $\overline{\mathsf{R}_p(D_V)}$. This operator is well defined and sectorial on $\overline{\mathsf{R}_p(D_V)}$, and $-\underline{L}$ generates a bounded analytic C_0-semigroup on this space (Theorem 5.6 and Definition 5.8).

The main results of this paper read as follows.

Theorem 2.1 (Domain of $\sqrt{L}$)**.** *Assume* (A1), (A2), (A3), *and let* $1 < p < \infty$. *The following assertions are equivalent:*

(1) $\mathsf{D}_p(\sqrt{L}) = \mathsf{D}_p(D_V)$ *with* $\|\sqrt{L} f\|_p \eqsim \|D_V f\|_p$ *for* $f \in \mathsf{D}_p(\sqrt{L})$*;*

(2) $\underline{L}$ *admits a bounded* H^∞*-functional calculus on* $\overline{\mathsf{R}_p(D_V)}$*;*
(3) $\mathsf{D}(\sqrt{A}) = \mathsf{D}(V)$ *with* $\|\sqrt{A}h\| \eqsim \|Vh\|$ *for* $h \in \mathsf{D}(\sqrt{A})$*;*
(4) $\underline{A}$ *admits a bounded* H^∞*-functional calculus on* $\overline{\mathsf{R}(V)}$*.*

For the precise definition of operators admitting a bounded H^∞-functional calculus we refer to Section 7. Moreover we shall see (Lemma 10.2) that if the equivalent conditions of Theorem 2.1 hold, then their analogues where B is replaced by B^* hold as well. Some further equivalent conditions are given at the end of Section 10.

Theorem 2.2 (Domain of L)**.** *Let* $1 < p < \infty$ *and let the equivalent conditions of Theorem 2.1 be satisfied. Then*

$$\mathsf{D}_p(L) = \mathsf{D}_p(D_V^2) \cap \mathsf{D}_p(D_A)$$

with equivalence of norms

$$\|f\|_p + \|Lf\|_p \eqsim \|f\|_p + \|D_V f\|_p + \|D_V^2 f\|_p + \|D_A f\|_p.$$

Here $D_A = AD$, where D is the Malliavin derivative in the direction of H. For the precise definitions of D_V^2 and D_A we refer to Section 11 where Theorem 2.2 is proved.

The conditions of Theorem 2.1 are automatically satisfied in each of the following two cases:

(i) B is self-adjoint. In this case A is self-adjoint and therefore (3) holds by the theory of symmetric forms (since A is associated with a closed symmetric form with domain $\mathsf{D}(V)$; see Section 4).
(ii) V has finite dimensional range. In this case (4) is satisfied; since $\underline{A}$ is injective on the (closed) range of V (see Lemma 5.4), the H^∞-functional calculus of $\underline{A}$ is given by the Dunford calculus.

In fact we shall prove the stronger result that one-sided inclusions in (1) and (3) of Theorem 2.1 hold if and only if $\underline{A}$ and/or $\underline{L}$ satisfies a corresponding square function estimate. In particular, L^p-boundedness of the Riesz transform

$$\|(D_V/\sqrt{L})f\|_p \lesssim \|f\|_p$$

is characterised by the square function estimate

$$\|u\| \lesssim \Big(\int_0^\infty \|t\underline{A}\,\underline{S}(t)u\|^2\,\frac{dt}{t}\Big)^{1/2}, \quad u \in \overline{\mathsf{R}(V)}.$$

Here $\underline{S}$ is the bounded analytic semigroup on $\underline{H}$ generated by $-\underline{A}$.

In Section 3 we present two applications of Theorem 2.1. The first gives an extension of the classical Meyer inequalities for the Ornstein-Uhlenbeck operator. The second concerns L^p-estimates for the square root of the L^p-realisation of generators of the second quantisation of analytic C_0-contraction semigroups on Hilbert spaces. We show that for such semigroups the square root property of Theorem 2.1 (3) is preserved under second quantisation.

The proof of Theorem 2.1 depends crucially on the following gradient bounds for the semigroup P generated by $-L$ and the first part of the Littlewood-Paley-Stein inequalities below.

Theorem 2.3 (Gradient bounds)**.** *Assume* (A1)*,* (A2)*,* (A3)*, and let* $1 < p < \infty$*.*

(1) *For all* $f \in \mathscr{F}C_{\mathrm{b}}(E)$ *and* $t > 0$ *we have, for* μ*-almost all* $x \in E$*,*

$$\sqrt{t}\|D_V P(t)f(x)\| \lesssim (P(t)|f|^2(x))^{1/2}.$$

(2) *The set* $\{\sqrt{t}D_V P(t) : \ t \geqslant 0\}$ *is* R*-bounded in* $\mathscr{L}(L^p, \underline{L}^p)$.

The notion of R-boundedness is a strengthening of the notion of uniform boundedness and is discussed in Section 6.

Theorem 2.4 (Littlewood-Paley-Stein inequalities)**.** *Assume* (A1), (A2), (A3), *and let* $1 < p < \infty$. *For all* $f \in L^p$ *we have the square function estimate*

$$\|f - P_{\mathsf{N}_p(L)} f\|_p \lesssim \Big\| \Big(\int_0^\infty \|\sqrt{t} D_V P(t) f\|^2 \, \frac{dt}{t} \Big)^{1/2} \Big\|_p \lesssim \|f\|_p,$$

where $P_{\mathsf{N}_p(L)}$ *is the projection onto* $\mathsf{N}_p(L)$ *along the direct sum decomposition* $L^p = \mathsf{N}_p(L) \oplus \overline{\mathsf{R}_p(L)}$.

Theorem 2.1 is proved in Sections 10 and 12, and Theorems 2.3 and 2.4 are proved in Section 8. At this point we emphasise that in the present nonsymmetric setting, it is not possible to derive the cases $p > 2$ from the cases $1 < p \leqslant 2$ by means of duality arguments (as is done, for example, in [12, 49]). New ideas are required; see Section 8.2.

It follows from [6, Proposition 2.2] that the following Hodge decompositions hold:

$$H = \overline{\mathsf{R}(V^*B)} \oplus \mathsf{N}(V), \quad \underline{H} = \overline{\mathsf{R}(V)} \oplus \mathsf{N}(V^*B). \tag{2.1}$$

Here V^*B is interpreted as a closed densely defined operator from $\underline{H}$ to H. The second decomposition, however, shows that the closures of the ranges of V^*B and its restriction to $\overline{\mathsf{R}(V)}$ are the same. Therefore, in the first decomposition we may just as well interpret V^*B as an unbounded operator from $\overline{\mathsf{R}(V)}$ to H. This observation is relevant for the formulation of the following Gaussian L^p-analogues of the above decompositions, which are proved in Section 12.

Theorem 2.5 (Hodge decompositions)**.** *Assume* (A1), (A2), (A3), *and let* $1 < p < \infty$. *One has the direct sum decomposition*

$$L^p = \overline{\mathsf{R}_p(D_V^* B)} \oplus \mathsf{N}_p(D_V),$$

where $D_V^* B$ *is interpreted a closed densely defined operator from* $\overline{\mathsf{R}_p(D_V)}$ *to* L^p. *If the equivalent conditions of Theorem 2.1 hold, then the above decomposition remains true when* $D_V^* B$ *is interpreted as a closed densely defined operator from* $\underline{L}^p$ *to* L^p. *In that case one has the direct sum decomposition*

$$\underline{L}^p = \overline{\mathsf{R}_p(D_V)} \oplus \mathsf{N}_p(D_V^* B),$$

where $D_V^* B$ *is interpreted as a closed densely defined operator from* $\underline{L}^p$ *to* L^p.

In the proofs of these theorems we use the Hodge-Dirac formalism introduced recently by Axelsson, Keith, and McIntosh [6] in the context of the Kato square root problem. In the spirit of this formalism, let us define the Hodge-Dirac operator Π associated with D_V and $D_V^* B$ by the operator matrix

$$\Pi := \begin{bmatrix} 0 & D_V^* B \\ D_V & 0 \end{bmatrix}.$$

Using Theorems 2.3, 2.4, and 2.5 we shall prove:

Theorem 2.6 (R-bisectoriality)**.** *Assume* (A1), (A2), (A3), *and let* $1 < p < \infty$. *The operator* Π *is* R*-bisectorial on* $L^p \oplus \overline{\mathsf{R}_p(D_V)}$. *If the equivalent conditions of Theorem 2.1 hold, then* Π *is* R*-bisectorial on* $L^p \oplus \underline{L}^p$.

For the definition of R-(bi)sectorial operators we refer to Section 7. The analogue of Theorem 2.6 for the more general framework considered in [6] generally fails for $p \neq 2$. It this therefore a non-trivial fact that the theorem does hold in the special case considered here. Its proof depends on Theorems 2.3, 2.4, and a delicate L^p-analysis of the operators D_V and $D_V^* B$, which is carried out in Section 9.

From the fact that on $L^p \oplus \overline{\mathsf{R}_p(D_V)}$ one has

$$\Pi^2 = \begin{bmatrix} L & 0 \\ 0 & \underline{L} \end{bmatrix}$$

we deduce that Π admits a bounded H^∞-functional calculus on $L^p \oplus \overline{\mathsf{R}_p(D_V)}$ if and only if $\underline{L}$ admits a bounded H^∞-functional calculus on $\overline{\mathsf{R}_p(D_V)}$, i.e., if and only if condition (2) in Theorem 2.1 is satisfied. An alternative proof of the implication (2)$\Rightarrow$(1) of Theorem 2.1 can now be derived from the H^∞-functional calculus of Π applied to the function $\operatorname{sgn}(z) = z/\sqrt{z^2}$; this is done in the final Section 12.

3. Consequences

Before we start with the proofs of our main results we discuss a number of situations where operators of the form studied in this paper arise naturally.

3.1. **The Ornstein-Uhlenbeck operator.** Let (A1) be satisfied. Taking $\underline{H} = H$ and $V = I$, the derivative D_V reduces to the Malliavin derivative in the direction of H. Assumption (A2) is then obviously satisfied. Let B be an arbitrary operator satisfying (A3). Since $\underline{A} = B$ is bounded and sectorial, condition (4) of Theorem 2.1 is satisfied.

For the special choice $B = \frac{1}{2}I$, the resulting operator $L = \frac{1}{2}D_V^* D_V$ is the classical Ornstein-Uhlenbeck operator of Malliavin calculus, and the two-sided L^p-estimate for $\sqrt{L}$ of Theorem 2.1 reduces to the celebrated Meyer inequalities.

3.2. **Linear stochastic evolution equations.** In this subsection we shall describe an application of our results to stochastic evolution equations. This application will be worked out in more detail in a forthcoming paper. For unexplained terminology and background material we refer to [16, 42].

Consider the following linear stochastic evolution equation in a Banach space E:

$$\begin{cases} dU(t) = \mathscr{A}U(t)\,dt + \sigma dW(t), & t \geqslant 0, \\ U(0) = x. \end{cases}$$

Here, $\mathscr{A}$ is assumed to generate a C_0-semigroup on E, σ is a bounded linear operator from a Hilbert space $\mathscr{H}$ to E, and W is an $\mathscr{H}$-cylindrical Brownian motion. For later reference we put $\underline{H} := \mathscr{H} \ominus \mathsf{N}(\sigma)$.

Let us assume now that for each initial value $x \in E$ the above problem admits a unique weak solution $U^x = (U^x(t))_{t \geqslant 0}$ and that these solutions admit an invariant measure; necessary and sufficient conditions for this to happen can be found in [16, 42, 43]. Under this assumption one has weak convergence $\lim_{t\to\infty} \mu_t = \mu$, where μ_t is the distribution of the E-valued centred Gaussian random variable $U^0(t)$ corresponding to the initial value $x = 0$. The limit measure μ is invariant as well; in a sense that can be made precise it is the minimal invariant measure associated with the above problem. The reproducing kernel Hilbert space associated with μ is denoted by H and the corresponding inclusion operator $H \hookrightarrow E$ by i.

Define, for bounded continuous functions $f : E \to \mathbb{R}$,

$$P(t)f(x) := \mathbb{E}f(U^x(t)), \quad t \geqslant 0,\ x \in E.$$

The operators $P(t)$ extend in a unique way to a C_0-contraction semigroup on $L^p(E,\mu)$ for all $1 \leqslant p < \infty$. It has been shown in [37] that if P is analytic for some (equivalently, for all) $1 < p < \infty$, then its infinitesimal generator $-L$ is of the form considered in Section 2. More precisely, there exists a unique coercive operator B on $\underline{H}$ such that

$$L = D_V^* B D_V,$$

where $V : \mathsf{D}(V) \subseteq H \to \underline{H}$ is the closed linear operator defined by

$$V(i^*x^*) := \sigma^* x^*, \quad x^* \in E^*.$$

It is easy to see that D_V is nothing but the Fréchet derivative on E in the direction of $\underline{H}$ (where we think of $\underline{H}$ as a Hilbert subspace of E under the identification $u \mapsto \sigma u$).

As a consequence of our main results we obtain the following result.

Theorem 3.1. *In the above situation, suppose that the transition semigroup P is analytic on $L^p(E,\mu)$ for some (all) $1 < p < \infty$.*

1. *The C_0-semigroup generated by A leaves $\underline{H}$ invariant and restricts to a bounded analytic C_0-semigroup on $\underline{H}$;*
2. *We have $\mathsf{D}_p(L) = \mathsf{D}_p(D_V)$ with equivalence of norms $\|Lf\|_p \eqsim \|D_V f\|_p$ if and only if the negative generator of the restricted semigroup admits a bounded H^∞-functional calculus on $\underline{H}$.*

3.3. **Second quantised operators.** Theorem 2.1 can be applied to the second quantisation of an arbitrary generator $-A$ of an analytic C_0-contraction semigroup on a Hilbert space H. The idea is to prove that such operators A can be represented as V^*BV for certain canonical choices of operators V and B satisfying (A2) and (A3). If E and μ are given such that (A1) holds, the second observation is that the generator of the second quantised semigroup on $L^p = L^p(E,\mu)$ equals the operator $-D_V^* B D_V$ and therefore Theorem 2.1 can be applied.

We begin with recalling the definition and elementary properties of second quantised operators. For more systematic discussions we refer to [24, 50]. We work over the real scalar field and complexify afterwards.

Let $H^{(\leqslant 0)} := \mathbb{R}\mathbf{1}$ and define $H^{(\leqslant n)}$ inductively as the closed linear span of $H^{(\leqslant (n-1))}$ together with all products of the form $\phi_{h_1} \cdot \ldots \cdot \phi_{h_n}$ with $h_1, \ldots, h_m \in H$. Then we let $H^{(0)} := \mathbb{R}\mathbf{1}$ and define $H^{(n)}$ as the orthogonal complement of $H^{(\leqslant (n-1))}$ in $H^{(\leqslant n)}$. The space $H^{(n)}$ is usually referred to as the n-th *Wiener-Itô chaos.* We have the orthogonal *Wiener-Itô decomposition*

$$L^2 = \bigoplus_{n=0}^{\infty} H^{(n)}.$$

It is well known that for all $1 \leqslant p \leqslant q < \infty$ there exist constants $C_{n,p,q} > 0$ such that

$$\|F\|_p \leqslant \|F\|_q \leqslant C_{n,p,q}\|F\|_p, \quad F \in H^{(n)}.$$

Denoting by I_n the orthogonal projection in L^2 onto $H^{(n)}$, we have the identity

$$[I_n(\phi_{h_1}\cdot\ldots\cdot\phi_{h_n}), I_n(\phi_{h'_1}\cdot\ldots\cdot\phi_{h'_n})] = \frac{1}{n!}\sum_{\sigma\in S_n}[h_1,h'_{\sigma(1)}]\cdot\ldots\cdot[h_n,h'_{\sigma(n)}],$$

where S_n is the permutation group on n elements. This shows that $H^{(n)}$ is canonically isometric to the n-fold symmetric tensor product $H^{\circledS n}$, the isometry being given explicitly by

$$I_n(\phi_{h_1}\cdot\ldots\cdot\phi_{h_n}) \mapsto \frac{1}{\sqrt{n!}}\sum_{\sigma\in S_n} h_{\sigma(1)}\otimes\cdots\otimes h_{\sigma(n)}.$$

Thus the Wiener-Itô decomposition induces a canonical isometry of L^2 and the (symmetric) *Fock space*

$$\Gamma(H) := \bigoplus_{n=0}^{\infty} H^{\circledS n}.$$

Let $T\in\mathscr{L}(H)$ be a contraction. We denote by $\Gamma(T)\in\mathscr{L}(\Gamma(H))$ the (symmetric) *second quantisation* of T, which is defined on $H^{\circledS n}$ by

$$\Gamma(T)\sum_{\sigma\in S_n} h_{\sigma(1)}\otimes\cdots\otimes h_{\sigma(n)} = \sum_{\sigma\in S_n} Th_{\sigma(1)}\otimes\cdots\otimes Th_{\sigma(n)}.$$

By the Wiener-Itô isometry, T induces a contraction on L^2 and we have

$$\Gamma(T)I_n(\phi_{h_1}\cdot\ldots\cdot\phi_{h_n}) = I_n(\phi_{Th_1}\cdot\ldots\cdot\phi_{Th_n}). \tag{3.1}$$

Moreover, $\Gamma(T)$ is a positive operator on L^2. We have the identities

$$\Gamma(I)=I,\quad \Gamma(T_1T_2)=\Gamma(T_1)\Gamma(T_2),\quad (\Gamma(T))^* = \Gamma(T^*). \tag{3.2}$$

For all $1\leqslant p\leqslant\infty$, $\Gamma(T)$ extends to a positive contraction on L^p and (3.2) continues to hold.

For later reference we collect some further properties of second quantised operators which will not be used in the present section. The following formula is known as *Mehler's formula* [45]: if $f=\varphi(\phi_{h_1},\ldots,\phi_{h_n})$ with $\varphi\in C_{\mathrm{b}}(\mathbb{R}^n)$ and $h_1,\ldots,h_n\in H$, then for μ-almost all $x\in E$ we have

(3.3)

$$\Gamma(T)f(x) = \int_E \varphi(\phi_{Th_1}(x)+\phi_{\sqrt{I-T^*T}h_1}(y),\ldots,\phi_{Th_n}(x)+\phi_{\sqrt{I-T^*T}h_n}(y))\,d\mu(y).$$

For $h\in H$ define

$$E_h = \sum_{n=0}^{\infty}\frac{1}{n!}I_n(\phi_h^n) = \exp(\phi_h - \tfrac12\|h\|^2). \tag{3.4}$$

This sum converges absolutely in L^p for all $1\leqslant p<\infty$, and the linear span of the functions E_h is dense in L^p [45, Chapter 1]. From a routine approximation argument using the closedness of D_V we obtain that $h\in\mathsf{D}(V)$ implies $E_h\in\mathsf{D}_p(D_V)$ and

$$D_VE_h = E_h\otimes Vh. \tag{3.5}$$

From (3.1) and (3.4) one has the identity

$$\Gamma(T)E_h = E_{Th}. \tag{3.6}$$

Let us now turn to the situation where $-A$ be the generator of an analytic C_0-contraction semigroup on a Hilbert space H. It is well known [46, Theorem 1.57, Theorem 1.58 and the remarks following these results] that A is associated with a sesquilinear form a on (the complexification of) H which is densely defined, closed and sectorial, i.e., there exists a constant $C \geqslant 0$ such that

$$|\mathrm{Im}\, a(h,h)| \leqslant C\mathrm{Re}\, a(h,h), \quad h \in \mathsf{D}(a).$$

The next result may be known to experts, but as we could not find an explicit reference we include a proof for the convenience of the reader.

Theorem 3.2. *There exists a Hilbert space $\underline{H}$, a closed operator $V : \mathsf{D}(V) \subseteq H \to \underline{H}$ with dense domain $\mathsf{D}(V) = \mathsf{D}(a)$ and dense range, and a bounded coercive operator $B \in \mathscr{L}(\underline{H})$ such that*

$$A = V^*BV.$$

More precisely, this identity means that we have $a(g,h) = [BVg, Vh]$ for all $g,h \in \mathsf{D}(V)$; cf. Section 4.

Proof. Writing $a(h) := a(h,h)$ by [46, Proposition 1.8] we have

$$|a(g,h)| \lesssim (\mathrm{Re}\, a(g))^{1/2}(\mathrm{Re}\, a(h))^{1/2}, \quad g,h \in \mathsf{D}(a).$$

We claim that $N := \{h \in \mathsf{D}(a) : \mathrm{Re}\, a(h) = 0\}$ is a closed subspace of $\mathsf{D}(a)$. Indeed, if $h_n \to h$ in $\mathsf{D}(a)$ and $\mathrm{Re}\, a(h_n) = 0$, then

$$\begin{aligned}|\mathrm{Re}\, a(h)| \leqslant |a(h) - a(h_n)| \leqslant &(\mathrm{Re}\, a(h))^{1/2}(\mathrm{Re}\, a(h-h_n))^{1/2} \\ &+ (\mathrm{Re}\, a(h-h_n))^{1/2}(\mathrm{Re}\, a(h_n))^{1/2},\end{aligned}$$

which becomes arbitrary small as $n \to \infty$.

On the quotient $\mathsf{D}(a)/N$ we define a sesquilinear form

$$[Vg, Vh] := \frac{1}{2}(a(g,h) + \overline{a(h,g)}), \qquad g,h \in \mathsf{D}(a),$$

where V denotes the canonical mapping from $\mathsf{D}(a)$ onto $\mathsf{D}(a)/N$. This form is well defined, since for $n, n' \in N$ we have

$$\begin{aligned}|a(g+n, h+n') - a(g,h)| &\leqslant (\mathrm{Re}\, a(n))^{1/2}(\mathrm{Re}\, a(h))^{1/2} \\ &\quad + (\mathrm{Re}\, a(g))^{1/2}(\mathrm{Re}\, a(n'))^{1/2} \\ &\quad + (\mathrm{Re}\, a(n))^{1/2}(\mathrm{Re}\, a(n'))^{1/2} \\ &= 0.\end{aligned}$$

Since $\mathrm{Re}\, a(h) = 0$ implies $[h] = [0]$, the form $[\cdot,\cdot]$ is an inner product on $\mathsf{D}(a)/N$. We put

$$\underline{H} := \overline{\mathsf{D}(a)/N},$$

where the completion is taken with respect to the norm induced by $[\cdot,\cdot]$. We interpret V as a linear operator from H into $\underline{H}$ with dense domain $\mathsf{D}(V) = \mathsf{D}(a)$ and dense range. To show that V is closed, we take a sequence $(h_n)_{n\geqslant 1}$ in $\mathsf{D}(a)$ such that $h_n \to h$ in H and $Vh_n \to u$ in $\underline{H}$. Since $\mathrm{Re}\, a(h_n - h_m) = \|V(h_n - h_m)\|^2 \to 0$ as $m,n \to \infty$, the sequence $(h_n)_{n\geqslant 1}$ is Cauchy in $\mathsf{D}(a)$. Thus the closedness of a implies that $(h_n)_{n\geqslant 1}$ has a limit in $D(a)$, which is h since $h_n \to h$ in H. Consequently, $\|Vh_n - Vh\|^2 = \mathrm{Re}\, a(h_n - h) \to 0$. We conclude that V is closed.

Now we define a sesquilinear form b on $\mathsf{R}(V)$ by

$$b(Vg, Vh) := a(g,h).$$

This is well defined, since $Vg = V\widetilde{g}$ and $Vh = V\widetilde{h}$ imply that

$$\begin{aligned}|a(g,h) - a(\widetilde{g},\widetilde{h})| &\leqslant |a(g-\widetilde{g},h)| + |a(\widetilde{g},h-\widetilde{h}))|\\ &\leqslant (\mathrm{Re}\,a(g-\widetilde{g})\,\mathrm{Re}\,a(h))^{1/2} + (\mathrm{Re}\,a(\widetilde{g})\,\mathrm{Re}\,a(h-\widetilde{h}))^{1/2}\\ &= \|V(g-\widetilde{g})\|\,\|Vh\| + \|V\widetilde{g}\|\,\|V(h-\widetilde{h})\| = 0.\end{aligned}$$

Moreover, the associated operator B extends to a bounded operator on $\underline{H}$, since

$$|b(Vg,Vh)| = |a(g,h)| \lesssim (\mathrm{Re}\,a(g))^{1/2}(\mathrm{Re}\,a(h))^{1/2} = \|Vg\|\,\|Vh\|.$$

We conclude that $a(g,h) = [BVg,Vh]$. By the identity

$$\|Vh\|^2 = \mathrm{Re}\,a(h) = \mathrm{Re}\,[BVh,Vh]$$

and the boundedness of B we infer that $\|u\|^2 = \mathrm{Re}\,[Bu,u]$ for all $u \in \underline{H}$, and the coercivity of B follows. $\square$

Although the triple $(\underline{H},V,B)$ is not unique, the next result implies that the statements in Theorem 2.1 do not depend on the choice of $(\underline{H},V,B)$.

Proposition 3.3. *Let $-A$ be the generator of an analytic C_0-contraction semigroup on H. Let $(\underline{H},V,B)$ and $(\underline{\widetilde{H}},\widetilde{V},\widetilde{B})$ be triples with the properties as stated in Theorem 3.2. Then:*

(i) *The coercivity constants k and $\widetilde{k}$ of B and $\widetilde{B}$ coincide;*
(ii) *$\mathsf{D}(V) = \mathsf{D}(\widetilde{V})$ with $\|Vh\| \eqsim \|\widetilde{V}h\|$.*

If in addition to the above assumptions (E,H,μ) is an abstract Wiener space, then for $1 \leqslant p < \infty$ we have

(iii) *$\mathsf{D}_p(D_V) = \mathsf{D}_p(D_{\widetilde{V}})$ with $\|D_V f\|_p \eqsim \|D_{\widetilde{V}} f\|_p$.*

Proof. (i): This follows from the identity $[BVh,Vh] = a(h,h) = [\widetilde{B}\widetilde{V}h,\widetilde{V}h]$ for $h \in \mathsf{D}(a)$ and the fact that V and $\widetilde{V}$ have dense range.

(ii): For $h \in \mathsf{D}(A)$ we have

$$k\|Vh\|^2 \leqslant \mathrm{Re}\,[BVh,Vh] = \mathrm{Re}\,[Ah,h] = \mathrm{Re}\,[\widetilde{B}\widetilde{V}h,\widetilde{V}h] \leqslant \|\widetilde{B}\|\,\|\widetilde{V}h\|^2.$$

Since $\mathsf{D}(A)$ is a core for both $\mathsf{D}(V)$ and $\mathsf{D}(\widetilde{V})$ the result follows.

(iii): Let D denote the Malliavin derivative, which is well defined as a densely defined closed operator from $L^p(E,\mu)$ to $L^p(E,\mu;H)$, $1 \leqslant p < \infty$. For $f \in \mathscr{F}C^1_{\mathrm{b}}(E;\mathsf{D}(V))$ we have, by (ii),

$$\|D_V f\|_p^p = \int_E \|VDf\|^p\,d\mu \eqsim \int_E \|\widetilde{V}Df\|^p\,d\mu = \|D_{\widetilde{V}} f\|_p^p.$$

The claim follows from this since $\mathscr{F}C^1_{\mathrm{b}}(E;\mathsf{D}(V))$ is a core for $\mathsf{D}_p(D_V)$ and $\mathsf{D}_p(D_{\widetilde{V}})$. $\square$

Let $N := \{h \in \mathsf{D}(a) : \mathrm{Re}\,a(h) = 0\}$ and let $\dot{\mathsf{D}}(a) := \underline{H}$ be defined as in the proof of Theorem 3.2, i.e., $\dot{\mathsf{D}}(a)$ is the completion of $\mathsf{D}(a)/N$ with respect to the norm

$$\|Vh\|_{\dot{\mathsf{D}}(a)} := \sqrt{\mathrm{Re}\,(a(h))},$$

where V denotes the canonical operator from $\mathsf{D}(a)$ onto $\mathsf{D}(a)/N$. In the proof of Theorem 3.2 we showed that V is a closed operator from H into $\dot{\mathsf{D}}(a)$ with dense domain and dense range. We also constructed a coercive operator $B \in \mathscr{L}(\dot{\mathsf{D}}(a))$ such that $a(g,h) = [BVg,Vh]$ for $g,h \in \mathsf{D}(a)$.

In Lemma 5.3 below we show that the semigroup S generated by $-A$ induces a bounded analytic C_0-semigroup $\underline{S}$ on $\dot{\mathsf{D}}(a)$ in the sense that $\underline{S}(t)Vh = VS(t)h$ for all $h \in \mathsf{D}(V)$. Its generator will be denoted by $-\underline{A}$.

Now let (E, H, μ) be an abstract Wiener space and let $D_V := VD$ as before. For cylindrical functions $f = f(\phi_{h_1}, \dots, \phi_{h_n})$ with $h_j \in \mathsf{D}(V) = \mathsf{D}(a)$ we have

$$D_V f = \sum_{j=1}^{n} \partial_j f(\phi_{h_1}, \dots, \phi_{h_n}) \otimes V h_j.$$

As will be shown in Section 4, the realisation on $L^2(E, \mu)$ of the operator $L := D_V^* B D_V$ extends to a sectorial operator L on $L^p(E, \mu)$ for $1 < p < \infty$, and $-L$ equals the generator of the second quantisation on $L^p(E, \mu)$ of the semigroup S generated by $-A$ on H, i.e.,

$$P(t) = \Gamma(S(t)), \quad t \geqslant 0.$$

As a consequence, Theorem 2.1 can be translated into the following result.

Theorem 3.4. *Assume* (A1) *and let $-A$ be the generator of an analytic C_0-contraction semigroup S on H. Let $1 < p < \infty$ and let $-L$ denote the realisation on $L^p(E, \mu)$ of the generator of the second quantisation of S. The following assertions are equivalent:*

- (1′) $\mathsf{D}_p(\sqrt{L}) = \mathsf{D}_p(D_V)$ *with* $\|\sqrt{L} f\|_p \eqsim \|D_V f\|_p$ *for* $f \in \mathsf{D}_p(\sqrt{L})$*;*
- (3′) $\mathsf{D}(\sqrt{A}) = \mathsf{D}(a)$ *with* $\|\sqrt{A} h\| \eqsim \sqrt{a(h)}$ *for* $h \in \mathsf{D}(a)$*;*
- (4′) *the realisation of A in $\dot{\mathsf{D}}(a)$ admits a bounded H^∞-functional calculus.*

The main equivalence here is (1′)⇔(3′). It asserts that the square root property with homogeneous norms is preserved when passing from H to $L^p(E, \mu)$ by means of second quantisation.

The equivalence (3′)⇔(4′) is probably known, although we could not find a reference for it. The related equivalence (3″)⇔(4″), with

- (3″) $\mathsf{D}(\sqrt{A}) = \mathsf{D}(a)$;
- (4″) *the realisation of A in $\mathsf{D}(a)$ admits a bounded H^∞-functional calculus,*

is stated explicitly in [2, Theorem 5.5.2].

We have already mentioned that Theorem 2.1 is obtained by combining two one-sided versions of it involving Riesz transforms. The same is true for Theorem 3.4.

4. The operator L

In this section we give a rigorous definition of the operator L as a closed and densely defined operator acting in $L^p := L^p(E, \mu)$, where $1 < p < \infty$.

We begin with an analysis in the space $L^2 := L^2(E, \mu)$. Associated with the (complexified) operators $B : \underline{H} \to \underline{H}$, $V : \mathsf{D}(V) \subseteq H \to \underline{H}$, and $D_V : \mathsf{D}(D_V) \subseteq L^2 \to \underline{L}^2$ are the sesquilinear forms a on H and l on L^2 defined by $\mathsf{D}(a) := \mathsf{D}(V)$ and

$$a(h_1, h_2) := [BVh_1, Vh_2],$$

and $\mathsf{D}(l) := \mathsf{D}(D_V)$ and

$$l(f_1, f_2) := [BD_V f_1, D_V f_2],$$

where in the second line we identify B with the operator $I \otimes B$. Here, and in what follows, we write

$$\mathsf{D}(D_V) := \mathsf{D}_2(D_V), \quad \mathsf{R}(D_V) := \mathsf{R}_2(D_V).$$

The forms a and l are easily seen to be closed, densely defined and sectorial in H and L^2, respectively. The operators associated with these forms are denoted by A and L, respectively; their domains will be denoted by $\mathsf{D}(A)$ and $\mathsf{D}(L)$. We may write

$$A = V^* B V, \quad L = D_V^* B D_V;$$

this notation is justified by the observation that

$$\begin{aligned} h \in \mathsf{D}(A) &\Longleftrightarrow [h \in \mathsf{D}(V) \text{ and } BVh \in \mathsf{D}(V^*)], \\ f \in \mathsf{D}(L) &\Longleftrightarrow [f \in \mathsf{D}(D_V) \text{ and } BD_V f \in \mathsf{D}(D_V^*)], \end{aligned}$$

in which case we have

$$Ah = V^*(BVh), \quad Lf = D_V^*(BD_V)f.$$

Let us also note (cf. [46, Lemma 1.25]) that $\mathsf{D}(A)$ and $\mathsf{D}(L)$ are cores for $\mathsf{D}(V)$ and $\mathsf{D}(D_V)$, respectively.

For later use we observe that if B satisfies (A3), then also B^* satisfies (A3) and we have

$$A^* = V^* B^* V, \quad L^* = D_V^* B^* D_V$$

with similar justifications.

The proof of the next lemma can be found in [19]. Recall that $\phi : H \to L^2$ is the isometric embedding defined in Section 2.

Lemma 4.1. *Let $1 < p < \infty$. For all $f \in \mathscr{F}C_{\rm b}^1(E; \mathsf{D}(V))$ and $u \in \mathsf{D}(V^*)$ we have*

$$f \otimes u \in \mathsf{D}_p(D_V^*) \text{ and } D_V^*(f \otimes u) = f\phi_{V^*u} - [D_V f, u].$$

Lemma 4.2. *Identifying H with its image $\phi(H)$ in L^2, A is the part of L in H.*

Proof. Suppose first that $h \in \mathsf{D}(A)$. Then, by the form definition of A, we have $h \in \mathsf{D}(V)$ and $BVh \in \mathsf{D}(V^*)$. Hence Lemma 4.1 gives us $\mathbf{1} \otimes BVh \in \mathsf{D}(D_V^*)$ and

$$D_V^*(\mathbf{1} \otimes BVh) = \phi_{V^*BVh} = \phi_{Ah}.$$

From this, combined with the identity

$$D_V \phi_h = \mathbf{1} \otimes Vh$$

which follows from the definition of D_V, we deduce that

$$[BD_V\phi_h, D_V f] = [\mathbf{1} \otimes BVh, D_V f] = [\phi_{Ah}, f]$$

for all $f \in \mathsf{D}(D_V)$. Therefore, $\phi_h \in \mathsf{D}(L)$ and $L\phi_h = \phi_{Ah}$. Denoting the part of L in H by L^H for the moment, this argument shows that $A \subseteq L^H$.

On the other hand, if $\phi_h \in \mathsf{D}(L^H)$, then $\phi_h \in \mathsf{D}(L)$ and $L\phi_h = \phi_{h'}$ for some $h' \in H$. Hence for all $g \in \mathsf{D}(V)$ we obtain

$$[BVh, Vg] = [BD_V\phi_h, D_V\phi_g] = l(\phi_h, \phi_g) = [L\phi_h, \phi_g] = [\phi_{h'}, \phi_g] = [h', g].$$

It follows that $h \in \mathsf{D}(A)$ and $[Ah, g] = [h', g]$. This shows that $Ah = h'$, and we have proved the opposite inclusion $A \supseteq L^H$. $\square$

It follows from the theory of forms (cf. [46, Proposition 1.51]) that $-A$ and $-L$ generate analytic C_0-contraction semigroups $S = (S(t))_{t\geqslant 0}$ and $P = (P(t))_{t\geqslant 0}$ on H and L^2, respectively. In fact we have the following more precise result. Recall that the constant $k > 0$ has been introduced in Assumption (A3).

Proposition 4.3. *The operators $-A$ and $-L$ generate analytic C_0-contraction semigroups on H and L^2 of angle $\arctan\gamma$, where $\gamma = \frac{1}{2k}\|B - B^*\|$.*

Proof. We prove this for L; the proof of A is similar (alternatively, the result for A follows from the result for L via Lemma 4.2).

By the Lumer-Phillips theorem it suffices to show that L has numerical range in the sector of angle $\arctan\gamma$. Using that B is a (complexified) real operator, for $f \in \mathsf{D}(L)$ we have, with $F = \mathrm{Re}\, D_V f$ and $G = \mathrm{Im}\, D_V f$,

$$\begin{aligned}\big|\mathrm{Im}\,[Lf, f]\big| &= \big|[(B - B^*)F, G]\big| \\ &\leqslant \frac{1}{2}\|B - B^*\| \,\big(\|F\|^2 + \|G\|^2\big) \\ &\leqslant \frac{1}{2k}\|B - B^*\|\big([BF, F] + [BG, G]\big) \\ &= \frac{1}{2k}\|B - B^*\|\,\mathrm{Re}\,[Lf, f].\end{aligned} \tag{4.1}$$

□

The first main result of this section identifies P as the second quantisation of S.

Theorem 4.4. *For all $t \geqslant 0$ we have $P(t) = \Gamma(S(t))$.*

Proof. We recall from Lemma 4.2 that $P(t)\phi_h = S(t)h$ for all $h \in H$.

First we check that for all $h \in \mathsf{D}(A)$, the functions $E_h \in L^2$ are in the domains of L and $\widetilde{L}$, where $-\widetilde{L}$ is the generator of $\Gamma(S)$, and that both generators agree on those functions. Using (3.5) and Lemma 4.1 we obtain

$$\begin{aligned}LE_h &= D_V^* B D_V E_h \\ &= D_V^*(E_h \otimes BVh) \\ &= E_h \phi_{V^*BVh} - [BVh, Vh]E_h \\ &= (\phi_{Ah} - [Ah, h])E_h,\end{aligned}$$

while on the other hand, using (3.4) and (3.6) combined with a simple approximation argument, we have

$$\begin{aligned}\widetilde{L}E_h &= \lim_{t\downarrow 0} \tfrac{1}{t}(E_{S(t)h} - E_h) \\ &= E_h \frac{d}{dt}\Big|_{t=0}\Big(\phi_{S(t)h} - \frac{1}{2}\|S(t)h\|^2\Big) \\ &= (\phi_{Ah} - [Ah, h])E_h.\end{aligned}$$

The set $\mathrm{lin}\{E_h : h \in \mathsf{D}(A)\}$ is dense in L^2 and invariant under the semigroup $\Gamma(S)$. As a consequence, this set is a core for $\mathsf{D}(\widetilde{L})$. It follows that $\mathsf{D}(\widetilde{L}) \subseteq \mathsf{D}(L)$. Since both $-\widetilde{L}$ and $-L$ are generators this implies $\mathsf{D}(\widetilde{L}) = \mathsf{D}(L)$ and therefore $\widetilde{L} = L$. □

So far we have considered P as a C_0-semigroup in L^2. Having identified P as a second quantised semigroup on L^2, we are in a position to prove that P extends to the spaces L^p.

Theorem 4.5. *For $1 \leqslant p < \infty$, the semigroup P extends to a C_0-semigroup of positive contractions on L^p satisfying $\|P(t)f\|_\infty \leqslant \|f\|_\infty$ for $f \in L^\infty$. The measure μ is an invariant measure for P, i.e.,*

$$\int_E P(t)f\,d\mu = \int_E f\,d\mu, \qquad f \in L^p,\ t \geqslant 0.$$

For $1 < p < \infty$, P is an analytic C_0-contraction semigroup on L^p.

Proof. The extendability to a C_0-contraction semigroup on L^p, as well as the L^∞-contractivity and positivity follow from general results on second quantisation. The invariance of μ follows from

$$\int_E P(t)f\,d\mu = \int_E f\,P^*(t)\mathbf{1}\,d\mu = \int_E f\,d\mu, \quad f \in L^p.$$

Here we use that P^* is a second quantised semigroup as well, and therefore satisfies $P^*(t)\mathbf{1} = \mathbf{1}$ for all $t \geqslant 0$.

It remains to prove the last statement. We have seen in Proposition 4.3 that P extends to an analytic C_0-contraction semigroup on L^2. The extension to an analytic C_0-contraction semigroup on L^p, $1 < p < \infty$, follows from a standard argument involving the Stein interpolation theorem and duality. □

Remark 4.6. We mention that a different argument to establish analyticity in L^p has been given in the case of Ornstein-Uhlenbeck semigroups in [10, 37]. This argument also works in the more general setting considered here and yields an angle of analyticity which is better than the one obtained by Stein interpolation. For Ornstein-Uhlenbeck semigroups (for which we have $B + B^* = I$, see [37]), this angle is optimal.

Definition 4.7. On L^p we define the operator L as the negative generator of the semigroup P.

Lemma 4.8. *For all $1 < p < \infty$, $\mathscr{F}C_{\mathrm{b}}^\infty(E;\mathsf{D}(A))$ is a P-invariant core for $\mathsf{D}_p(L)$. Moreover, for $f, g \in \mathscr{F}C_{\mathrm{b}}^\infty(E;\mathsf{D}(A))$ and $\psi \in C_{\mathrm{b}}^\infty(\mathbb{R})$ we have*

(1) (Product rule) $fg \in \mathscr{F}C_{\mathrm{b}}^\infty(E;\mathsf{D}(A))$ *and*

$$L(fg) = fLg + gLf - [(B + B^*)D_V f, D_V g];$$

(2) (Chain rule) $\psi \circ f \in \mathscr{F}C_{\mathrm{b}}^\infty(E;\mathsf{D}(A))$ *and*

$$L(\psi \circ f) = (\psi' \circ f)Lf - (\psi'' \circ f)[BD_V f, D_V f].$$

Proof. First we show that $\mathscr{F}C_{\mathrm{b}}^\infty(E;\mathsf{D}(A))$ is contained in $\mathsf{D}_p(L)$; we thank Vladimir Bogachev for pointing out an argument which simplifies our original proof. Pick a function $f \in \mathscr{F}C_{\mathrm{b}}^\infty(E;\mathsf{D}(A))$ and notice that $f \in \mathsf{D}(L) \cap L^p$. The space L^p being reflexive, by a standard result from semigroup theory (cf. [9]) it suffices to show that

$$\limsup_{t \downarrow 0} \frac{1}{t}\|P(t)f - f\|_p < \infty.$$

Using that $L = D_V^* B D_V$ in L^2, an explicit calculation using Lemma 4.1 shows that $Lf \in L^2 \cap L^p$. Moreover, in L^2 we have the identity

$$\frac{1}{t}(P(t)f - f) = \frac{1}{t}\int_0^t P(s)Lf\,ds.$$

Since $Lf \in L^p$, the right-hand side can be interpreted as a Bochner integral in L^p, which for $0 < t \leqslant 1$ can be estimated in L^p by

$$\Big\| \frac{1}{t} \int_0^t P(s) Lf \, ds \Big\|_p \leqslant \|Lf\|_p.$$

This gives the desired bound for the limes superior.

To show that $\mathscr{F}C_{\rm b}^\infty(E; \mathsf{D}(A))$ is invariant under P, we take f of the form

$$f = \varphi(\phi_{h_1}, \dots, \phi_{h_n}),$$

with $\varphi \in C_{\rm b}^\infty(\mathbb{R}^n)$ and $h_1, \dots, h_n \in \mathsf{D}(A)$. Let $R(t) := \sqrt{I - S^*(t)S(t)}$. By Mehler's formula, for μ-almost all $x \in E$ we have

$$\begin{aligned} P(t)f(x) &= \int_E \varphi(\phi_{S(t)h_1}(x) + \phi_{R(t)h_1}(y), \dots \\ &\qquad\qquad \dots, \phi_{S(t)h_n}(x) + \phi_{R(t)h_n}(y)) \, d\mu(y) \\ &= \psi_t(\phi_{S(t)h_1}(x), \dots, \phi_{S(t)h_n}(x)), \end{aligned} \tag{4.2}$$

where

$$\psi_t(\xi_1, \dots, \xi_n) = \int_E \varphi(\xi_1 + \phi_{R(t)h_1}(y), \dots, \xi_n + \phi_{R(t)h_n}(y)) \, d\mu(y).$$

Since $\psi_t \in C_{\rm b}^\infty(\mathbb{R}^n)$ and $S(t)h_j \in \mathsf{D}(A)$ for $j = 1, \dots, n$, it follows that the subspace $\mathscr{F}C_{\rm b}^\infty(E; \mathsf{D}(A))$ is invariant under P. Since it is dense in L^p and contained in $\mathsf{D}_p(L)$, it is a core for $\mathsf{D}_p(L)$.

The identities (1) and (2) follow by direct computation, using the identity $L = D_V^* B D_V$ and Lemma 4.1. $\square$

Remark 4.9. The same proof shows that $\mathscr{F}C_{\rm b}^\infty(E; \mathsf{D}(A^k))$ is a P-invariant core for $\mathsf{D}_p(L)$ for every $k \geqslant 1$.

5. The operator $\underline{L}$

Our next aim is to give a rigorous description of the operator $\underline{L}$ on the spaces $\overline{\mathsf{R}_p(D_V)}$, $1 < p < \infty$, where the closure is taken in $\underline{L}^p := L^p(E, \mu; \underline{H})$.

On $\underline{H}$ and $\underline{L}^2$ we consider the sesquilinear forms $\underline{a} : \mathsf{D}(V^*) \times \mathsf{D}(V^*) \to \mathbb{C}$,

$$\underline{a}(u_1, u_2) := [V^* u_1, V^* u_2]$$

and $\underline{l} : \mathsf{D}(D_V^*) \times \mathsf{D}(D_V^*) \to \mathbb{C}$,

$$\underline{l}(F_1, F_2) := [D_V^* F_1, D_V^* F_2].$$

Here, $D_V^* : \mathsf{D}(D_V^*) \subseteq \underline{L}^2 \to L^2$ is the adjoint of the operator $D_V : \mathsf{D}(D_V) \subseteq L^2 \to \underline{L}^2$. The forms a and l are closed, densely defined and sectorial. The operators associated with these forms are denoted by $\underline{A}_I$ and $\underline{L}_I$ respectively, with domains $\mathsf{D}(\underline{A}_I)$ and $\mathsf{D}(\underline{L}_I)$. We may write

$$\underline{A}_I = VV^*, \qquad \underline{L}_I = D_V D_V^*$$

with similar justifications as before. These operators are self-adjoint; see e.g. [46, Proposition 1.31]. We introduce next the operators

$$\begin{aligned} \mathsf{D}(\underline{A}) &:= \{h \in \underline{H} : \ Bh \in \mathsf{D}(\underline{A}_I)\}, \qquad & \underline{A} &:= \underline{A}_I B; \\ \mathsf{D}(\underline{L}) &:= \{F \in \underline{L}^2 : \ BF \in \mathsf{D}(\underline{L}_I)\}, \qquad & \underline{L} &:= \underline{L}_I B. \end{aligned}$$

Note that

$$\underline{A} = VV^*B, \qquad \underline{L} = D_V D_V^* B.$$

It follows from standard operator theory [6, Lemma 4.1] that $\underline{A}$ and $\underline{L}$ are closed and densely defined and satisfy

$$\underline{A} = (B^*\underline{A}_I)^*, \qquad \underline{L} = (B^*\underline{L}_I)^*.$$

Proposition 5.1. *The operators $\underline{A}$ and $\underline{L}$ are sectorial on $\underline{H}$ and $\underline{L}^2$ of angle* $\arctan\gamma$, *where $\gamma := \frac{1}{2k}\|B - B^*\|$. For all $u \in \mathsf{D}(\underline{A})$ we have $\mathbf{1}\otimes u \in \mathsf{D}(\underline{L})$ and*

$$\underline{L}(\mathbf{1}\otimes u) = \mathbf{1}\otimes \underline{A}u.$$

Proof. Writing $v := \mathrm{Re}\, u$ and $w := \mathrm{Im}\, u$, by estimating as in (4.1) we obtain

$$|\mathrm{Im}\,[Bu,u]| \leqslant \frac{1}{2k}\|B - B^*\|\mathrm{Re}\,[Bu,u].$$

This shows that the numerical range of B is contained in the closed sector around $\mathbb{R}_+$ of angle $\arctan\gamma$. The same is true for the operator B as an operator acting on $\underline{L}^2$. Hence it follows from [5, Proposition 7.1] (in which 'positive' may be weakened to 'non-negative') that the operators $\underline{A} = \underline{A}_I B$ and $\underline{L} = \underline{L}_I B$ are sectorial of angle $\arctan\gamma$. The final identity follows from

$$\underline{L}(\mathbf{1}\otimes u) = D_V D_V^*(\mathbf{1}\otimes Bu) = D_V(\phi_{V^*Bu}) = \mathbf{1}\otimes VV^*Bu = \mathbf{1}\otimes \underline{A}u.$$

□

As a consequence, $-\underline{A}$ and $-\underline{L}$ generate bounded analytic C_0-semigroups of angle $\operatorname{arccot}\gamma$ on $\underline{H}$ and $\underline{L}^2$. In what follows we denote these semigroups by $\underline{S}$ and $\underline{P}$.

Lemma 5.2. *If $h \in \mathsf{D}(A)$ and $Ah \in \mathsf{D}(V)$, then $Vh \in \mathsf{D}(\underline{A})$ and*

$$\underline{A}Vh = VAh.$$

Proof. Since $h \in \mathsf{D}(A)$, the definition of A as the operator associated with the form $(h,g) \mapsto [BVh, Vg]$ implies that $h \in \mathsf{D}(V)$, $BVh \in \mathsf{D}(V^*)$, and $Ah = V^*(BVh)$.

To check that we have $Vh \in \mathsf{D}(\underline{A})$, in view of the identity $\underline{A} = (B^*\underline{A}_I)^*$ we must find $h' \in \underline{H}$ such that $[B^*\underline{A}_I g, Vh] = [g, h']$ for all $g \in \mathsf{D}(\underline{A}_I)$. But $h' := VAh$ does the job, since $[g, VAh] = [g, VV^*BVh] = [B^*\underline{A}_I g, Vh]$; this implies that $Vh \in \mathsf{D}(\underline{A})$ and $\underline{A}Vh = VAh$. □

Lemma 5.3. *For all $h \in \mathsf{D}(V)$ and $t \geqslant 0$ we have $S(t)h \in \mathsf{D}(V)$ and*

$$VS(t)h = \underline{S}(t)Vh.$$

Proof. We may assume that $t > 0$.

First let $g \in \mathsf{D}(A^2)$. Then $Ag \in \mathsf{D}(A) \subseteq \mathsf{D}(V)$, and therefore Lemma 5.2 implies that $Vg \in \mathsf{D}(\underline{A})$ and $\underline{A}Vg = VAg$. For $\lambda > 0$ it follows that $(I + \lambda\underline{A})Vg = V(I + \lambda A)g$. Applying this to $g = (I + \lambda A)^{-1}h$ with $h \in \mathsf{D}(A)$ we obtain

$$V(I + \lambda A)^{-1}h = (I + \lambda\underline{A})^{-1}Vh.$$

Taking $\lambda = \frac{t}{n}$ and repeating this argument n times we obtain, for all $h \in \mathsf{D}(A)$,

$$V(I + \tfrac{t}{n}A)^{-n}h = (I + \tfrac{t}{n}\underline{A})^{-n}Vh.$$

Taking limits $n \to \infty$ and using the closedness of V, we obtain $S(t)h \in D(V)$ and

$$VS(t)h = \underline{S}(t)Vh.$$

We are still assuming that $h \in \mathsf{D}(A)$. However, this assumption may now be removed by recalling the fact that $\mathsf{D}(A)$ is a core for $\mathsf{D}(V)$. □

Lemma 5.4. *For all $t \geqslant 0$ we have $\underline{S}(t)\overline{\mathsf{R}(V)} \subseteq \overline{\mathsf{R}(V)}$. Moreover, the part of $\underline{A}$ in $\overline{\mathsf{R}(V)}$ is injective.*

Proof. The first assertion follows from Lemma 5.3. Suppose that $\underline{A}u = VV^*Bu = 0$ for some u belonging to the domain of the part of $\underline{A}$ in $\overline{\mathsf{R}(V)}$. Then $\|V^*Bu\|^2 = 0$, so $Bu \in \mathsf{N}(V^*)$. Thus $[Bu, Vh] = 0$ for all $h \in \mathsf{D}(V)$. Since $u \in \overline{\mathsf{R}(V)}$ it follows that $[Bu, u] = 0$, and therefore $u = 0$ by the coercivity of B on $\overline{\mathsf{R}(V)}$. $\square$

Next we show that the semigroups $\underline{P}$ and $P \otimes \underline{S}$ agree on $\overline{\mathsf{R}(D_V)}$. We need two lemmas which are formulated, for later reference, for the L^p-setting.

Lemma 5.5. *For $1 < p < \infty$, $\mathscr{F}C_{\mathrm{b}}^{\infty}(E; \mathsf{D}(A))$ is a core for $\mathsf{D}_p(D_V)$.*

Proof. First let $f = \varphi(\phi_{h_1}, \dots, \phi_{h_n})$ with $\varphi \in C_{\mathrm{b}}^1(\mathbb{R}^n)$ and $h_1, \dots, h_n \in \mathsf{D}(V)$. Choose sequences $(h_{jk})_{k\geqslant 1}$ in $\mathsf{D}(A)$ with $h_{jk} \to h_j$ in $\mathsf{D}(V)$ as $k \to \infty$. Then $f_k \to f$ in L^p and $D_V f_k \to D_V f$ in $\underline{L}^p$, where $f_k = \varphi(\phi_{h_{1k}}, \dots . \phi_{h_{nk}})$. Since $\mathscr{F}C_{\mathrm{b}}^1(E; \mathsf{D}(V))$ is a core for $\mathsf{D}_p(D_V)$, this proves that $\mathscr{F}C_{\mathrm{b}}^1(E; \mathsf{D}(A))$ is a core for $\mathsf{D}_p(D_V)$. Now a standard mollifier argument, convolving φ with a smooth function of compact support, shows that $\mathscr{F}C_{\mathrm{b}}^{\infty}(E; \mathsf{D}(A))$ is a core for $\mathsf{D}_p(D_V)$. $\square$

The next result is well known in the context of Ornstein-Uhlenbeck semigroups; see, e.g., [12, Lemma 2.7], [36, Proposition 3.5].

Theorem 5.6. *For all $1 < p < \infty$, the semigroup $P \otimes \underline{S}$ restricts to a bounded analytic C_0-semigroup on $\overline{\mathsf{R}_p(D_V)}$. For $f \in \mathsf{D}_p(D_V)$ and $t \geqslant 0$ we have $P(t)f \in \mathsf{D}_p(D_V)$ and*

$$D_V P(t)f = (P(t) \otimes \underline{S}(t))D_V f.$$

Proof. First we show that for all $f \in \mathsf{D}_p(D_V)$ we have $P(t)f \in \mathsf{D}_p(D_V)$ and $D_V P(t)f = (P(t) \otimes \underline{S}(t))D_V f$. Since D_V is closed and $\mathscr{F}C_{\mathrm{b}}^{\infty}(E; \mathsf{D}(A))$ is a core for $\mathsf{D}_p(D_V)$ by Lemma 5.5, it suffices to check this for functions $f \in \mathscr{F}C_{\mathrm{b}}^{\infty}(E; \mathsf{D}(A))$.

We use the notations of Lemma 4.8. By (4.2) and Lemma 5.3, for functions $f = \varphi(\phi_{h_1}, \dots, \phi_{h_n})$ we have, for μ-almost all $x \in E$,

$$\begin{aligned}
D_V P(t)f(x) &= \sum_{j=1}^{n} \partial_j \psi_t(\phi_{S(t)h_1}(x), \dots, \phi_{S(t)h_n}(x)) \otimes VS(t)h_j \\
&= \sum_{j=1}^{n} \int_E \partial_j \varphi(\phi_{S(t)h_1}(x) + \phi_{R(t)h_1}(y), \ \dots \\
&\qquad\qquad \dots \ , \phi_{S(t)h_n}(x) + \phi_{R(t)h_n}(y))\, d\mu(y) \otimes \underline{S}(t)Vh_j \\
&= (P(t) \otimes \underline{S}(t))D_V f(x).
\end{aligned}$$

This identity shows that $P(t) \otimes \underline{S}$ maps $\mathsf{R}_p(D_V)$ into itself, and therefore $P \otimes \underline{S}$ restricts to a bounded C_0-semigroup on $\overline{\mathsf{R}_p(D_V)}$. The invariance of $\overline{\mathsf{R}_p(D_V)}$ under the operators $P(z) \otimes \underline{S}(z)$, where $z \in \mathbb{C}$ is in the sector of bounded analyticity of P, follows by uniqueness of analytic continuation (consider the quotient mapping from $\underline{L}^p$ to $\underline{L}^p/\overline{\mathsf{R}_p(D_V)}$). $\square$

In the next result we return to the L^2-setting and show that the semigroups $P \otimes \underline{S}$ and $\underline{P}$ on $\underline{L}^2$ agree on $\overline{\mathsf{R}(D_V)}$.

Theorem 5.7. *Both $\underline{P}$ and $P \otimes \underline{S}$ restrict to bounded analytic C_0-semigroups on $\overline{\mathsf{R}(D_V)}$, and their restrictions coincide:*

$$\underline{P}(t)F = P(t) \otimes \underline{S}(t)F, \quad F \in \overline{\mathsf{R}(D_V)}.$$

Proof. The invariance of $\overline{\mathsf{R}(D_V)}$ under $P \otimes \underline{S}$ follows from the previous theorem. Let us write $-N$ for the generator of $P \otimes \underline{S}$ on $\overline{\mathsf{R}(D_V)}$. From $V(\mathsf{D}(A^2)) \subseteq \mathsf{D}(\underline{A})$ (cf. the proof of Lemma 5.3) and $\mathscr{F}C_{\rm b}^\infty(E; \mathsf{D}(A^2)) \otimes \mathsf{D}(\underline{A}) \subseteq \mathsf{D}(L) \otimes \mathsf{D}(\underline{A})$ we see that the subspace $U := \{D_V f : f \in \mathscr{F}C_{\rm b}^\infty(E; \mathsf{D}(A^2))\}$ is contained in $\mathsf{D}(N)$. This subspace is dense in $\overline{\mathsf{R}(D_V)}$ since $\mathscr{F}C_{\rm b}^\infty(E; \mathsf{D}(A^2))$ is a core for $\mathsf{D}(L)$ (by Lemma 4.8 and the remark following it) and $\mathsf{D}(L)$ is a core for $\mathsf{D}(D_V)$. Since $(P \otimes \underline{S})U \subseteq U$ by Theorem 5.6, it follows that U is a core for $\mathsf{D}(N)$.

For functions $f \in \mathscr{F}C_{\rm b}^\infty(E; \mathsf{D}(A^2))$ we obtain

$$N D_V f = D_V L f = \underline{L} D_V f.$$

The first identity follows from Theorem 5.6 and the second from a direct computation. Alternatively, the second identity can be deduced from the analogue of Lemma 5.2 for D_V and L.

Thus $N = \underline{L}$ on the core U of $\mathsf{D}(N)$. It follows that $\mathsf{D}(N) \subseteq \mathsf{D}(\underline{L})$ and $N = \underline{L}$ on $\mathsf{D}(N)$. Let $\lambda > 0$. Multiplying the identity $\lambda + N = \lambda + \underline{L}$ from the right with $(\lambda + N)^{-1}$ and from the left with $(\lambda + \underline{L})^{-1}$, we obtain $(\lambda + N)^{-1} = (\lambda + \underline{L})^{-1}$ on $\overline{\mathsf{R}(D_V)}$. In particular, $(\lambda + \underline{L})^{-1}$ maps $\overline{\mathsf{R}(D_V)}$ into itself. As in Lemma 5.3 it follows that $\underline{P}$ leaves $\overline{\mathsf{R}(D_V)}$ invariant and that the restriction of $\underline{P}$ to $\overline{\mathsf{R}(D_V)}$ equals the semigroup generated by $-N$, which is $P \otimes \underline{S}|_{\overline{\mathsf{R}(D_V)}}$. □

Definition 5.8. Let $1 < p < \infty$. On $\overline{\mathsf{R}_p(D_V)}$ we define $\underline{P} := P \otimes \underline{S}|_{\overline{\mathsf{R}_p(D_V)}}$. The negative generator of $\underline{P}$ is denoted by $\underline{L}$.

By Theorem 5.7, for $p = 2$ this definition is consistent with the one given at the beginning of this section.

6. Intermezzo I: R-boundedness and radonifying operators

Before proceeding with the proofs of the main results we insert a section containing a concise discussion of the notions of R-boundedness, radonifying operators, and square functions. For more information and further results we refer to the excellent sources [17, 31] as well as the papers [42, 41] and the references given therein. The notations in this section will be independent of those in the previous ones.

6.1. **R-boundedness.** Throughout this section, unless otherwise stated (M, μ) is an arbitrary σ-finite measure space and $\underline{H}$ is an arbitrary Hilbert space. In analogy to previous notations we write $L^p := L^p(M, \mu)$ and $\underline{L}^p := L^p(M, \mu; \underline{H})$.

Let X and Y be Banach spaces and let $(r_j)_{j \geqslant 1}$ be a sequence of independent *Rademacher variables*, i.e., $\mathbb{P}(r_j = 1) = \mathbb{P}(r_j = -1) = \frac{1}{2}$ for each j.

A collection of bounded linear operators $\mathscr{T} \subseteq \mathscr{L}(X, Y)$ is said to be *R-bounded* if there exists $C \geqslant 0$ such that for all $k = 1, 2, \ldots$ and all choices of $x_1, \ldots, x_k \in X$ and $T_1, \ldots, T_k \in \mathscr{T}$ we have

$$\mathbb{E}\Big\| \sum_{j=1}^k r_j T_j x_j \Big\|^2 \leqslant C^2 \mathbb{E}\Big\| \sum_{j=1}^k r_j x_j \Big\|^2.$$

The smallest constant C for which this inequality holds is denoted by $R(\mathscr{T})$. By the Kahane-Khintchine inequalities one may replace the exponents 2 by arbitrary $p \in [1, \infty)$; this only changes the value of the constant C. Every bounded subset of operators on a Hilbert space is R-bounded. If $\mathscr{T}$ is R-bounded, then the closure with respect to the strong operator topology of the absolutely convex hull of $\mathscr{T}$ is R-bounded as well, with constant at most C (in the real case) or $2C$ (in the complex case). A useful consequence of this is the following result [31, Corollary 2.14] which we formulate for real spaces X and Y (in the complex case an extra constant 2 appears).

Proposition 6.1. *Let $\mathscr{T} \subseteq \mathscr{L}(X, Y)$ be R-bounded, and let $f : M \to \mathscr{L}(X, Y)$ be a function with values in $\mathscr{T}$ such that $\xi \mapsto f(\xi)x$ is strongly μ-measurable for all $x \in E$. For $\phi \in L^1$ define*

$$T_{\phi,f}x := \int_M \phi(t)f(t)x\, d\mu(t), \qquad x \in X.$$

Then the collection $\{T_{\phi,f} : \ \|\phi\|_{L^1} \leqslant 1\}$ is R-bounded in $\mathscr{L}(X, Y)$.

The next result may be known to specialists, but since we couldn't find a reference for it we include a proof.

Proposition 6.2. *Let $1 \leqslant p < \infty$. If $\mathscr{T} \subseteq \mathscr{L}(L^p)$ is R-bounded and $\mathscr{S} \subseteq \mathscr{L}(\underline{H})$ is bounded, then $\mathscr{T} \otimes \mathscr{S} \subseteq \mathscr{L}(\underline{L}^p)$ is R-bounded.*

Proof. Since $T \otimes S = (T \otimes I)(I \otimes S)$ and $I \otimes \mathscr{S}$ is R-bounded by the Fubini theorem, it suffices to show that $\mathscr{T} \otimes I$ is R-bounded.

Let $(h_i)_{i=1}^n$ be an orthonormal system in $\underline{H}$ and let $F_1, \dots, F_k$ be functions in $\underline{L}^p$ of the form $F_j := \sum_{i=1}^n f_{ij} \otimes h_i$. Let $(r_i)_{i\geqslant 1}$ and $(\widetilde{r}_i)_{i\geqslant 1}$ be independent Rademacher sequences. Then, putting $g_i := \sum_{j=1}^k r_j T_j f_{ij}$,

$$\begin{aligned}
\mathbb{E}\Big\| \sum_{j=1}^k r_j (T_j \otimes I) F_j \Big\|_p^p &= \mathbb{E}\Big\| \sum_{j=1}^k r_j \sum_{i=1}^n T_j f_{ij} \otimes h_i \Big\|_p^p \\
&= \mathbb{E} \int_M \Big\| \sum_{j=1}^k r_j \sum_{i=1}^n T_j f_{ij} \otimes h_i \Big\|^p d\mu \\
&= \mathbb{E} \int_M \Big\| \sum_{i=1}^n g_i \otimes h_i \Big\|^p d\mu \\
&\eqsim \mathbb{E} \int_M \widetilde{\mathbb{E}} \Big| \sum_{i=1}^n \widetilde{r}_i g_i \Big|^p d\mu \\
&= \widetilde{\mathbb{E}}\mathbb{E} \Big\| \sum_{j=1}^k r_j T_j \Big(\sum_{i=1}^n \widetilde{r}_i f_{ij} \Big) \Big\|_p^p \\
&\lesssim \widetilde{\mathbb{E}}\mathbb{E} \Big\| \sum_{j=1}^k r_j \Big(\sum_{i=1}^n \widetilde{r}_i f_{ij} \Big) \Big\|_p^p \\
&\eqsim \mathbb{E} \Big\| \sum_{j=1}^k r_j F_j \Big\|_p^p .
\end{aligned}$$

The last step follows by performing the computation in reverse order. The result follows by an application of the Kahane-Khintchine inequalities. □

We need the following duality result for R-bounded families [26, Proposition 3.5]. Let $I_1 \in \mathscr{L}(L^2(E,\mu))$ be the orthogonal projection defined in Section 3.3 and let I_X be the identity operator on a Banach space X. Then X is said to be K-convex if the operator $I_1 \otimes I_X$ on $L^2(E,\mu) \otimes X$ extends to a bounded operator on the Lebesgue-Bochner space $L^2(E,\mu;X)$ (see, e.g., [18, 48]).

Proposition 6.3. *If X and Y are K-convex Banach spaces, then a family $\mathscr{T} \subseteq \mathscr{L}(X,Y)$ is R-bounded if and only if the adjoint family $\mathscr{T}^* \subseteq \mathscr{L}(Y^*,X^*)$ is R-bounded.*

We shall apply this proposition to the K-convex spaces $X = L^p$ and $Y = \underline{L}^p$ for $1 < p < \infty$.

6.2. **Radonifying operators.** It will be convenient to exploit the connection between square functions and radonifying norms. Let $(\gamma_n)_{n\geqslant 1}$ be a Gaussian sequence, i.e., a sequence of independent standard normal random variables. If $\mathscr{H}$ is a Hilbert space and X is a Banach space, we denote by $\gamma(\mathscr{H},X)$ the completion of the finite rank operators from $\mathscr{H}$ to X with respect to the norm

$$\Big\| \sum_{j=1}^k h_j \otimes x_j \Big\|^2_{\gamma(\mathscr{H},X)} = \mathbb{E}\Big\| \sum_{j=1}^k \gamma_j x_j \Big\|^2,$$

where it is assumed that the vectors $h_1,\dots,h_k$ are orthonormal in $\mathscr{H}$. We have a continuous inclusion $\gamma(\mathscr{H},X) \hookrightarrow \mathscr{L}(\mathscr{H},X)$. Operators in $\mathscr{L}(\mathscr{H},X)$ belonging to $\gamma(\mathscr{H},X)$ are called *radonifying*; this terminology is explained by the fact that an operator $T \in \mathscr{L}(\mathscr{H},X)$ is radonifying if and only if there exists a centred Gaussian Radon measure on X whose covariance operator equals TT^*.

We continue with an observation about repeated radonifying norms which follows from the Kahane-Khintchine inequalities and Fubini's theorem. For a proof see, e.g., [44].

Proposition 6.4. *Let (S_1,σ_1), (S_2,σ_2) be σ-finite measure spaces, and let $1 \leqslant p < \infty$. The mapping*

$$f_1 \otimes (f_2 \otimes g) \mapsto (f_1 \otimes f_2) \otimes g, \quad f_i \in L^2(S_i,\sigma_i),\ g \in \underline{L}^p,$$

extends uniquely to an isomorphism of Banach spaces

$$\gamma(L^2(S_1,\sigma_1), \gamma(L^2(S_2,\sigma_2), \underline{L}^p)) \simeq \gamma(L^2(S_1 \times S_2, \sigma_1 \otimes \sigma_2), \underline{L}^p).$$

The next multiplier result is a slight extension of a result due to Kalton and Weis [27] and can be proved in the same way.

Proposition 6.5. *Let X and Y be Banach spaces, and let $K : M \to \mathscr{L}(X,Y)$ be a function such that $K(\cdot)x$ is strongly μ-measurable for all $x \in X$. If the set $\mathscr{T}_K = \{K(\xi) : \ \xi \in M\}$ is R-bounded, then the mapping*

$$T_K : f(\cdot) \otimes x \mapsto f(\cdot) \otimes K(\cdot)x, \quad f \in L^2, x \in X,$$

extends uniquely to a bounded operator T_K from $\gamma(L^2,X)$ to $\gamma(L^2,Y)$ of norm $\|T_K\| \leqslant R(\mathscr{T}_K)$.

6.3. **Square functions.** In this subsection we recall how R-bounded families in L^p-spaces and radonifying operators into L^p-spaces can be characterised by square functions.

The first result follows from a standard application of the Kahane-Khintchine inequalities; see [31].

Proposition 6.6. *A family $\mathscr{T} \subseteq \mathscr{L}(L^p, \underline{L}^p)$ is R-bounded if and only if there exists a constant C such that for all $T_1, \dots, T_N \in \mathscr{T}$ and $f_1, \dots, f_N \in L^p$,*

$$\Big\| \Big(\sum_{n=1}^N \|T_n f_n\|^2 \Big)^{1/2} \Big\|_p \leqslant C \Big\| \Big(\sum_{n=1}^N |f_n|^2 \Big)^{1/2} \Big\|_p.$$

The next result is another consequence of the Kahane-Khintchine inequalities; see [8, 41].

Proposition 6.7. *Let (S, σ) be a σ-finite measure space and let $1 \leqslant p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Let $k : S \to \underline{L}^p$ be a strongly σ-measurable function such that for all $g \in \underline{L}^q$ the function $s \mapsto \langle k(s), g\rangle$ is square integrable. The following assertions are equivalent:*

(1) *The operator $I_k \in \mathscr{L}(L^2(S, \sigma), \underline{L}^p)$ defined by*

$$\langle I_k f, g\rangle = \int_S f(s)\langle k(s), g\rangle \, d\sigma(s), \quad f \in L^2(S, \sigma), \; g \in \underline{L}^q,$$

is radonifying;

(2) *The square function $\big(\int_S \|k(s)\|^2 \, d\sigma(s)\big)^{1/2}$ defines an element of L^p.*

In this situation we have an equivalence of norms

$$\|I_k\|_{\gamma(L^2(S,\sigma), \underline{L}^p)} \eqsim \Big\| \Big(\int_S \|k(s)\|^2 \, d\sigma(s)\Big)^{1/2} \Big\|_p.$$

In what follows we always identify k with the operator I_k.

7. Intermezzo II: H^∞-functional calculi

In this section we recall some basic facts concerning H^∞-functional calculi. For more information we refer to the monographs [17, 23], the lecture notes [1, 31], and the references given therein.

For $\omega \in (0, \pi)$ we consider the open sector

$$\Sigma_\omega^+ := \{z \in \mathbb{C} : \; z \neq 0, \; |\arg z| < \omega\}.$$

A closed operator A acting on a Banach space X is said to be *sectorial* of angle $\omega \in (0, \pi)$ if $\sigma(A) \subseteq \overline{\Sigma_\omega^+}$ and the set $\{\lambda(\lambda - A)^{-1} : \lambda \notin \overline{\Sigma_\theta^+}\}$ is bounded for all $\theta \in (\omega, \pi)$. The least angle of sectoriality is denoted by $\omega^+(A)$. If A is sectorial and the set $\{\lambda(\lambda - A)^{-1} : \lambda \notin \overline{\Sigma_\theta^+}\}$ is R-bounded for all $\theta \in (\omega, \pi)$, then A is said to be *R-sectorial* of angle $\omega \in (0, \pi)$. The least angle of R-sectoriality is denoted by $\omega_R^+(A)$.

We will frequently use the fact [23, Proposition 2.1.1(h)] that a sectorial operator A on a reflexive Banach space X induces a direct sum decomposition

$$X = \mathsf{N}(A) \oplus \overline{\mathsf{R}(A)}. \tag{7.1}$$

Let $H^\infty(\Sigma_\theta^+)$ be the space of all bounded holomorphic functions on Σ_θ^+, and let $H_0^\infty(\Sigma_\theta^+)$ denote the linear subspace of all $\psi \in H^\infty(\Sigma_\theta^+)$ which satisfy an estimate

$$|\psi(z)| \leqslant C\Big(\frac{|z|}{1+|z|^2}\Big)^\alpha, \quad z \in \Sigma_\theta^+,$$

for some $\alpha > 0$ and $C \geqslant 0$. If A is a sectorial operator and ψ is a function in $H_0^\infty(\Sigma_\theta^+)$ with $0 < \omega^+(A) < \theta' < \theta < \pi$, we may define the bounded operator $\psi(A)$ on X by the Dunford integral

$$\psi(A)x := \frac{1}{2\pi i}\int_{\partial\Sigma_{\theta'}^+} \psi(z)(z-A)^{-1}x\,dz, \quad x \in X,$$

where $\partial_{\Sigma_{\theta'}^+}$ is parametrised counter-clockwise. By Cauchy's theorem this definition does not depend on the choice of θ'.

A sectorial operator A on X is said to admit a *bounded $H^\infty(\Sigma_\theta^+)$-functional calculus*, or a *bounded H^∞-functional calculus of angle θ*, if there exists a constant $C_\theta \geqslant 0$ such that for all $\psi \in H_0^\infty(\Sigma_\theta^+)$ and all $x \in X$ we have

$$\|\psi(A)x\| \leqslant C_\theta\|\psi\|_\infty\|x\|,$$

where $\|\psi\|_\infty = \sup_{z\in\Sigma_\theta^+}|\psi(z)|$. The infimum over all possible angles θ is denoted $\omega_{H^\infty}^+(A)$. We say that a sectorial operator A admits a *bounded H^∞-functional calculus* if it admits a bounded $H^\infty(\Sigma_\theta^+)$-functional calculus for some $0 < \theta < \pi$.

The following result is well known; see, e.g., [31, Theorem 2.20].

Lemma 7.1. *Let A be R-sectorial of angle $\omega_R^+(A) < \frac{1}{2}\pi$ on X, and let S be the bounded analytic C_0-semigroup generated by $-A$. The family $\{S(t) : t \geqslant 0\}$ is R-bounded in $\mathscr{L}(X)$.*

In the remainder of this section we work in an L^p-setting and use the notations of the previous section. As before we write $L^p = L^p(M,\mu)$ and $\underline{L}^p = L^p(M,\mu;\underline{H})$, where (M,μ) is a σ-finite measure space and $\underline{H}$ is a Hilbert space.

The following result is taken from [15, 33] where the result is proved for scalar-valued L^p-spaces. An extension to more a general class of Banach spaces can be found in [26].

Proposition 7.2. *Let $1 < p < \infty$ and let A be an R-sectorial operator on $\underline{L}^p$. Let $\omega_R^+(A) < \theta < \pi$. For all non-zero $\varphi, \psi \in H_0^\infty(\Sigma_\theta^+)$ we have*

$$\Big\|\Big(\int_0^\infty \|\varphi(tA)F\|^2\,\frac{dt}{t}\Big)^{1/2}\Big\|_p \eqsim \Big\|\Big(\int_0^\infty \|\psi(tA)F\|^2\,\frac{dt}{t}\Big)^{1/2}\Big\|_p,$$

with implied constants independent of F. Moreover, the following assertions are equivalent:

1. *A admits a bounded H^∞-calculus;*
2. *For some (equivalently, for all) non-zero $\psi \in H_0^\infty(\Sigma_\theta^+)$ we have*

$$\|F - P_{\mathsf{N}(A)}F\|_p \lesssim \Big\|\Big(\int_0^\infty \|\psi(tA)F\|^2\,\frac{dt}{t}\Big)^{1/2}\Big\|_p \lesssim \|F\|_p, \quad F \in \underline{L}^p.$$

In (2), $P_{\mathsf{N}(A)}$ is the projection onto $\mathsf{N}(A)$ with kernel $\overline{\mathsf{R}(A)}$ along the decomposition (7.1). If these equivalent conditions are fulfilled, then $\omega_R^+(A) = \omega_{H^\infty}^+(A)$.

In the next result we let $1 < p < \infty$ and consider two R-sectorial operators L and $\underline{A}$. We assume that $-L$ and $-\underline{A}$ generate R-bounded analytic C_0-semigroups P and $\underline{S}$ on L^p and $\underline{H}$. We denote by $-\underline{L}$ the generator of the tensor product C_0-semigroup $\underline{P} = P \otimes \underline{S}$ on $\underline{L}^p$. This operator is R-sectorial of angle $\max\{\omega_R^+(L), \omega^+(\underline{A})\} < \frac{1}{2}\pi$ on $\underline{L}^p$.

We consider the following three square function norms:

$$\|u\|_{\underline{A}} := \Big(\int_0^\infty \|t\underline{A}\,\underline{S}(t)u\|^2 \,\frac{dt}{t}\Big)^{1/2}, \quad u \in \underline{H};$$

$$\|f\|_{p,L} := \Big\|\Big(\int_0^\infty |tLP(t)f|^2 \,\frac{dt}{t}\Big)^{1/2}\Big\|_p, \quad f \in L^p;$$

$$\|F\|_{p,\underline{L}} := \Big\|\Big(\int_0^\infty \|t\underline{L}\,\underline{P}(t)F\|^2 \,\frac{dt}{t}\Big)^{1/2}\Big\|_p, \quad F \in \underline{L}^p.$$

Proposition 7.3. *Under the above assumptions we have:*

(1) *If $\|u\|_{\underline{A}} \lesssim \|u\|$ for all $u \in \underline{H}$ and $\|f\|_{p,L} \lesssim \|f\|_p$ for all $f \in L^p$, then $\|F\|_{p,\underline{L}} \lesssim \|F\|_p$ for all $F \in \underline{L}^p$.*

(2) *If $\|u\|_{\underline{A}} \gtrsim \|(I - P_{\mathsf{N}(\underline{A})})u\|$ for all $u \in \underline{H}$ and $\|f\|_{p,L} \gtrsim \|(I - P_{\mathsf{N}(L)})f\|_p$ for all $f \in L^p$, then $\|F\|_{p,\underline{L}} \gtrsim \|(I - P_{\mathsf{N}(\underline{L})})F\|_p$ for all $F \in \underline{L}^p$.*

As a consequence, if $\underline{A}$ and L have bounded H^∞-functional calculi of angles less than $\frac{1}{2}\pi$, then $\underline{L}$ has a bounded H^∞-functional calculus of angle less than $\frac{1}{2}\pi$.

Proof. Let us first show that (1) implies (2). It is well known that the assumptions of (2) imply the dual estimates $\|u\|_{\underline{A}^*} \lesssim \|u\|$ and $\|f\|_{q,L^*} \lesssim \|f\|_q$, where $\frac{1}{p} + \frac{1}{q} = 1$. By (1) we find that $\|F\|_{q,\underline{L}^*} \lesssim \|F\|_q$ and by duality we obtain the conclusion of (2).

The final assertion follows by combining (1) and (2) with Proposition 7.2.

It remains to prove (1). We proceed in three steps.

Step 1: We prove that

$$\|t(I \otimes \underline{A})(I \otimes \underline{S}(t))F\|_{\gamma(L^2(\mathbb{R}_+, \frac{dt}{t}), \underline{L}^p)} \lesssim \|F\|_p.$$

For $F \in \underline{L}^p$ we have, for μ-almost all $x \in M$,

$$\Big(\int_0^\infty \|t(I \otimes \underline{A})(I \otimes \underline{S}(t))F(x)\|^2 \frac{dt}{t}\Big)^{1/2} \lesssim \|F(x)\|.$$

Integrating this estimate over M yields

$$\Big\|\Big(\int_0^\infty \|t(I \otimes \underline{A})(I \otimes \underline{S}(t))F\|^2 \frac{dt}{t}\Big)^{1/2}\Big\|_p \leqslant \|F\|_p.$$

Step 2: We prove that

$$\|t(L \otimes I)(P(t) \otimes I)F\|_{\gamma(L^2(\mathbb{R}_+, \frac{dt}{t}), \underline{L}^p)} \lesssim \|F\|_p.$$

Let $(h_j)_{j=1}^k$ be a finite orthonormal system in $\underline{H}$ and pick $F := \sum_{j=1}^k f_j \otimes h_j \in \underline{L}^p$. For $f \in L^p$ let $(Uf)(t) := tLP(t)f$, and notice that U is a bounded operator from L^p into $\gamma(L^2(\mathbb{R}_+, \frac{dt}{t}), L^p)$ by the assumption in (1) and Proposition 6.7.

Let $(r'_j)_{j\geqslant 1}$ and $(\gamma'_j)_{j\geqslant 1}$ be a Rademacher and a Gaussian sequence respectively on a probability space $(\Omega', \mathbb{P}')$. Noting the pointwise equality

$$\|t(L \otimes I)(P(t) \otimes I)F\|^2 = \sum_{j=1}^k |Uf_j(t)|^2$$

we have

$$
\begin{aligned}
\Big\|\Big(\int_0^\infty \|t(L\otimes I)(P(t)\otimes I)F\|^2\frac{dt}{t}\Big)^{1/2}\Big\|_p &= \Big\|\Big(\int_0^\infty \sum_{j=1}^k |Uf_j(t)|^2\frac{dt}{t}\Big)^{1/2}\Big\|_p\\
&= \Big\|\Big(\int_0^\infty \mathbb{E}'\Big|\sum_{j=1}^k r_j' Uf_j(t)\Big|^2\frac{dt}{t}\Big)^{1/2}\Big\|_p\\
&\eqsim \Big\|\sum_{j=1}^k r_j' Uf_j\Big\|_{\gamma(L^2(\mathbb{R}_+\times\Omega',\frac{dt}{t}\otimes\mathbb{P}'),L^p)}\\
&\overset{(*)}{\eqsim} \Big\|U\sum_{j=1}^k r_j' f_j\Big\|_{\gamma(L^2(\Omega',\mathbb{P}'),\gamma(L^2(\mathbb{R}_+,\frac{dt}{t}),L^p))}\\
&\overset{(**)}{\lesssim} \Big\|\sum_{j=1}^k r_j' f_j\Big\|_{\gamma(L^2(\Omega',\mathbb{P}'),L^p)}\\
&\overset{(***)}{=} \Big(\mathbb{E}'\Big\|\sum_{j=1}^k \gamma_j' f_j\Big\|_{L^p}^2\Big)^{1/2}\\
&\eqsim \Big(\mathbb{E}'\Big\|\sum_{j=1}^k \gamma_j' f_j\Big\|_{L^p}^p\Big)^{1/p}\\
&= \Big\|\Big(\mathbb{E}'\Big|\sum_{j=1}^k \gamma_j' f_j\Big|^p\Big)^{1/p}\Big\|_{L^p}\\
&\eqsim \Big\|\Big(\sum_{j=1}^k |f_j|^2\Big)^{1/2}\Big\|_{L^p}\\
&= \|F\|_p.
\end{aligned}
$$

In $(*)$ we used Proposition 6.4, in $(**)$ we used the boundedness of U from L^p into $\gamma(L^2(\mathbb{R}_+,\frac{dt}{t}),L^p)$, and in $(***)$ the definition of the radonifying norm.

Step 3: We combine the previous estimates. By Lemma 7.1 the family $\{P(t):t\geqslant 0\}$ is R-bounded on L^p. Hence by Proposition 6.2 the family $\{P(t)\otimes I:\ t\geqslant 0\}$ is R-bounded on $\underline{L}^p$. Also, by a simple application of Fubini's theorem, $\{I\otimes\underline{S}(t):t\geqslant 0\}$ is R-bounded. Combining these facts with Proposition 6.5, for $F\in\underline{L}^p$ we obtain

$$
\begin{aligned}
\Big\|\Big(\int_0^\infty \|t\underline{L}\,\underline{P}F\|^2\frac{dt}{t}\Big)^{1/2}\Big\|_p &\eqsim \|t\underline{L}\,\underline{P}F\|_{\gamma(L^2(\mathbb{R}_+,\frac{dt}{t}),\underline{L}^p)}\\
&\lesssim \|(I\otimes S(t))t(L\otimes I)(P(t)\otimes I)F\|_{\gamma(L^2(\mathbb{R}_+,\frac{dt}{t}),\underline{L}^p)}\\
&\qquad+\|(P(t)\otimes I)t(I\otimes\underline{A})(I\otimes\underline{S}(t))F\|_{\gamma(L^2(\mathbb{R}_+,\frac{dt}{t}),\underline{L}^p)}\\
&\lesssim \|t(L\otimes I)(P(t)\otimes I)F\|_{\gamma(L^2(\mathbb{R}_+,\frac{dt}{t}),\underline{L}^p)}\\
&\qquad+\|t(I\otimes\underline{A})(I\otimes\underline{S}(t))F\|_{\gamma(L^2(\mathbb{R}_+,\frac{dt}{t}),\underline{L}^p)}\\
&\lesssim \|F\|_p.
\end{aligned}
$$

□

Remark 7.4. The final assertion in Proposition 7.3 is due to Lancien, Lancien, and Le Merdy [32, Theorem 1.4] who proved it using operator-valued H^∞-functional calculi.

The next proposition has been proved in [33, Theorem 3.5, Remark 3.6] (for $\underline{H} = \mathbb{C}$) and can be extended to a more general class of Banach spaces including the spaces $\underline{L}^p$ [22, 44] (where a generalisation of the crucial ingredient [33, Proposition 3.3] is obtained).

Proposition 7.5. *Let L be R-sectorial on L^p of angle $\omega_R^+(L) < \frac{1}{2}\pi$, and let P be the bounded analytic C_0-semigroup P on L^p generated by $-L$. Let $U : \mathsf{D}_p(L) \to \underline{L}^p$ be a linear operator, bounded with respect to the graph norm of $\mathsf{D}_p(L)$. Consider the following statements.*

(1) $\Big\| \Big(\displaystyle\int_0^\infty \|\sqrt{t} UP(t)f\|^2 \, \frac{dt}{t} \Big)^{1/2} \Big\|_p \lesssim \|f\|_p, \qquad f \in \mathsf{D}_p(L);$

(2) *The family $\{\sqrt{t} UP(t) : \ t > 0\}$ is R-bounded in $\mathscr{L}(L^p, \underline{L}^p)$.*

Then (1) *implies* (2)*. If L satisfies the square function estimate*

$$\Big\| \Big(\int_0^\infty |tLP(t)f|^2 \, \frac{dt}{t} \Big)^{1/2} \Big\|_p \lesssim \|f\|_p,$$

then (2) *implies* (1)*.*

Remark 7.6. In [33] and other works in the mathematical systems theory literature, condition (2) is replaced by the following equivalent condition:

($2'$) The family $\{tU(I + t^2 L)^{-1} : \ t > 0\}$ is R-bounded.

That (2) implies ($2'$) follows by taking Laplace transforms and the opposite direction is observed in [33, (3.12)]. Since our computations involve semigroups rather than resolvents we find it more natural to use (2).

Below we shall also need the notion of an (R-)*bisectorial* operator, which is analogous to that of an (R-)sectorial operator, the only difference being that the sector Σ_θ^+ is replaced by the bisector $\Sigma_\theta = \Sigma_\theta^+ \cup \Sigma_\theta^-$, where $\Sigma_\theta^- = -\Sigma_\theta^+$. Many results in the literature on (R-)sectorial operators carry over to (R-)bisectorial operators, with only minor changes in the proofs. We refer to the lecture notes [4] for more details.

8. Proof of Theorems 2.3 and 2.4

We return to the main setting of the paper and take up our study of the operators L and $\underline{L}$ introduced in Sections 4 and 5. Notations are again as in these sections.

For functions $f \in \mathscr{F}C_{\rm b}^\infty(E; \mathsf{D}(A))$ we consider the Littlewood-Paley-Stein square functions

$$\mathscr{H} f(x) := \Big(\int_0^\infty \|\sqrt{t} D_V P(t) f(x)\|^2 \, \frac{dt}{t} \Big)^{1/2}, \quad x \in E,$$

$$\mathscr{G} f(x) := \Big(\int_0^\infty \|t D_V Q(t) f(x)\|^2 \, \frac{dt}{t} \Big)^{1/2}, \quad x \in E.$$

where Q denotes the analytic C_0-semigroup generated by $-\sqrt{L}$.

The functions $t \mapsto D_V P(t)f$ are analytic in a sector containing $\mathbb{R}_+$, and therefore a well-known result of Stein [51] allows us to select a pointwise version $(t,x) \mapsto D_V P(t)f(x)$ which is analytic in t for every fixed x. Using such a version, we see that $\mathscr{H} f$ is well defined almost everywhere (but possibly infinite). The square function $\mathscr{G} f$ is well defined by similar reasoning.

In Section 10 we shall need the following inequality. The argument is taken from [13].

Lemma 8.1. *For all $f \in \mathscr{F}C_{\mathrm{b}}^\infty(E;\mathsf{D}(A))$ we have $\mathscr{G} f \leqslant \mathscr{H} f$ μ-almost everywhere.*

Proof. Using the representation

$$Q(t)f = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} P\Big(\frac{t^2}{4u}\Big) f \, du$$

and the closedness of D_V,

$$\begin{aligned}\mathscr{G}^2 f(x) &= \int_0^\infty \|tD_V Q(t) f(x)\|^2 \frac{dt}{t} \\ &\leqslant \frac{1}{\pi} \int_0^\infty \Big(\int_0^\infty \Big\|tD_V P\Big(\frac{t^2}{4u}\Big) f(x)\Big\| \frac{e^{-u}}{\sqrt{u}} \, du \Big)^2 \frac{dt}{t}.\end{aligned}$$

Since $\int_0^\infty \frac{e^{-u}}{\sqrt{u}} \, du = \sqrt{\pi}$ we may apply Jensen's inequality to obtain

$$\begin{aligned}\mathscr{G}^2 f(x) &\leqslant \frac{1}{\sqrt{\pi}} \int_0^\infty \Big(\int_0^\infty \Big\|tD_V P\Big(\frac{t^2}{4u}\Big) f(x)\Big\|^2 \frac{e^{-u}}{\sqrt{u}} \, du \Big) \frac{dt}{t} \\ &= \frac{1}{\sqrt{\pi}} \int_0^\infty \Big(\int_0^\infty \Big\|tD_V P\Big(\frac{t^2}{4u}\Big) f(x)\Big\|^2 \frac{dt}{t} \Big) \frac{e^{-u}}{\sqrt{u}} \, du \\ &= \frac{2}{\sqrt{\pi}} \int_0^\infty \Big(\int_0^\infty \big\|\sqrt{s} D_V P(s) f(x)\big\|^2 \frac{ds}{s} \Big) \sqrt{u} e^{-u} \, du \\ &= \mathscr{H}^2 f(x).\end{aligned}$$

□

The main results of this section are the following two theorems, which together imply part (2) of Theorem 2.3 as well as Theorem 2.4. Part (1) of Theorem 2.3 is contained in Theorem 8.10.

Theorem 8.2 (R-Gradient bounds)**.** *Assume* (A1), (A2), (A3), *and let* $1 < p < \infty$. *Then $\mathsf{D}_p(L)$ is a core for $\mathsf{D}_p(D_V)$ and the families*

$$\big\{\sqrt{t} D_V P(t) : \ t > 0\big\} \quad \text{and} \quad \big\{tD_V(I + t^2 L)^{-1} : \ t > 0\big\}$$

are R-bounded in $\mathscr{L}(L^p, \underline{L}^p)$.

Theorem 8.3 (Littlewood-Paley-Stein inequalities)**.** *Assume* (A1), (A2), (A3), *and let $1 < p < \infty$. For all $f \in \mathscr{F}C_{\mathrm{b}}^\infty(E;\mathsf{D}(A))$ we have the square function estimate*

$$\|f - P_{\mathsf{N}_p(L)} f\|_p \lesssim \|\mathscr{H} f\|_p \lesssim \|f\|_p,$$

where $P_{\mathsf{N}_p(L)}$ is the projection onto $\mathsf{N}_p(L)$ along the direct sum decomposition $L^p = \mathsf{N}_p(L) \oplus \overline{\mathsf{R}_p(L)}$.

By Theorem 8.3 the square function $\mathscr{H} f$ is actually well-defined for arbitrary $f \in L^p$, and by approximation Theorem 8.3 extends to all of L^p. Since we do not need these observations we leave the details to the reader.

For the proofs of both theorems we distinguish between the cases $1 < p \leqslant 2$ and $2 < p < \infty$. For $1 < p \leqslant 2$ we show by a direct argument that $\mathscr{H}$ is L^p-bounded and deduce from this that $\mathsf{D}_p(L)$ is a core for $\mathsf{D}_p(D_V)$. Theorem 8.2 is then a consequence of Proposition 7.5. For $2 < p < \infty$ we first derive Theorem 8.2 from a pointwise gradient bound and a duality argument involving maximal functions. Since L has a bounded H^∞-calculus of angle $< \frac{1}{2}\pi$ by Lemma 8.4, the right-hand side estimate of Theorem 8.3 then follows by an application of Proposition 7.5. Finally, the left-hand side inequality of Theorem 8.3 is proved, for $1 < p < \infty$, by a duality argument.

We begin with an easy observation.

Lemma 8.4. *Let $1 < p < \infty$. The operator L is R-sectorial and admits a bounded H^∞-calculus on L^p of angle $\omega^+_{H^\infty}(L) = \omega^+_R(L) < \frac{1}{2}\pi$. Moreover,*

(1) *The family $\{P(t) : t \geqslant 0\}$ is R-bounded in $\mathscr{L}(L^p)$;*
(2) *The family $\{\underline{P}(t) : t \geqslant 0\}$ is R-bounded in $\mathscr{L}(\overline{\mathsf{R}_p(D_V)})$.*

Proof. Since $-L$ generates an analytic C_0-semigroup of positive contractions on L^p for all $1 < p < \infty$, the first part follows from [28, Corollary 5.2 and Theorem 5.3]. Assertion (1) follows from Lemma 7.1, and assertion (2) follows by combining (1) with the identity $\underline{P} = P \otimes \underline{S}$ and Proposition 6.2. □

We continue with a simple extension of a well-known result of Cowling [14, Theorem 7] (see also [52]). For the convenience of the reader we give a sketch of the proof.

Proposition 8.5. *Let (M, μ) be a σ-finite measure space and let T be an analytic C_0-semigroup of positive operators on $L^2 := L^2(M, \mu)$ satisfying $\|T(t)f\|_p \leqslant \|f\|_p$ for all $f \in L^2 \cap L^p$, $t \geqslant 0$ and $1 \leqslant p \leqslant \infty$. Let*

$$T_\star f(x) := \sup_{t>0} |T(t)f(x)|.$$

Then for $1 < p < \infty$ we have

$$\|T_\star f\|_p \lesssim \|f\|_p, \quad f \in L^p.$$

Proof. Let $-L$ denote the generator of T in L^p. By [28, Corollary 5.2], L has a bounded H^∞-calculus of angle $\omega < \frac{1}{2}\pi$. The key idea of the proof is to write

$$T(t)f = \frac{1}{t}\int_0^t T(s)f\,ds + \Big(T(t)f - \frac{1}{t}\int_0^t T(s)f\,ds\Big) = \frac{1}{t}\int_0^t T(s)f\,ds + m(tL)f,$$

where $m(z) := e^{-z} - \int_0^1 e^{-sz}\,ds$.

By the Hopf-Dunford-Schwartz ergodic theorem [30, Theorem 6.12] we have

$$\Big\|\sup_{t>0}\Big|\frac{1}{t}\int_0^t T(s)f\,ds\Big|\Big\|_p \lesssim \|f\|_p,$$

so that it remains to prove that $\big\|\sup_{t>0}|m(tL)f|\,\big\|_p \lesssim \|f\|_p$.

Let $n := m \circ \exp$ and let $\hat{n}$ be its Fourier transform. Using the identities

$$m(z) = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{n}(u) z^{iu}\,du, \qquad \hat{n}(u) = (1 - (1+iu)^{-1})\Gamma(iu),$$

and the estimate $|\hat{n}(u)| \leqslant Ce^{-\frac{1}{2}\pi|u|}$ (see [14]) we obtain

$$\sup_{t>0}|m(tL)F| \leqslant \sup_{t>0}\frac{1}{2\pi}\int_{\mathbb{R}}|\hat{n}(u)|\,|(tL)^{iu}F|\,du \lesssim \frac{1}{2\pi}\int_{\mathbb{R}} e^{-\frac{1}{2}\pi|u|}|L^{iu}F|\,du.$$

From the H^∞-calculus of L we obtain $\|L^{iu}f\|_p \lesssim e^{\omega|u|}\|f\|_p$. Taking L^p-norms we obtain

$$\Big\|\sup_{t>0}|m(tL)F|\,\Big\|_p \lesssim \frac{1}{2\pi}\int_{\mathbb{R}} e^{-\frac12\pi|u|}\|L^{iu}F\|_p\,du \lesssim \frac{1}{2\pi}\int_{\mathbb{R}} e^{(\omega-\frac12\pi)|u|}\|F\|_p\,du \lesssim \|F\|_p.$$

□

8.1. **The case** $1 < p \leqslant 2$**.** We begin with some preliminary observations.

Lemma 8.6. *For $h \in \mathsf{D}(V)$ we have*

$$\int_0^\infty \|\underline{S}(t)Vh\|^2\,dt \leqslant (2k)^{-1}\|h\|^2.$$

Proof. Let $t > 0$. Using Lemma 5.3 and the fact that $S(t)h \in \mathsf{D}(A)$ by analyticity, we obtain

$$\begin{aligned}\|\underline{S}(t)Vh\|^2 = \|VS(t)h\|^2 &\leqslant k^{-1}[BVS(t)h, VS(t)h]\\ &= k^{-1}[AS(t)h, S(t)h]\\ &= -(2k)^{-1}\frac{d}{dt}\|S(t)h\|^2.\end{aligned}$$

Hence

$$\begin{aligned}\int_0^\infty \|\underline{S}(t)Vh\|^2\,dt &\leqslant (2k)^{-1}\limsup_{T\to\infty}\int_0^T -\frac{d}{dt}\|S(t)h\|^2\,dt\\ &= (2k)^{-1}\Big(\|h\|^2 - \liminf_{T\to\infty}\|S(T)h\|^2\Big)\\ &\leqslant (2k)^{-1}\|h\|^2.\end{aligned}$$

□

Lemma 8.7. *Let $f \in \mathscr{F}C_{\mathrm{b}}^\infty(E;\mathsf{D}(A))$ and $F \in \mathscr{F}C_{\mathrm{b}}^\infty(E;\mathsf{D}(A))\otimes\mathsf{D}(A)$ be such that $D_V f = (I\otimes V)F$. Then for all $1 < p < \infty$ we have $\mathscr{H}f \in L^p$ and $\|\mathscr{H}f\|_p \lesssim \|F\|_p$.*

Proof. By Proposition 6.2 and Lemma 7.1, the set $\{P(t)\otimes I : \ t \geqslant 0\}$ is R-bounded in $\mathscr{L}(\underline{L}^p)$. Hence, by Propositions 6.5, 6.7, and Lemma 8.6,

$$\begin{aligned}\|\mathscr{H}f\|_p &= \Big\|\Big(\int_0^\infty \|\underline{P}(t)D_V f\|^2\,dt\Big)^{1/2}\Big\|_p\\ &= \Big\|\Big(\int_0^\infty \|(P(t)\otimes I)(I\otimes\underline{S}(t))(I\otimes V)F\|^2\,dt\Big)^{1/2}\Big\|_p\\ &\lesssim \Big\|\Big(\int_0^\infty \|(I\otimes\underline{S}(t))(I\otimes V)F\|^2\,dt\Big)^{1/2}\Big\|_p\\ &\leqslant (2k)^{-1/2}\|F\|_p.\end{aligned}$$

□

The following proof is based on a classical argument which goes back to Stein [51]. The same idea has been applied in the related works [12, 13, 36, 49]. For the convenience of the reader we include a proof.

Proof of the first part of Theorem 8.3, $1 < p \leqslant 2$. First we show that it suffices to prove the estimate for functions $f \in \mathscr{F}C_{\mathrm{b}}^\infty(E;\mathsf{D}(A))$ satisfying $f \geqslant \varepsilon$ for some $\varepsilon > 0$.

Fix $f = \varphi(\phi_{h_1},\dots,\phi_{h_k}) \in \mathscr{F}C_{\rm b}^\infty(E;\mathsf{D}(A))$ of the usual form. Pick functions $m_n \in C_{\rm b}^\infty(\mathbb{R}^k)$ satisfying $m_n \geqslant 0$, $\mathrm{supp}(m_n) \subseteq [-\frac1n,\frac1n]^k$, and $\|m_n\|_1 = 1$, and put

$$\begin{aligned}\psi_{n,\pm} &:= \big(\varphi^{\pm} + \frac{1}{n}\big) * m_n,\\ g_{n,\pm} &:= \psi_{n,\pm}(\phi_{h_1},\dots,\phi_{h_k}),\\ g_{n,\pm,j} &:= \partial_j\psi_{n,\pm}(\phi_{h_1},\dots,\phi_{h_k}).\end{aligned}$$

Clearly $g_{n,\pm} \in \mathscr{F}C_{\rm b}^\infty(E;\mathsf{D}(A))$ satisfy $\frac1n \leqslant g_{n,\pm} \leqslant \|\varphi\|_\infty + 1$, and

$$\big\|\big(f^{\pm} + \frac{1}{n}\big) - g_{n,\pm}\big\|_p \to 0$$

by dominated convergence. From Lemma 8.7 it follows that

$$\begin{aligned}\|\mathscr{H}f - \mathscr{H}(g_{n,+} - g_{n,-})\|_p &\leqslant \|\mathscr{H}(f - (g_{n,+} - g_{n,-}))\|_p\\ &\lesssim \Big\|\sum_{j=1}^k (f_j - (g_{n,+,j} - g_{j,n,-,j}))\otimes h_j\Big\|_p,\end{aligned}$$

where $f_j = \partial_j\varphi(\phi_{h_1},\dots,\phi_{h_k})$. Since the functions

$$g_{n,\pm,j} = \big(\partial_j\varphi(\phi_{h_1},\dots,\phi_{h_k})\mathbf{1}_{\{\pm\varphi(\phi_{h_1},\dots,\phi_{h_k})>0\}}\big) * m_n,$$

belong to L^∞ uniformly in n, we conclude by dominated convergence that $\|f_j - (g_{n,+,j} - g_{n,-,j})\|_p \to 0$. Therefore $\mathscr{H}(g_{n,+} - g_{n,-}) \to \mathscr{H}f$ in L^p as $n\to\infty$. Hence if $\|\mathscr{H}g_{n,\pm}\|_p \lesssim \|g_{n,\pm}\|_p$ with constants not depending on n, then

$$\begin{aligned}\|\mathscr{H}f\|_p &= \lim_{n\to\infty}\|\mathscr{H}(g_{n,+} - g_{n,-})\|_p\\ &\leqslant \limsup_{n\to\infty}(\|\mathscr{H}g_{n,+}\|_p + \|\mathscr{H}g_{n,-}\|_p)\\ &\lesssim \limsup_{n\to\infty}(\|g_{n,+}\|_p + \|g_{n,-}\|_p)\\ &= \|f^+\|_p + \|f^-\|_p\\ &\leqslant 2\|f\|_p.\end{aligned}$$

Thus it suffices to prove the result for $f \in \mathscr{F}C_{\rm b}^\infty(E;\mathsf{D}(A))$ satisfying $f \geqslant \varepsilon$ for some $\varepsilon > 0$. Set

$$u(t,x) := P(t)f(x), \qquad x \in E,\ t > 0,$$

and notice that by Mehler's formula (3.3) we have $u(t,x) \geqslant \varepsilon$ for all $x \in E$ and $t \geqslant 0$. By Lemma 4.8 we have $u(t,\cdot) \in \mathscr{F}C_{\rm b}^\infty(E;\mathsf{D}(A)) \subseteq \mathsf{D}_p(L)$ for all $t \geqslant 0$. Arguing as in [12, 13, 49], for $1 < p \leqslant 2$ we use Lemma 4.8 and a truncation argument to obtain that $u(t,\cdot)^p \in \mathsf{D}_p(L)$ and

$$\begin{aligned}(\partial_t + L)u(t,x)^p &= pu(t,x)^{p-1}(\partial_t + L)u(t,x)\\ &\quad - p(p-1)u(t,x)^{p-2}[BD_Vu(t,x), D_Vu(t,x)]\\ &= -p(p-1)u(t,x)^{p-2}[BD_Vu(t,x), D_Vu(t,x)].\end{aligned}$$

Hence, using the coercivity Assumption (A3),

$$\begin{aligned}\|D_Vu(t,x)\|^2 &\leqslant k^{-1}[BD_Vu(t,x), D_Vu(t,x)]\\ &= -\frac{1}{kp(p-1)}u(t,x)^{2-p}(\partial_t + L)u(t,x)^p.\end{aligned}$$

Now we set

$$K(x) := -\int_0^\infty (\partial_t + L)u(t,x)^p \, dt$$

and

$$u_\star(x) := \sup_{t>0} u(t,x)$$

to obtain

$$\begin{aligned}\mathscr{H}f(x)^2 &= \int_0^\infty \|D_V u(t,x)\|^2 \, dt \\ &\leqslant -C_{p,k}\int_0^\infty u(t,x)^{2-p}(\partial_t + L)u(t,x)^p \, dt \\ &\leqslant C_{p,k}u_\star(x)^{2-p}K(x).\end{aligned}$$

Hölder's inequality with exponents $\frac{2}{2-p}$ and $\frac{2}{p}$ implies

$$\begin{aligned}\int_E \mathscr{H}f(x)^p \, d\mu(x) &\leqslant C_{p,k}^{\frac{p}{2}}\int_E u_\star(x)^{\frac{(2-p)p}{2}}K(x)^{\frac{p}{2}} \, d\mu(x) \\ &\leqslant C_{p,k}^{\frac{p}{2}}\Big(\int_E u_\star(x)^p \, d\mu(x)\Big)^{\frac{2-p}{2}}\Big(\int_E K(x) \, d\mu(x)\Big)^{\frac{p}{2}}.\end{aligned} \tag{8.1}$$

Using the invariance of μ and the L^p-contractivity of P we obtain

$$\begin{aligned}\int_E K(x) \, d\mu(x) &= -\int_0^\infty\int_E (\partial_t + L)u(t,x)^p \, d\mu(x) \, dt \\ &= -\int_0^\infty\int_E \partial_t u(t,x)^p \, d\mu(x) \, dt \\ &= -\int_0^\infty \partial_t\int_E u(t,x)^p \, d\mu(x) \, dt \\ &\leqslant \limsup_{t\to\infty}\big(\|f\|_p^p - \|u(t,\cdot)\|_p^p\big) \\ &\leqslant \|f\|_p^p,\end{aligned} \tag{8.2}$$

where the use of Fubini's theorem is justified by the non-negativity of the integrand K, and the interchange of differentiation and integration by the fact that $f \in \mathscr{F}C_{\mathrm{b}}^\infty(E;\mathsf{D}(A))$.

Combining (8.1), (8.2) and Proposition 8.5 we conclude that

$$\|\mathscr{H}f\|_p^p \lesssim \|u_\star\|_p^{\frac{(2-p)p}{2}}\|f\|_p^{\frac{p^2}{2}} \lesssim \|f\|_p^p.$$

□

Proof of Theorem 8.2, $1 < p \leqslant 2$. First we show that $\mathsf{D}_p(L)$ is contained in $\mathsf{D}_p(D_V)$. Once we know this, Lemmas 4.8 and 5.5 imply that $\mathsf{D}_p(L)$ is even a core for $\mathsf{D}_p(D_V)$.

Fix a function $f \in \mathscr{F}C_{\mathrm{b}}^\infty(E;\mathsf{D}(A))$. From Theorem 5.6 it follows that $s \mapsto e^{-s}D_V P(s)f = e^{-s}\underline{P}(s)D_V f$ is Bochner integrable in $\underline{L}^p$ and

$$\int_0^\infty e^{-s}D_V P(s)f \, ds = (I + \underline{L})^{-1}D_V f.$$

Since $s \mapsto e^{-s}P(s)f$ is Bochner integrable in L^p, the closedness of D_V implies that $(I+L)^{-1}f = \int_0^\infty e^{-s}P(s)f \, ds \in \mathsf{D}_p(D_V)$ and

$$D_V(I+L)^{-1}f = D_V\int_0^\infty e^{-s}P(s)f \, ds = \int_0^\infty e^{-s}D_V P(s)f \, ds.$$

Moreover, by the Cauchy-Schwarz inequality,

$$\begin{aligned}\|D_V(I+L)^{-1}f\|_p &\leqslant \Big\| \int_0^\infty e^{-s}\|D_V P(s)f\|\,ds\Big\|_p \\ &\leqslant \frac{1}{\sqrt{2}}\Big\|\Big(\int_0^\infty \|D_V P(s)f\|^2\,ds\Big)^{1/2}\Big\|_p \\ &= \frac{1}{\sqrt{2}}\|\mathscr{H}f\|_p \lesssim \|f\|_p.\end{aligned}$$

It follows that $D_V(I+L)^{-1}$ extends to a bounded operator from L^p to $\underline{L}^p$. In view of the closedness of D_V and Lemma 4.8, the desired inclusion follows from this. This concludes the proof that $\mathsf{D}_p(L)$ is a core for $\mathsf{D}_p(D_V)$.

The R-boundedness assertions follow from Proposition 7.5 and Remark 7.6. □

8.2. **The case** $2 < p < \infty$**.** In case that P is symmetric it is possible to use a variant of a duality argument of Stein [51] to prove the boundedness of $\mathscr{H}$. This approach has been taken in [12], but the proof breaks down if L is non-symmetric and we have to proceed in a different way.

First we derive an explicit formula for the semigroup P which allows us to prove suitable gradient bounds. Having obtained those gradient bounds we give a general argument involving a maximal inequality for P^* to prove the R-boundedness of the collection $\{\sqrt{t}D_V P(t) : \ t > 0\}$. Since L has a bounded H^∞-calculus, we obtain the boundedness of $\mathscr{H}$ by an appeal to Proposition 7.5.

We begin with some preliminary observations. For $0 < t < \infty$ we define the operators $Q_t \in \mathscr{L}(E^*, E)$ by

$$Q_t x^* := ii^*x^* - iS^*(t)S(t)i^*x^*,$$

where $i : H \hookrightarrow E$ is the inclusion operator. The operators Q_t are positive and symmetric, i.e., for all $x^*, y^* \in E^*$ we have $\langle Q_t x^*, x^*\rangle \geqslant 0$ and $\langle Q_t x^*, y^*\rangle = \langle Q_t y^*, x^*\rangle$. Let H_t be the reproducing kernel Hilbert space associated with Q_t and let $i_t : H_t \hookrightarrow E$ be the inclusion mapping. Then,

$$i_t i_t^* = Q_t.$$

Since $\langle Q_t x^*, x^*\rangle \leqslant \langle ii^*x^*, x^*\rangle$ for all $x^* \in E^*$, the operators Q_t are covariances of centred Gaussian measures μ_t on E; see, e.g., [20]. This estimate also implies that we have a continuous inclusion $H_t \hookrightarrow H$ and that the mapping

$$V_t : i^*x^* \mapsto i_t^* x^*, \quad x^* \in E^*,$$

is well defined and extends to a contraction from H into H_t. It is easy to check that the adjoint operator V_t^* is the inclusion from H_t into H.

Let us also note that for $s \leqslant t$ and $x^* \in E^*$ we have

$$\langle Q_s x^*, x^*\rangle = \|i^*x\|^2 - \|S(s)i^*x^*\|^2 \leqslant \|i^*x\|^2 - \|S(t)i^*x^*\|^2 = \langle Q_t x^*, x^*\rangle$$

by the contractivity of S.

In the next proposition we fix $t > 0$ and $h \in H_t$ and denote by $\phi_h^{\mu_t} : E \to \mathbb{R}$ the (μ_t-essentially unique; see Section 2) μ_t-measurable linear extension of the function $\phi_h^{\mu_t}(i_t g) := [g, h]_{H_t}$.

Proposition 8.8. *For all $f = \varphi(\phi_{h_1}, \dots, \phi_{h_n}) \otimes h \in \mathscr{F}C_{\mathrm{b}}(E) \otimes \mathscr{H}$, where $\mathscr{H}$ is some Hilbert space, the following identity holds for μ-almost all $x \in E$:*

$$(P(t) \otimes I)f(x) = \int_E \varphi(\phi_{S(t)h_1}(x) + \phi^{\mu_t}_{V_t h_1}(y), \dots, \phi_{S(t)h_n}(x) + \phi^{\mu_t}_{V_t h_n}(y))h \, d\mu_t(y).$$

Proof. Defining $\psi : E \times \mathbb{R}^n \to \mathscr{H}$ by

$$\psi(x, \xi) := \varphi(\phi_{S(t)h_1}(x) + \xi_1, \dots, \phi_{S(t)h_n}(x) + \xi_n)h,$$

we have

$$\begin{aligned}
&\int_E \varphi(\phi_{S(t)h_1}(x) + \phi^{\mu_t}_{V_t h_1}(y), \dots, \phi_{S(t)h_n}(x) + \phi^{\mu_t}_{V_t h_n}(y))h \, d\mu_t(y) \\
&\quad = \int_E \psi\big(x, (\phi^{\mu_t}_{V_t h_1}(y), \dots, \phi^{\mu_t}_{V_t h_n}(y))\big) \, d\mu_t(y) \\
&\quad = \int_{\mathbb{R}^n} \psi(x, \xi) \, d\gamma_t(\xi),
\end{aligned}$$

where γ_t is the centred Gaussian measure on $\mathbb{R}^n$ whose covariance matrix equals $\big([V_t h_i, V_t h_j]\big)_{i,j=1}^n$.

On the other hand, writing $R(t) = \sqrt{I - S^*(t)S(t)}$, by Mehler's formula (3.3) we have

$$\begin{aligned}
(P(t) \otimes I)f(x) &= \int_E \varphi(\phi_{S(t)h_1}(x) + \phi_{R(t)h_1}(y), \dots, \phi_{S(t)h_n}(x) + \phi_{R(t)h_n}(y))h \, d\mu(y) \\
&= \int_E \psi\big(x, (\phi_{R(t)h_1}(y), \dots, \phi_{R(t)h_n}(y))\big) \, d\mu(y) \\
&= \int_{\mathbb{R}^n} \psi(x, \xi) \, d\tilde{\gamma}_t(\xi),
\end{aligned}$$

where $\tilde{\gamma}_t$ is the centred Gaussian measure on $\mathbb{R}^n$ whose covariance matrix equals $\big([R(t)h_i, R(t)h_j]\big)_{i,j=1}^n$.

The result follows from the observation that

$$[V_t h_i, V_t h_j] = [h_i, h_j] - [S(t)h_i, S(t)h_j] = [R(t)h_i, R(t)h_j].$$

∎

Lemma 8.9. *For all $u \in \underline{H}$ and $t > 0$ we have $\underline{S}^*(t)u \in \mathsf{D}(V^*)$, $V^*\underline{S}^*(t)u \in H_t$, and*

$$\|V^*\underline{S}^*(t)u\|_{H_t} \lesssim \frac{1}{\sqrt{t}}\|u\|.$$

Proof. First we observe that $S(s)$ maps H into $\mathsf{D}(A) \subseteq \mathsf{D}(V)$ for $s > 0$. For $t > 0$ we claim that

$$J_t : V_t h \mapsto VS(\cdot)h$$

extends to a bounded operator from H_t into $L^2(0, t; \underline{H})$ of norm $\leqslant \frac{1}{\sqrt{2k}}$.

Indeed, by the coercivity of B and the definition of H_t, we obtain for $h \in H$,

$$\begin{aligned}\int_0^t \|VS(s)h\|^2\,ds &\leqslant \frac{1}{k}\int_0^t [BVS(s)h, VS(s)h]\,ds\\ &= -\frac{1}{2k}\int_0^t \frac{d}{ds}\|S(s)h\|^2\,ds\\ &= \tfrac{1}{2k}\big(\|h\|^2 - \|S(t)h\|^2\big)\\ &= \tfrac{1}{2k}\|V_t h\|_{H_t}^2.\end{aligned}$$

Recall that V_t^* is the inclusion mapping $H_t \hookrightarrow H$. Noting that $\underline{S}^*(t)$ maps $\underline{H}$ into $\mathsf{D}(\underline{A}^*) \subseteq \mathsf{D}(V^*)$ and using Lemma 5.3, the adjoint mapping $J_t^* : L^2(0,t;\underline{H}) \to H_t$ is given by

$$V_t^* J_t^* f = \int_0^t V^*\underline{S}^*(s) f(s)\,ds, \quad f \in L^2(0,t;\underline{H}).$$

The resulting identity $V^*\underline{S}^*(t)u = \frac{1}{t}V_t^*J_t^*\big(\underline{S}^*(t-\cdot)u\big)$ shows that $V^*\underline{S}^*(t)u$ can be identified with the element $\frac{1}{t}J_t^*\big(\underline{S}^*(t-\cdot)u\big)$ of H_t and we obtain

$$\begin{aligned}\|V^*\underline{S}^*(t)u\|_{H_t} &= \frac{1}{t}\|J_t^*\big(\underline{S}^*(t-\cdot)u\big)\|_{H_t}\\ &\leqslant \frac{1}{t\sqrt{2k}}\|\underline{S}^*(t-\cdot)u\|_{L^2(0,t;\underline{H})}\\ &\leqslant \frac{1}{\sqrt{2kt}}\sup_{s\geqslant 0}\|\underline{S}^*(s)\|_{\mathscr{L}(\underline{H})}\|u\|.\end{aligned}$$

□

The following pointwise gradient bound is included for reasons of completeness. We shall only need the special case corresponding to $r = 2$, for which a simpler proof can be given; see Remark 8.11.

Theorem 8.10 (Pointwise gradient bounds)**.** *Let $1 < r < \infty$. For $f \in \mathscr{F}C_{\rm b}(E)$ and $t > 0$ we have, for μ-almost all $x \in E$,*

$$\sqrt{t}\|D_V P(t)f(x)\| \lesssim (P(t)|f|^r(x))^{1/r}.$$

Proof. For notational simplicity we take f of the form $f = \varphi(\phi_h)$ with $\varphi \in C_{\rm b}(\mathbb{R})$ and $h \in H$. It is immediate to check that the argument carries over to general cylindrical functions in $\mathscr{F}C_{\rm b}(E)$.

By Lemma 8.9 we have $S^*(t)V^*u \in H_t$ for $u \in \mathsf{D}(V^*)$ and therefore, for all $h \in H$,

$$\phi_{S(t)h}(iV^*u) = [S(t)h, V^*u] = [h, S^*(t)V^*u] = \phi^{\mu_t}_{V_t h}(iS^*(t)V^*u).$$

By Proposition 8.8 (with $\mathscr{H} = \mathbb{R}$) we find that for all $g \in H$,

$$\begin{aligned}P(t)f(x+iV^*u) &= \int_E \varphi(\phi_{S(t)h}(x+iV^*u) + \phi^{\mu_t}_{V_t h}(y))\,d\mu_t(y)\\ &= \int_E \varphi(\phi_{S(t)h}(x) + \phi^{\mu_t}_{V_t h}(y + iS^*(t)V^*u))\,d\mu_t(y).\end{aligned}$$

Recalling that D denotes the Malliavin derivative we have, for all $u \in \mathsf{D}(V^*)$,

$$\begin{aligned}
[D_V P(t)f(x), u] &= [DP(t)f(x), V^*u] \\
&= \lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon}\Big(P(t)f(x+\varepsilon i V^*u) - P(t)f(x)\Big) \\
&= \lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon}\int_E \varphi(\phi_{S(t)h}(x) + \phi^{\mu_t}_{V_t h}(y+\varepsilon i S^*(t)V^*u)) \\
&\qquad\qquad - \varphi(\phi_{S(t)h}(x) + \phi^{\mu_t}_{V_t h}(y))\,d\mu_t(y).
\end{aligned}$$

Using Lemma 8.9 and the Cameron-Martin formula [7, Corollary 2.4.3] we obtain

$$[D_V P(t)f(x), u] = \lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon}\int_E \big(E^{\mu_t}_{\varepsilon S^*(t)V^*u}(y) - 1\big)\varphi(\phi_{S(t)h}(x) + \phi^{\mu_t}_{V_t h}(y))\,d\mu_t(y),$$

where $E^{\mu_t}_h(y) = \exp\big(\phi^{\mu_t}_h(y) - \frac{1}{2}\|h\|^2_{H_t}\big)$. It is easy to see that for each $h \in H_t$ the family $\big(\frac{1}{\varepsilon}(E^{\mu_t}_{\varepsilon h} - 1)\big)_{0<\varepsilon<1}$ is uniformly bounded in $L^2(E,\mu_t)$, and therefore uniformly integrable in $L^1(E,\mu_t)$. Passage to the limit $\varepsilon \downarrow 0$ now gives

$$[D_V P(t)f(x), u] = \int_E \phi^{\mu_t}_{S^*(t)V^*u}(y)\varphi(\phi_{S(t)h}(x) + \phi^{\mu_t}_{V_t h}(y))\,d\mu_t(y).$$

By Hölder's inequality with $\frac{1}{q} + \frac{1}{r} = 1$, using the Gaussianity of $\phi^{\mu_t}_{S^*(t)V^*u}$ on (E,μ_t) and the Kahane-Khintchine inequality, Proposition 8.8, and Lemma 8.9 we find that

$$\begin{aligned}
&|[D_V P(t)f(x), u]| \\
&\quad\leqslant \Big(\int_E |\phi^{\mu_t}_{S^*(t)V^*u}(y)|^q\,d\mu_t(y)\Big)^{1/q}\Big(\int_E |\varphi(\phi_{S(t)h}(x) + \phi^{\mu_t}_{V_t h}(y))|^r\,d\mu_t(y)\Big)^{1/r} \\
&\quad\lesssim \Big(\int_E |\phi^{\mu_t}_{S^*(t)V^*u}(y)|^2\,d\mu_t(y)\Big)^{1/2}(P(t)|f|^r(x))^{\frac{1}{r}} \\
&\quad= \|S^*(t)V^*u\|_{H_t}(P(t)|f|^r(x))^{\frac{1}{r}} \\
&\quad\lesssim \frac{1}{\sqrt{t}}\|u\|(P(t)|f|^r(x))^{\frac{1}{r}}.
\end{aligned}$$

The desired estimate is obtained by taking the supremum over all $u \in \mathsf{D}(V^*)$ with $\|u\| \leqslant 1$. □

Remark 8.11. There is a well-known elementary trick which we learned from [34, p.328]) which can be used to prove Theorem 8.10 for $r = 2$. Using the product rule from Lemma 4.8, the fact that $\|Bu\| \geqslant k\|u\|$ for $u \in \overline{\mathsf{R}(V)}$, and the positivity of $P(s)$, we obtain

$$\begin{aligned}
P(t)f^2 - (P(t)f)^2 &= \int_0^t \partial_s\big(P(s)\big(|P(t-s)f|^2\big)\big)\,ds \\
&= -\int_0^t P(s)\big(L(P(t-s)f)^2 - 2P(t-s)f \cdot LP(t-s)f\big)\,ds \\
&= 2\int_0^t P(s)\big(\|BD_V P(t-s)f\|^2\big)\,ds \\
&\geqslant 2k\int_0^t P(s)\big(\|D_V P(t-s)f\|^2\big)\,ds.
\end{aligned}$$

Next we estimate, for μ-almost all $x \in E$,

$$\begin{aligned} M^2 P(r)(\|D_V f\|^2)(x) &\geqslant P(r)(\|\underline{S}(r) D_V f\|^2)(x) \\ &\overset{(*)}{\geqslant} \|(P(r)\otimes I)(\underline{S}(r) D_V f)\|^2(x) \\ &= \|\underline{P}(r) D_V f(x)\|^2 \\ &= \|D_V P(r) f(x)\|^2, \end{aligned}$$

where $M := \sup_{t\geqslant 0} \|\underline{S}(t)\|$ and $(*)$ follows from Proposition 8.8 (with $\mathscr{H} = \underline{H}$) and Jensen's inequality. The case $r = 2$ of Theorem 8.10 follows from these two estimates.

The next result is in some sense the dual version of a maximal inequality. It could be compared with the dual version of the non-commutative Doob inequality of [25].

Proposition 8.12. *Let (M, μ) be a σ-finite measure space, $1 \leqslant p < \infty$, and let $(T(t))_{t>0}$ be a family of positive operators on $L^p := L^p(M, \mu)$. Suppose that the maximal function $T^*_\star f := \sup_{t>0} |T^*(t)f|$ is measurable and L^q-bounded, where $\frac{1}{p} + \frac{1}{q} = 1$. Then, for all $f_1, \dots, f_n \in L^p$ and all $t_1, \dots, t_n > 0$,*

$$\Big\| \sum_{k=1}^n T(t_k)|f_k| \Big\|_p \lesssim \Big\| \sum_{k=1}^n |f_k| \Big\|_p.$$

Proof. Taking the supremum over all $g = (g_k)_{k=1}^n \in L^q(\ell^\infty_n)$ of norm one we obtain

$$\begin{aligned} \Big\| \sum_{k=1}^n T(t_k)|f_k| \Big\|_p &= \|\big(T(t_{(\cdot)})|f_{(\cdot)}|\big)\|_{L^p(\ell^1_n)} \\ &= \sup_g \int_E \sum_{k=1}^n T(t_k)|f_k| \cdot g_k \, d\mu \\ &= \sup_g \int_E \sum_{k=1}^n |f_k| \cdot T^*(t_k) g_k \, d\mu \\ &\leqslant \|\big(|f_{(\cdot)}|\big)\|_{L^p(\ell^1_n)} \sup_g \|\big(T^*(t_{(\cdot)}) g_{(\cdot)}\big)\|_{L^q(\ell^\infty_n)}. \end{aligned}$$

Using the positivity of T^* on L^q to obtain $\sup_{1\leqslant k\leqslant n} T^*_\star |g_k| \leqslant T^*_\star(\sup_{1\leqslant k\leqslant n} |g_k|)$ we estimate

$$\begin{aligned} \big\|\big(T^*(t_{(\cdot)}) g_{(\cdot)}\big)\big\|_{L^q(\ell^\infty_n)} &= \big\| \sup_{1\leqslant k\leqslant n} |T^*(t_k) g_k| \,\big\|_{L^q} \\ &\leqslant \big\| \sup_{1\leqslant k\leqslant n} T^*_\star |g_k| \,\big\|_{L^q} \\ &\leqslant \big\| T^*_\star (\sup_{1\leqslant k\leqslant n} |g_k|) \,\big\|_{L^q} \\ &\lesssim \big\| \sup_{1\leqslant k\leqslant n} |g_k| \,\big\|_{L^q} \\ &= \|(g_k)\|_{L^q(\ell^\infty_n)}. \end{aligned}$$

This completes the proof. □

The previous two results are now combined to prove:

Proof of Theorem 8.2 (for $2 < p < \infty$). Let $\frac{2}{p} + \frac{1}{q} = 1$. Proposition 8.5 implies that the maximal function

$$P^*_\star f := \sup_{t>0} |P^*(t)f|$$

is bounded on L^q. Using Theorem 8.10 (for $r = 2$) and Proposition 8.12 we obtain, for all $f_1, \ldots, f_n \in \mathscr{F}C_{\rm b}(E)$,

$$\begin{aligned}
\Big\| \Big(\sum_{k=1}^n \|\sqrt{t_k} D_V P(t_k) f_k\|^2 \Big)^{1/2} \Big\|_p &\lesssim \Big\| \Big(\sum_{k=1}^n P(t_k)|f_k|^2 \Big)^{1/2} \Big\|_p \\
&= \Big\| \sum_{k=1}^n P(t_k)|f_k|^2 \Big\|_{p/2}^{1/2} \\
&\lesssim \Big\| \sum_{k=1}^n |f_k|^2 \Big\|_{p/2}^{1/2} \\
&= \Big\| \Big(\sum_{k=1}^n |f_k|^2 \Big)^{1/2} \Big\|_p .
\end{aligned}$$

By an approximation argument this estimate extends to arbitrary $f_1, \ldots, f_n \in L^p$. Now Proposition 6.6 implies the R-boundedness of $\{\sqrt{t} D_V P(t) : \ t > 0\}$.

Taking Laplace transforms and using Proposition 6.1, it follows that $\mathsf{D}_p(L) \subseteq \mathsf{D}_p(D_V)$ and that the collection $\{tD_V(I + t^2 L)^{-1} \ : t > 0\}$ is R-bounded from L^p into $\underline{L}^p$. As in the case $1 < p \leqslant 2$, Lemmas 4.8 and 5.5 imply that $\mathsf{D}_p(L)$ is even a core for $\mathsf{D}_p(D_V)$. □

Proof of the first part of Theorem 8.3 (for $2 < p < \infty$). By Lemma 8.4, L has a bounded H^∞-calculus of angle $< \frac{1}{2}\pi$, and the result follows from Proposition 7.5. □

8.3. **Completion of the proof of Theorem 8.3.** It remains to prove, for $1 < p < \infty$, the left-hand side inequality of Theorem 8.3. We adapt a standard duality argument (see, e.g., [3, Section 7, Step 8]).

It is enough to prove the estimate for $f \in \mathsf{R}_p(L)$; for such f we have $f - P_{\mathsf{N}_p(L)} f = f$. First let $f = Lg$ with $g \in \mathsf{D}_p(L^2)$. Then by [31, Lemma 9.13],

$$\lim_{t\to\infty} P(t)Lg - Lg = -\lim_{t\to\infty} \int_0^t P(s)L^2 g \, ds = -\lim_{t\to\infty} \int_0^t \psi(sL)Lg \, \frac{ds}{s} = -Lg,$$

where $\psi(z) = ze^{-z}$. Hence, $\lim_{t\to\infty} P(t)Lg = 0$. By a density argument, this implies

$$\lim_{t\to\infty} P(t)f = 0, \quad f \in \overline{\mathsf{R}(L)}.$$

Fix $f \in L^2 \cap \overline{\mathsf{R}_p(L)}$ and $g \in L^2 \cap L^q$ with $\frac{1}{p} + \frac{1}{q} = 1$. For any $t > 0$,

$$\begin{aligned}
-\partial_t \int_E (P(t)f) g \, d\mu &= \int_E L(P(t)f) g \, d\mu \\
&= \int_E L(P(\tfrac{1}{2}t)f) P^*(\tfrac{1}{2}t) g \, d\mu \\
&= \int_E [BD_V P(\tfrac{1}{2}t)f, D_V P^*(\tfrac{1}{2}t)g] \, d\mu.
\end{aligned}$$

Since $\int_E f\,d\mu = 0$ we obtain, using Theorem 8.3 applied to the adjoint semigroup P^* (which is generated by $L^* = D_V^* B^* D_V$) in L^q,

$$\begin{aligned}
\int_E fg\,d\mu &= \int_E fg\,d\mu - \int_E f\,d\mu \int_E g\,d\mu \\
&= \lim_{\varepsilon\downarrow 0} \int_E (P(\varepsilon)f)g\,d\mu - \lim_{t\to\infty} \int_E (P(t)f)g\,d\mu \\
&= -\int_0^\infty \partial_t \int_E (P(t)f)g\,d\mu\,dt \\
&= \int_0^\infty \int_E [BD_V P(\tfrac12 t)f, D_V P^*(\tfrac12 t)g]\,d\mu\,dt \\
&\leqslant \|B\| \Big\| \Big(\int_0^\infty \|\sqrt{t} D_V P(\tfrac12 t)f\|^2 \frac{dt}{t}\Big)^{1/2}\Big\|_p \Big\| \Big(\int_0^\infty \|\sqrt{t} D_V P^*(\tfrac12 t)g\|^2 \frac{dt}{t}\Big)^{1/2}\Big\|_q \\
&\lesssim \Big\| \Big(\int_0^\infty \|\sqrt{t} D_V P(t)f\|^2 \frac{dt}{t}\Big)^{1/2}\Big\|_p \|g\|_q.
\end{aligned}$$

This implies that

$$\|f\|_p \lesssim \Big\| \Big(\int_0^\infty \|\sqrt{t} D_V P(t)f\|^2 \frac{dt}{t}\Big)^{1/2}\Big\|_p.$$

So far we have assumed that $f \in L^2 \cap \overline{\mathsf{R}_p(L)}$. The extension to general $f \in \overline{\mathsf{R}_p(L)}$ follows by a density argument (using the first part of the theorem to see that the right hand side can be approximated as well).

9. The operators D_V and $D_V^* B$

In this section we study some L^p-properties of the operators D_V and $D_V^* B$ and provide a rigorous interpretation of the identities $L = D_V^* B D_V$ and $\underline{L} = D_V D_V^* B$ in L^p and $\overline{\mathsf{R}_p(D_V)}$. From these operators we build operator matrices which will play an important role in the proofs of Theorems 2.1, 2.5, and 2.6.

Throughout this section we fix $1 < p < \infty$. The operator $D_V^* B$ is closed and densely defined as an operator from $\underline{L}^p$ to L^p with domain

$$\mathsf{D}_p(D_V^* B) = \{F \in \underline{L}^p :\ BF \in \mathsf{D}_p(D_V^*)\}.$$

Moreover, since $\|BF\|_p \eqsim \|F\|_p$ for $F \in \overline{\mathsf{R}_p(D_V)}$,

$$D_V^* B = (B^* D_V)^*,$$

where $B^* D_V$ is interpreted as a operator from L^q to $\underline{L}^q$, $\frac1p + \frac1q = 1$.

For the next result we recall that $\mathscr{C} := \mathscr{F}C_{\rm b}^\infty(E; \mathsf{D}(A))$ is a P-invariant core for $\mathsf{D}_p(L)$. We set $\mathscr{C}^* := \mathscr{F}C_{\rm b}^\infty(E; \mathsf{D}(A^*))$; this is a P^*-invariant core for $\mathsf{D}_p(L^*)$.

Proposition 9.1. *In L^p we have $L = (D_V^* B)D_V$. More precisely, $f \in \mathsf{D}_p(L)$ if and only if $f \in \mathsf{D}_p(D_V)$ and $D_V f \in \mathsf{D}_p(D_V^* B)$, in which case we have $Lf = (D_V^* B)D_V f$.*

Proof. First note that for all $f, g \in \mathscr{C}$ we have $\langle Lf, g\rangle = \langle D_V f, B^* D_V g\rangle$. Since $\mathscr{C}$ is a core for $\mathsf{D}_p(L)$, and $\mathsf{D}_p(L)$ is core for $\mathsf{D}_p(D_V)$ by the first part of Theorem 8.2, this identity extends to all $f \in \mathsf{D}_p(L)$ and $g \in \mathsf{D}_p(D_V)$. This implies that $D_V f \in \mathsf{D}_p((B^* D_V)^*)$ and $(B^* D_V)^* D_V f = Lf$. Since $(B^* D_V)^* = D_V^* B$, we find that $L \subseteq (D_V^* B)D_V$.

To prove the other inclusion we take $f \in \mathsf{D}_p(D_V)$ such that $D_V f \in \mathsf{D}_p(D_V^* B)$. We have $\langle f, L^* g\rangle = \langle D_V f, B^* D_V g\rangle = \langle (D_V^* B) D_V f, g\rangle$ for all $g \in \mathscr{C}^*$, where the second identity follows from $D_V f \in \mathsf{D}_p(D_V^* B) = \mathsf{D}_p((B^* D_V)^*)$. Since $\mathscr{C}^*$ is a core for $\mathsf{D}_q(L^*)$ this implies that $f \in \mathsf{D}_p(L)$ and $Lf = (D_V^* B) D_V f$. $\square$

We shall be interested in the restriction $D_V^* B|_{\overline{\mathsf{R}_p(D_V)}}$ of $D_V^* B$ to $\overline{\mathsf{R}_p(D_V)}$. As its domain we take

$$\mathsf{D}_p(D_V^* B|_{\overline{\mathsf{R}_p(D_V)}}) := \{F \in \overline{\mathsf{R}_p(D_V)} :\ BF \in \mathsf{D}_p(D_V^*)\} = \mathsf{D}_p(D_V^* B) \cap \overline{\mathsf{R}_p(D_V)}.$$

In the middle expression, as before we consider D_V^* as a densely defined operator from $\underline{L}^p$ to L^p.

Corollary 9.2. *The restriction $D_V^* B|_{\overline{\mathsf{R}_p(D_V)}}$ is closed and densely defined.*

Proof. Let $f \in \mathsf{D}_p(D_V)$. By the first part of Theorem 8.2 there exist functions $f_n \in \mathsf{D}_p(L)$ such that $f_n \to f$ in $\mathsf{D}_p(D_V)$. Proposition 9.1 implies that $D_V f_n \in \mathsf{D}_p(D_V^* B|_{\overline{\mathsf{R}_p(D_V)}})$. This shows that $D_V^* B|_{\overline{\mathsf{R}_p(D_V)}}$ is densely defined on $\overline{\mathsf{R}_p(D_V)}$. Closedness is clear. $\square$

Proposition 9.3. *The domain $\mathsf{D}_p(\underline{L})$ is a core for $\mathsf{D}_p(D_V^* B|_{\overline{\mathsf{R}_p(D_V)}})$. Moreover, for all $t > 0$ the operators $(I + t^2 L)^{-1} D_V^* B|_{\overline{\mathsf{R}_p(D_V)}}$ and $P(t) D_V^* B|_{\overline{\mathsf{R}_p(D_V)}}$ (initially defined on $\mathsf{D}_p(D_V^* B|_{\overline{\mathsf{R}_p(D_V)}})$) extend uniquely to bounded operators from $\overline{\mathsf{R}_p(D_V)}$ to L^p, and for all $F \in \overline{\mathsf{R}_p(D_V)}$ we have*

$$(I + t^2 L)^{-1} D_V^* B F = D_V^* B (I + t^2 \underline{L})^{-1} F$$

and

$$P(t) D_V^* B F = D_V^* B \underline{P}(t) F.$$

Proof. We split the proof into four steps.

Step 1 - By Proposition 9.1, for all $f \in \mathsf{D}_p(L)$ we have $f \in \mathsf{D}_p(D_V)$ and $D_V f \in \mathsf{D}_p(D_V^* B|_{\overline{\mathsf{R}_p(D_V)}})$, and for all $t > 0$ we have

$$P(t)(D_V^* B) D_V f = P(t) L f = L P(t) f = (D_V^* B) D_V P(t) f = D_V^* B \underline{P}(t) D_V f.$$

By taking Laplace transforms and using the closedness of $D_V^* B$, this gives $(I + t^2 \underline{L})^{-1} D_V f \in \mathsf{D}_p(D_V^* B|_{\overline{\mathsf{R}(D_V)}})$ and

$$(I + t^2 L)^{-1} (D_V^* B) D_V f = D_V^* B (I + t^2 \underline{L})^{-1} D_V f. \tag{9.1}$$

Step 2 - By Theorem 8.2, for all $t > 0$ the operator $T(t) := B^* D_V (I + t^2 L^*)^{-1}$ is bounded from L^q into $\underline{L}^q$, $\frac{1}{p} + \frac{1}{q} = 1$. For all $F \in \mathsf{D}_p(D_V^* B|_{\overline{\mathsf{R}_p(D_V)}})$ and $g \in L^q$ we have

$$\langle F, T(t) g\rangle = \langle F, B^* D_V (I + t^2 L^*)^{-1} g\rangle = \langle (I + t^2 L)^{-1} D_V^* B F, g\rangle. \tag{9.2}$$

Now let $F \in \overline{\mathsf{R}_p(D_V)}$ be arbitrary and take a sequence $(F_n)_{n \geqslant 1} \subseteq \mathsf{D}_p(D_V^* B|_{\overline{\mathsf{R}_p(D_V)}})$ converging to F in $\overline{\mathsf{R}_p(D_V)}$. By Proposition 9.1 and the fact that $\mathsf{D}_p(L)$ is a core for $\mathsf{D}_p(D_V)$ we may take the F_n of the form $D_V f_n$ with $f_n \in \mathsf{D}_p(L)$. Then $(I + t^2 \underline{L})^{-1} F_n \to (I + t^2 \underline{L})^{-1} F$, and from (9.1) we obtain

$$D_V^* B (I + t^2 \underline{L})^{-1} F_n = (I + t^2 L)^{-1} D_V^* B F_n = T^*(t) F_n \to T^*(t) F.$$

The closedness of D_V^*B implies that $(I+t^2\underline{L})^{-1}F \in \mathsf{D}_p(D_V^*B|_{\overline{\mathsf{R}_p(D_V)}})$. This proves the domain inclusion $\mathsf{D}_p(\underline{L}) \subseteq \mathsf{D}_p(D_V^*B|_{\overline{\mathsf{R}_p(D_V)}})$, along with the identity

$$D_V^*B(I+t^2\underline{L})^{-1}F = T^*(t)F, \quad F \in \overline{\mathsf{R}_p(D_V)}.$$

Note that for $F \in \mathsf{D}_p(D_V^*B)$, from (9.2) we also obtain

$$D_V^*B(I+t^2\underline{L})^{-1}F = T^*(t)F = (I+t^2L)^{-1}D_V^*BF. \tag{9.3}$$

Step 3 - By Step 2 the operator $D_V^*B(I+t^2\underline{L})^{-1}$ is bounded from $\overline{\mathsf{R}_p(D_V)}$ to L^p. Therefore, by (9.1), the operator $(I+t^2L)^{-1}D_V^*B$ (initially defined on the dense domain $\mathsf{D}_p(D_V^*B|_{\overline{\mathsf{R}_p(D_V)}})$) uniquely extends to a bounded operator from $\overline{\mathsf{R}_p(D_V)}$ to L^p, and for this extension we obtain the identity

$$(I+t^2L)^{-1}D_V^*B = D_V^*B(I+t^2\underline{L})^{-1}.$$

On $\mathsf{D}_p(D_V^*B|_{\overline{\mathsf{R}_p(D_V)}})$, the identity $D_V^*B\underline{P}(t) = P(t)D_V^*B$ follows from (9.3) by real Laplace inversion (cf. the proof of Lemma 5.3). The existence of a unique bounded extension of $P(t)D_V^*B$ is proved in the same way as before.

Step 4 – It remains to prove that $\mathsf{D}_p(\underline{L})$ is a core for $\mathsf{D}_p(D_V^*B|_{\overline{\mathsf{R}_p(D_V)}})$. Take $F \in \mathsf{D}_p(D_V^*B|_{\overline{\mathsf{R}_p(D_V)}})$. Then $\lim_{t\to 0}(I+t^2\underline{L})^{-1}F = F$ in $\overline{\mathsf{R}_p(D_V)}$ and, by (9.3) $\lim_{t\to 0} D_V^*B(I+t^2\underline{L})^{-1}F = \lim_{t\to 0}(I+t^2L)^{-1}D_V^*BF = D_V^*BF$ in L^p. This gives the result. □

Proposition 9.4. *For all $F \in \mathsf{D}_p(\underline{L})$ we have $F \in \mathsf{D}_p(D_V^*B)$, $D_V^*BF \in \mathsf{D}_p(D_V)$, and $D_V(D_V^*B)F = \underline{L}F$.*

Proof. Since $\mathsf{D}_p(L)$ is a core for $\mathsf{D}_p(D_V)$, the set $\mathscr{P} := \{D_V(I+L)^{-1}g : g \in \mathsf{D}_p(D_V)\}$ is a $\underline{P}$-invariant dense subspace of $\overline{\mathsf{R}_p(D_V)}$. To see that $\mathscr{P}$ is contained in $\mathsf{D}_p(\underline{L})$, note that if $g \in \mathsf{D}_p(D_V)$, then $f := (I+L)^{-1}g \in \mathsf{D}_p(L)$ and $D_Vf = D_V(1+L)^{-1}g = (1+\underline{L})^{-1}D_Vg \in \mathsf{D}_p(\underline{L})$ as claimed. It follows that $\mathscr{P}$ is a core for $\mathsf{D}_p(\underline{L})$, and hence a core for $\mathsf{D}_p(D_V^*B|_{\overline{\mathsf{R}_p(D_V)}})$ by Proposition 9.3. Moreover, $(1+\underline{L})D_Vf = D_Vg = D_V(I+L)f$, and therefore $\underline{L}D_Vf = D_VLf$.

For $F \in \mathscr{P}$, say $F = D_Vf$ with $f = (I+L)^{-1}g$ for some $g \in \mathsf{D}_p(D_V)$, we then have

$$\underline{L}F = \underline{L}D_Vf = D_VLf = D_V((D_V^*B)D_V)f = (D_V(D_V^*B))D_Vf = D_V(D_V^*B)F.$$

To see that this above identity extends to arbitrary $F \in \mathsf{D}_p(\underline{L})$, let $F_n \to F$ in $\mathsf{D}_p(\underline{L})$ with all F_n in $\mathscr{P}$. It follows from Proposition 9.3 that $F_n \to F$ in $\mathsf{D}_p(D_V^*B)$. In particular, $D_V^*BF_n \to D_V^*BF$ in L^p. Since $D_V(D_V^*B)F_n = \underline{L}F_n \to \underline{L}F$ in $\overline{\mathsf{R}_p(D_V)}$, the closedness of D_V then implies that $D_V^*BF \in \mathsf{D}_p(D_V)$ and $D_V(D_V^*B)F = \underline{L}F$. □

In the remainder of this section we consider D_V^*B as a closed and densely defined operator from $\overline{\mathsf{R}_p(D_V)}$ to L^p and write D_V^*B instead of using the more precise notation $D_V^*B|_{\overline{\mathsf{R}_p(D_V)}}$.

For the proof of Proposition 9.5 we need the first part of Theorem 2.6. Its proof uses the Hodge-Dirac formalism, introduced by Axelsson, Keith, and M^cIntosh [6] in their study of the Kato square root problem. It was by using this formalism that the main results of this paper suggested themselves naturally.

On the Hilbertian direct sum $H \oplus \overline{\mathsf{R}(V)}$ we consider the closed and densely defined operator

$$T := \begin{bmatrix} 0 & V^*B \\ V & 0 \end{bmatrix}. \tag{9.4}$$

By [5, Theorem 8.3] T is bisectorial on $H \oplus \overline{\mathsf{R}(V)}$.

On the direct sum $L^p \oplus \overline{\mathsf{R}_p(D_V)}$ we introduce the closed and densely defined operator Π by

$$\Pi := \begin{bmatrix} 0 & D_V^*B \\ D_V & 0 \end{bmatrix}.$$

Proof of Theorem 2.6, first part. By Theorems 4.5 and 5.6, L and $\underline{L}$ are sectorial on L^p and $\overline{\mathsf{R}_p(D_V)}$, respectively. From this it is easy to see that on $L^p \oplus \overline{\mathsf{R}_p(D_V)}$ we have $i\mathbb{R} \setminus \{0\} \subseteq \rho(\Pi)$ and

$$(I - it\Pi)^{-1} = \begin{bmatrix} (1+t^2L)^{-1} & it(I+t^2L)^{-1}D_V^*B \\ itD_V(I+t^2L)^{-1} & (I+t^2\underline{L})^{-1} \end{bmatrix}, \quad t \in \mathbb{R} \setminus \{0\};$$

the rigorous interpretation of this identity is provided by the above propositions. Note that the off-diagonal entries are well defined and bounded by Theorem 8.2 and Proposition 9.3; the proof of the latter result also shows that $(I+t^2L)^{-1}D_V^*B$ is the adjoint of $B^*D_V(I+t^2L^*)^{-1}$.

We check the R-boundedness of the entries of the right-hand side matrix for $t \in \mathbb{R} \setminus \{0\}$. For the upper left and the lower right entry this follows from the R-sectoriality of L and $\underline{L}$ on L^p and $\overline{\mathsf{R}_p(D_V)}$ respectively. Theorem 8.2 ensures the R-boundedness of the lower left entry, and the R-boundedness of the upper right entry follows from Proposition 6.3 (applied with B and L replaced by B^* and L^*). ☐

As a consequence of the bisectoriality of Π, the operator Π^2 is sectorial. Moreover,

$$\Pi^2 = \begin{bmatrix} (D_V^*B)D_V & 0 \\ 0 & D_V(D_V^*B) \end{bmatrix} = \begin{bmatrix} L & 0 \\ 0 & \underline{L} \end{bmatrix}.$$

To justify the latter identity, we appeal to Propositions 9.1 and 9.4 to obtain the inclusion $\begin{bmatrix} L & 0 \\ 0 & \underline{L} \end{bmatrix} \subseteq \Pi^2$. Since both operators are sectorial of angle $< \frac{1}{2}\pi$, they are in fact equal.

Proposition 9.5. *On L^p and $\overline{\mathsf{R}_p(D_V)}$ the following identities hold:*

$$\overline{\mathsf{R}_p(L)} = \overline{\mathsf{R}_p(D_V^*B)}, \qquad \mathsf{N}_p(L) = \mathsf{N}_p(D_V),$$
$$\overline{\mathsf{R}_p(\underline{L})} = \overline{\mathsf{R}_p(D_V)}, \qquad \mathsf{N}_p(\underline{L}) = \mathsf{N}_p(D_V^*B) = \{0\}.$$

Moreover, $L^p = \overline{\mathsf{R}_p(D_V^*B)} \oplus \mathsf{N}_p(D_V)$.

We recall that D_V^*B is interpreted as a densely defined closed operator from $\overline{\mathsf{R}_p(D_V)}$ to L^p. In the final section we will show that under the assumptions of Theorem 2.1 we have $\overline{\mathsf{R}_p(D_V^*B)} = \overline{\mathsf{R}_p(D_V^*)}$ and that in this situation the space $\overline{\mathsf{R}_p(D_V^*B)}$ does not change if we consider D_V^*B as an unbounded operator from $\underline{L}^p$ to L^p.

Proof. The bisectoriality of Π on $L^p \oplus \overline{\mathsf{R}_p(D_V)}$ implies that

$$\overline{\mathsf{R}_p(\Pi^2)} = \overline{\mathsf{R}_p(\Pi)} \quad \text{and} \quad \mathsf{N}_p(\Pi^2) = \mathsf{N}_p(\Pi).$$

The result follows from this by considering both coordinates separately. The fact that $\mathsf{N}_p(D_V^* B) = \{0\}$ follows from the bisectorial decomposition $L^p \oplus \overline{\mathsf{R}_p(D_V)} = \overline{\mathsf{R}_p(\Pi)} \oplus \mathsf{N}_p(\Pi)$ and considering the second coordinate. The final identity follows by inspecting the first coordinate of the same decomposition. $\square$

10. Proof of Theorem 2.1

The main effort in this section is directed towards proving the following comprehensive version of Theorem 2.1.

Theorem 10.1. *Assume* (A1), (A2), (A3), *and let* $1 < p < \infty$.

(a) *The following assertions are equivalent:*
  (a1) $\mathsf{D}_p(\sqrt{L}) \subseteq \mathsf{D}_p(D_V)$ *with* $\|D_V f\|_p \lesssim \|\sqrt{L} f\|_p$;
  (a2) $\underline{L}$ *satisfies a square function estimate on* $\overline{\mathsf{R}_p(D_V)}$:

$$\|F\|_p \lesssim \Big\| \Big(\int_0^\infty \|t \underline{L}\, \underline{P}(t) F\|^2 \, \frac{dt}{t} \Big)^{1/2} \Big\|_p;$$

  (a3) $\mathsf{D}(\sqrt{A}) \subseteq \mathsf{D}(V)$ *with* $\|Vh\| \lesssim \|\sqrt{A} h\|$;
  (a4) $\underline{A}$ *satisfies a square function estimate on* $\overline{\mathsf{R}(V)}$:

$$\|u\| \lesssim \Big(\int_0^\infty \|t \underline{A}\, \underline{S}(t) u\|^2 \, \frac{dt}{t} \Big)^{1/2}.$$

(b) *The same result holds with '$\lesssim$' and '$\subseteq$' replaced by '$\gtrsim$' and '$\supseteq$'.*

(c) *The following assertions are equivalent:*
  (c1) $\mathsf{D}_p(\sqrt{L}) = \mathsf{D}_p(D_V)$ *with* $\|D_V f\|_p \eqsim \|\sqrt{L} f\|_p$;
  (c2) $\underline{L}$ *admits a bounded* H^∞*-functional calculus on* $\overline{\mathsf{R}_p(D_V)}$;
  (c3) $\mathsf{D}(\sqrt{A}) = \mathsf{D}(V)$ *with* $\|Vh\| \eqsim \|\sqrt{A} h\|$;
  (c4) $\underline{A}$ *admits a bounded* H^∞*-functional calculus on* $\overline{\mathsf{R}(V)}$.

The plan of the proof is as follows. First we consider (a). The equivalence of (a3) and (a4) will be proved in Lemma 10.2, while the implications (a1) $\Rightarrow$ (a3) and (a2) $\Rightarrow$ (a4) follow by considering functions of the form $f = \phi_h$ and $F = \mathbf{1} \otimes u$ respectively, and using the equivalence of L^p-norms on the first Wiener-Itô chaos. In Proposition 9.5 we have shown that $\underline{L}$ is injective on $\overline{\mathsf{R}_p(D)}$, and then Proposition 7.3 asserts that (a4) implies (a2), so that it remains to show that (a2) implies (a1).

Next we turn to part (b). The equivalence of (b3) and (b4) follows from Lemma 10.2, and the implications (b1) $\Rightarrow$ (b3) and (b2) $\Rightarrow$ (b4) follow as in part (a). Proposition 7.3 asserts that (b4) implies (b2), so that it suffices to show that (b4) implies (b1).

Finally, part (c) follows by putting together the estimates obtained in (a) and (b) and appealing to Proposition 7.2.

The next lemma is a variation of Theorem 10.1 in [5]. We are grateful to Alan M$^{\text{c}}$Intosh for showing us the argument below. Keeping in mind that B satisfies (A3) if and only if B^* satisfies (A3), we write $\underline{A}_* := VV^*B^*$ and we denote the semigroup generated by $-\underline{A}_*$ by $\underline{S}_*$.

Lemma 10.2. *Assume* (A2) *and* (A3). *For* $h \in \mathsf{D}(A)$ *we have*

$$\Big(\int_0^\infty \|t\underline{A}\,\underline{S}(t)Vh\|^2\,\frac{dt}{t}\Big)^{1/2} \eqsim \|\sqrt{A}h\|. \tag{10.1}$$

As a first consequence, the following assertions are equivalent:

(1) $\mathsf{D}(\sqrt{A}) \subseteq \mathsf{D}(V)$ *with* $\|\sqrt{A}h\| \gtrsim \|Vh\|$, $h \in \mathsf{D}(\sqrt{A})$;

(2) $\underline{A}$ *satisfies a square function estimate on* $\overline{\mathsf{R}(V)}$:
$$\Big(\int_0^\infty \|t\underline{A}\,\underline{S}(t)u\|^2\,\frac{dt}{t}\Big)^{1/2} \gtrsim \|u\|;$$

(3) $\mathsf{D}(\sqrt{A^*}) \supseteq \mathsf{D}(V)$ *with* $\|\sqrt{A^*}h\| \lesssim \|Vh\|$, $h \in \mathsf{D}(V)$;

(4) $\underline{A}_*$ *satisfies a square function estimate on* $\overline{\mathsf{R}(V)}$:
$$\Big(\int_0^\infty \|t\underline{A}_*\underline{S}_*(t)u\|^2\,\frac{dt}{t}\Big)^{1/2} \lesssim \|u\|.$$

As a second consequence, the following assertions are equivalent:

(1′) $\mathsf{D}(\sqrt{A}) = \mathsf{D}(V)$ *with equivalence of norms* $\|\sqrt{A}h\| \eqsim \|Vh\|$;

(2′) $\underline{A}$ *admits a bounded* H^∞*-functional calculus on* $\overline{\mathsf{R}(V)}$;

(3′) $\mathsf{D}(\sqrt{A^*}) = \mathsf{D}(V)$ *with* $\|\sqrt{A^*}h\| \eqsim \|Vh\|$;

(4′) $\underline{A}_*$ *admits a bounded* H^∞*-functional calculus on* $\overline{\mathsf{R}(V)}$.

Proof. To prove (10.1), let $\omega \in (\omega(T), \frac12\pi)$, where T is defined by (9.4). By [5, Proposition 8.1] we have for all $\psi \in H_0^\infty(\Sigma_\omega)$ and $\tilde\psi \in H_0^\infty(\Sigma_{2\omega}^+)$,

$$\int_0^\infty \|\psi(tT)u\|^2\,\frac{dt}{t} \eqsim \int_0^\infty \|\tilde\psi(tT^2)u\|^2\,\frac{dt}{t}, \quad u \in \overline{\mathsf{R}(T)}.$$

Using this, the fact that A has a bounded H^∞-calculus, and the fact that $\varphi := \mathrm{sgn}\cdot\psi \in H_0^\infty(\Sigma_\omega)$, for $h \in \mathsf{D}(A)$ we obtain

$$\begin{aligned}
\|\sqrt{A}h\|^2 &\eqsim \int_0^\infty \|\tilde\psi(tA)\sqrt{A}h\|^2\frac{dt}{t}\\
&= \int_0^\infty \Big\|\tilde\psi(tT^2)\sqrt{T^2}\begin{bmatrix} h\\ 0\end{bmatrix}\Big\|^2\frac{dt}{t}\\
&\eqsim \int_0^\infty \Big\|\psi(tT)\sqrt{T^2}\begin{bmatrix} h\\ 0\end{bmatrix}\Big\|^2\frac{dt}{t}\\
&= \int_0^\infty \Big\|\varphi(tT)T\begin{bmatrix} h\\ 0\end{bmatrix}\Big\|^2\frac{dt}{t}\\
&= \int_0^\infty \Big\|\varphi(tT)\begin{bmatrix} 0\\ Vh\end{bmatrix}\Big\|^2\frac{dt}{t}\\
&\eqsim \int_0^\infty \Big\|\tilde\psi(tT^2)\begin{bmatrix} 0\\ Vh\end{bmatrix}\Big\|^2\frac{dt}{t}\\
&\eqsim \int_0^\infty \|\tilde\psi(t\underline{A})Vh\|^2\frac{dt}{t}.
\end{aligned}$$

The equivalence of (1) and (2) follows immediately by taking $\tilde\psi(z) = ze^{-z}$. Replacing B by B^* we obtain the equivalence of (3) and (4). Finally, the equivalence of (1) and (3) is a well-known consequence of the duality theory of forms [29, 35] (see also [5, Theorem 10.1]).

The equivalence of the primed statements follows in the same way (or can alternatively be deduced from the equivalence of the un-primed statements). □

Proof of Theorem 10.1. Fix $1 < p < \infty$ and let $\frac{1}{p} + \frac{1}{q} = 1$.

Part (a): It remains to prove that (a2) implies (a1). Perhaps the shortest proof of this implication is based on a lower bound for the square function associated with the semigroup $\underline{Q}$ generated by $-\sqrt{\underline{L}}$. Alternatively, one could adapt the argument in Lemma 10.2 to the L^p-setting.

Consider the functions $\varphi(z) = ze^{-z}$ and $\psi(z) = \sqrt{z}e^{-\sqrt{z}}$. These functions belong to $H_0^\infty(\Sigma_\theta^+)$ for $\theta < \frac{1}{2}\pi$. Substituting $t = s^2$ we obtain, from Proposition 7.2,

$$\Big\| \Big(\int_0^\infty \|\psi(t\underline{L})F\|^2 \,\frac{dt}{t}\Big)^{1/2} \Big\|_p = \sqrt{2} \Big\| \Big(\int_0^\infty \|s\sqrt{\underline{L}}\underline{Q}(s)F\|^2 \,\frac{ds}{s}\Big)^{1/2} \Big\|_p.$$

Using (a2) and the first part of Proposition 7.2, the identity of Theorem 5.6 (which extends to the semigroup Q generated by $-\sqrt{L}$), and Lemma 8.1 and Theorem 8.3, for all $f \in \mathsf{D}_p(L)$ we obtain

$$\begin{aligned} \|D_V f\|_p &\lesssim \Big\| \Big(\int_0^\infty \|t\underline{L}\,\underline{P}(t)D_V f\|^2 \,\frac{dt}{t}\Big)^{1/2} \Big\|_p \\ &\eqsim \Big\| \Big(\int_0^\infty \|s\sqrt{\underline{L}}\underline{Q}(s)D_V f\|^2 \,\frac{ds}{s}\Big)^{1/2} \Big\|_p \\ &= \|\mathscr{G}(\sqrt{L}f)\|_p \\ &\leqslant \|\mathscr{H}(\sqrt{L}f)\|_p \\ &\lesssim \|\sqrt{L}f\|_p. \end{aligned}$$

Since $\mathsf{D}_p(L)$ is a core for both $\mathsf{D}_p(\sqrt{L})$ and $\mathsf{D}_p(D_V)$, the desired domain inclusion follows and the norm estimate holds for all $f \in \mathsf{D}_p(\sqrt{L})$.

Part (b): It remains to show that (b4) implies (b1).

By Lemma 10.2 (applied with the roles of B and B^* reversed), (b4) implies the estimate

$$\Big(\int_0^\infty \|t\underline{A}_{B^*}\underline{S}_{B^*}(t)u\|^2 \,\frac{dt}{t}\Big)^{1/2} \gtrsim \|u\|, \quad u \in \overline{\mathsf{R}(V)}.$$

It follows from Part (a) (with B replaced by B^*) that $\mathsf{D}_q(\sqrt{L^*}) \subseteq \mathsf{D}_q(D_V)$ and, for $f \in \mathsf{D}_q(\sqrt{L^*})$,

$$\|D_V f\|_q \lesssim \|\sqrt{L^*}f\|_q, \quad 1 < q < \infty.$$

We will use next a standard duality argument to prove the estimate $\|\sqrt{L}g\|_p \lesssim \|D_V g\|_p$ where $\frac{1}{p} + \frac{1}{q} = 1$. For $g \in \mathscr{F}C_{\rm b}^\infty(E; \mathsf{D}(A))$ we have

$$\|\sqrt{L}g\|_p = \sup_{\|\widetilde{f}\|_q \leqslant 1} |\langle \sqrt{L}g, \widetilde{f}\rangle|.$$

The sectoriality of $\sqrt{L^*}$ allows us to use the decomposition $\widetilde{f} = \widetilde{f}_0 + \widetilde{f}_1 \in \mathsf{N}(\sqrt{L^*}) \oplus \overline{\mathsf{R}(\sqrt{L^*})} = L^q$, and since $\mathscr{F}C_{\rm b}^\infty(E; \mathsf{D}(A^*))$ is a core for $\mathsf{D}_q(\sqrt{L^*})$, it suffices to consider $\widetilde{f}$ of the form $\widetilde{f} = \widetilde{f}_0 + \sqrt{L^*}f$, with $f \in \mathscr{F}C_{\rm b}^\infty(E; \mathsf{D}(A^*))$ and $\|\widetilde{f}\|_q \leqslant 1$.

Since $\widetilde{f}_0 \in \mathsf{N}(\sqrt{L^*})$ and $\|\sqrt{L^*}f\|_p \leqslant \|P_{\overline{\mathsf{R}(\sqrt{L^*})}}\|_p\|\widetilde{f}\|_p$, we obtain

$$\begin{aligned}
\|\sqrt{L}g\|_p &= \sup_{\|\widetilde{f}_0+\sqrt{L^*}f\|_q\leqslant 1} |\langle \sqrt{L}g, \widetilde{f}_0 + \sqrt{L^*}f\rangle| \\
&\lesssim \sup_{\|\sqrt{L^*}f\|_q\leqslant 1} |\langle \sqrt{L}g, \sqrt{L^*}f\rangle| \\
&= \sup_{\|\sqrt{L^*}f\|_q\leqslant 1} |\langle Lg, f\rangle| \\
&\lesssim \sup_{\|D_V f\|_q\leqslant 1} |\langle Lg, f\rangle| \\
&= \sup_{\|D_V f\|_q\leqslant 1} |\langle BD_V g, D_V f\rangle| \\
&\leqslant \sup_{\|D_V f\|_q\leqslant 1} \|B\|\, \|D_V g\|_p \|D_V f\|_q \\
&= \|B\|\, \|D_V g\|_p.
\end{aligned}$$

Since $\mathscr{F}C_{\mathrm{b}}^\infty(E;\mathsf{D}(A))$ is a core for $\mathsf{D}_p(L)$, the result of Step 2 follows.

Part (c): The equivalences follow immediately from (a) and (b) combined with Proposition 7.2. $\square$

We finish this section by pointing out two further equivalences to the ones of Theorem 2.1 and their one-sided extensions in Theorem 10.1.

The conditions (1)-(4) of Theorem 2.1 are equivalent to

(5) $\mathsf{D}_p(\sqrt{\underline{L}}) = \mathsf{D}_p(D_V^* B)$ with $\|\sqrt{\underline{L}}F\|_p \eqsim \|D_V^* BF\|_p$ for $F \in \mathsf{D}_p(\sqrt{\underline{L}})$;
(6) $\mathsf{D}(\sqrt{\underline{A}}) = \mathsf{D}(V^* B)$ with $\|\sqrt{\underline{A}}u\|_p \eqsim \|V^* Bu\|_p$ for $u \in \mathsf{D}(\sqrt{\underline{A}})$.

Here, in the spirit of Theorem 2.1, we interpret $\underline{A}$ as an operator in $\overline{\mathsf{R}(V)}$. This is immaterial, however, in view of the definition $\underline{A} = VV^*B$ and the (not necessarily orthogonal) Hodge decomposition $\underline{H} = \overline{\mathsf{R}(V)} \oplus \mathsf{N}(V^*B)$ (see (2.1)) by virtue of which (6) also holds on the full space $\underline{H}$.

To see that (1) implies (5), note that for $f \in \mathsf{D}_p(L)$ we have

$$\|(D_V^* B)D_V f\|_p = \|Lf\|_p \eqsim \|D_V \sqrt{L}f\|_p = \|\sqrt{\underline{L}}D_V f\|_p.$$

Since $D_V(\mathsf{D}_p(L))$ is a core for both $\mathsf{D}_p(D_V^* B)$ and $\mathsf{D}_p(\sqrt{\underline{L}})$, (5) follows. The converse implication that (5) implies (1) is proved similarly. The equivalence (3)$\Leftrightarrow$(6) is proved in the same way. It is clear from the proofs that the one-sided versions of these implications hold as well.

11. Proof of Theorem 2.2

We continue with the proof of Theorem 2.2. It will be a standing assumption that the equivalent conditions of Theorem 2.1 are satisfied. As we have already observed (in Lemma 10.2, see also the discussion below Theorem 2.1), the corresponding equivalences obtained by replacing B with B^* then also hold.

Below, for $k = 1, 2$ we will use the bounded analytic C_0-semigroups

$$\underline{P}_{(k)}(t) := P(t) \otimes \underline{S}^{\otimes k}(t),$$

which are defined on the spaces

$$\underline{L}^p_{(k)} := L^p(E, \mu; \underline{H}^{\otimes k}).$$

Note that $\underline{L}^p_{(1)} = \underline{L}^p$ and $\underline{P}_{(1)}$ coincides with $\underline{P}$ on the closed subspace $\overline{\mathsf{R}_p(D_V)}$. The generators of $\underline{P}_{(k)}$ will be denoted by $-\underline{L}_{(k)}$. The semigroups generated by $-\sqrt{I+\underline{L}_{(k)}}$ will be denoted by $\underline{Q}_{(k)}$.

We also consider the operator $D_V \otimes I$, initially defined on the algebraic tensor product $\mathsf{D}_p(D_V)\otimes \underline{H}$, which is viewed as a dense subspace of $\underline{L}^p_{(1)}$. Using that D_V is a closed operator from L^p into $\underline{L}^p$, it is straightforward to check that $D_V \otimes I$ extends to a closed operator

$$\underline{D}_V : \mathsf{D}_p(\underline{D}_V) \subseteq \underline{L}^p_{(1)} \to \underline{L}^p_{(2)}.$$

On the algebraic tensor product $L^p \otimes \underline{H}$, for $t > 0$ we define the operators

$$\begin{aligned}\underline{D}_V \underline{P}_{(1)}(t) &:= (D_V P(t)) \otimes \underline{S}(t),\\ \underline{D}_V \underline{P}^*_{(1)}(t) &= (D_V P^*(t)) \otimes \underline{S}^*(t).\end{aligned}$$

By Theorem 8.2 these operators extend uniquely to bounded operators from $\underline{L}^p_{(1)}$ to $\underline{L}^p_{(2)}$.

Proposition 11.1. *Let $1 < p < \infty$.*

(i) *The collections $\{\sqrt{t}\underline{D}_V \underline{P}_{(1)}(t) : t > 0\}$ and $\{\sqrt{t}\underline{D}_V \underline{P}^*_{(1)}(t) : t > 0\}$ are R-bounded in $\mathscr{L}(\underline{L}^p_{(1)}, \underline{L}^p_{(2)})$.*

(ii) *The following square function estimates hold for $F \in \underline{L}^p_{(1)}$:*

$$\begin{aligned}\Big\|\Big(\int_0^\infty \|\sqrt{t}\underline{D}_V \underline{P}_{(1)}(t)F\|^2 \,\frac{dt}{t}\Big)^{1/2}\Big\|_p &\lesssim \|F\|_p,\\ \Big\|\Big(\int_0^\infty \|\sqrt{t}\underline{D}_V \underline{P}^*_{(1)}(t)F\|^2 \,\frac{dt}{t}\Big)^{1/2}\Big\|_p &\lesssim \|F\|_p.\end{aligned}$$

(iii) *The domain inclusions $\mathsf{D}_p(\sqrt{\underline{L}_{(1)}}) \subseteq \mathsf{D}_p(\underline{D}_V)$ and $\mathsf{D}_p(\sqrt{\underline{L}^*_{(1)}}) \subseteq \mathsf{D}_p(\underline{D}_V)$ hold with norm estimates*

$$\begin{aligned}\|\underline{D}_V F\|_p &\lesssim \|F\|_p + \|\sqrt{\underline{L}_{(1)}}F\|_p\\ \textit{and}\quad \|\underline{D}_V F\|_p &\lesssim \|F\|_p + \|\sqrt{\underline{L}^*_{(1)}}F\|_p.\end{aligned}$$

Proof. (i): The R-boundedness is a consequence from (an easy Hilbert space-valued extension of) Proposition 6.2 combined with (11) and Theorem 8.2.

(ii): Since $\underline{A}$ has a bounded H^∞-calculus on $\underline{H}$ of angle $< \frac12\pi$, the same holds for $\underline{A}^*$. Proposition 7.3 implies that $\underline{L}_{(1)}$ and $\underline{L}^*_{(1)}$ have bounded H^∞-functional calculi on $\underline{L}^p_{(1)}$ of angle $< \frac12\pi$. The domain inclusions $\mathsf{D}_p(\underline{L}_{(1)}) \subseteq \mathsf{D}_p(\underline{D}_V)$ and $\mathsf{D}_p(\underline{L}^*_{(1)}) \subseteq \mathsf{D}_p(\underline{D}_V)$ follow from (i) by taking Laplace transforms. By combining (i) and Proposition 7.5 we obtain the desired result.

(iii): Combining the fact that $\sqrt{I+\underline{L}_{(2)}}$ has a bounded H^∞-calculus of angle $< \frac12\pi$ with Proposition 7.2, the commutation relation $\underline{D}_V\underline{P}_{(1)}(t) = \underline{P}_{(2)}(t)\underline{D}_V$, the $\underline{H}$-valued analogue of Lemma 8.1, and the first estimate of (ii), for all $F \in \mathsf{D}_p(\underline{L}_{(1)})$

we obtain

$$\begin{aligned}
\|\underline{D}_V F\|_p &\lesssim \Big\|\Big(\int_0^\infty \|t\sqrt{I+\underline{L}_{(2)}}\underline{Q}_{(2)}(t)\underline{D}_V F\|^2\,\frac{dt}{t}\Big)^{1/2}\Big\|_p \\
&= \Big\|\Big(\int_0^\infty \|t\underline{D}_V\underline{Q}_{(1)}(t)\sqrt{I+\underline{L}_{(1)}}F\|^2\,\frac{dt}{t}\Big)^{1/2}\Big\|_p \\
&\leqslant \Big\|\Big(\int_0^\infty \|\sqrt{t}\underline{D}_V e^{-t}\underline{P}_{(1)}(t)\sqrt{I+\underline{L}_{(1)}}F\|^2\,\frac{dt}{t}\Big)^{1/2}\Big\|_p \\
&\leqslant \Big\|\Big(\int_0^\infty \|\sqrt{t}\underline{D}_V\underline{P}_{(1)}(t)\sqrt{I+\underline{L}_{(1)}}F\|^2\,\frac{dt}{t}\Big)^{1/2}\Big\|_p \\
&\lesssim \big\|\sqrt{I+\underline{L}_{(1)}}F\big\|_p \\
&\eqsim \|F\|_p + \big\|\sqrt{\underline{L}_{(1)}}F\big\|_p.
\end{aligned}$$

This gives the first estimate. Since $\mathsf{D}_p(\underline{L}_{(1)})$ is a core for $\mathsf{D}_p(\sqrt{\underline{L}_{(1)}})$, the domain inclusion follows as well.

To prove the second estimate we put $T := P^*\otimes\underline{S}_*\otimes\underline{S}^*$, where $\underline{S}_*$ is the bounded analytic semigroup generated by $-VV^*B^*$; this notation is as in Section 10. Note that the negative generator C of T has a bounded H^∞-calculus of angle $< \frac12\pi$; this follows from the fact that if Theorem 2.1 holds for B, then it also holds for B^* (see Lemma 10.2) and therefore the negative generators of $\underline{S}^*$ and $\underline{S}_*$ both have bounded H^∞-calculi of angle $< \frac12\pi$. Let R be the semigroup generated by $-\sqrt{I+C}$. Using the identity

$$\underline{D}_V\underline{P}^*_{(1)}(t)F = T(t)\underline{D}_V F,$$

and arguing as above, for all $F\in\mathsf{D}_p(\underline{L}^*_{(1)})$ we obtain

$$\begin{aligned}
\|\underline{D}_V F\|_p &\lesssim \Big\|\Big(\int_0^\infty \|t\sqrt{I+C}R(t)\underline{D}_V F\|^2\,\frac{dt}{t}\Big)^{1/2}\Big\|_p \\
&= \Big\|\Big(\int_0^\infty \|t\underline{D}_V\underline{Q}^*_{(1)}(t)\sqrt{I+\underline{L}^*_{(1)}}F\|^2\,\frac{dt}{t}\Big)^{1/2}\Big\|_p \\
&\leqslant \Big\|\Big(\int_0^\infty \|\sqrt{t}\underline{D}_V e^{-t}\underline{P}^*_{(1)}(t)\sqrt{I+\underline{L}^*_{(1)}}F\|^2\,\frac{dt}{t}\Big)^{1/2}\Big\|_p \\
&\leqslant \Big\|\Big(\int_0^\infty \|\sqrt{t}\underline{D}_V\underline{P}^*_{(1)}(t)\sqrt{I+\underline{L}^*_{(1)}}F\|^2\,\frac{dt}{t}\Big)^{1/2}\Big\|_p \\
&\lesssim \big\|\sqrt{I+\underline{L}^*_{(1)}}F\big\|_p \\
&\eqsim \|F\|_p + \big\|\sqrt{\underline{L}^*_{(1)}}F\big\|_p.
\end{aligned}$$

The second domain inclusion now follows from the fact that $\mathsf{D}_p(\underline{L}^*_{(1)})$ is a core for $\mathsf{D}_p(\sqrt{\underline{L}^*_{(1)}})$. □

In the following theorem we give a characterisation of $\mathsf{D}_p(\sqrt{\underline{L}_{(1)}})$. Since $\sqrt{\underline{L}} = \sqrt{\underline{L}_{(1)}}$ on $\overline{\mathsf{R}_p(D_V)}$, this gives a further equivalence of norms for $\sqrt{\underline{L}}$ on $\overline{\mathsf{R}_p(D_V)}$, different from the one in Theorem 2.1. In the proof of Theorem 2.2 we use both equivalences to determine the domain of L.

First we need a simple lemma.

Lemma 11.2. *Let $1 < p < \infty$. The semigroup $\underline{Q}_{(1)}$ restricts to C_0-semigroups on the space $\mathsf{D}_p(\underline{D}_V) \cap \mathsf{D}_p(\sqrt{I \otimes \underline{A}})$.*

Proof. It suffices to prove the result with $\underline{Q}_{(1)}$ replaced by $\underline{P}_{(1)}$; the latter is readily seen to restrict to a C_0-semigroup on $\mathsf{D}_p(\underline{D}_V) \cap \mathsf{D}_p(\sqrt{I \otimes \underline{A}})$ by the identities $\underline{D}_V P_{(1)}(t) = \underline{P}_{(2)}(t)\underline{D}_V$ and $\sqrt{I \otimes \underline{A}}\,\underline{P}_{(1)}(t) = \underline{P}_{(1)}(t)\sqrt{I \otimes \underline{A}}$. $\square$

Theorem 11.3. *Let $1 < p < \infty$. We have equality of domains*

$$\mathsf{D}_p(\sqrt{\underline{L}_{(1)}}) = \mathsf{D}_p(\underline{D}_V) \cap \mathsf{D}_p(\sqrt{I \otimes \underline{A}}),$$

with equivalence of norms

$$\|F\|_p + \big\|\sqrt{\underline{L}_{(1)}}F\big\|_p \eqsim \|F\|_p + \|\underline{D}_V F\|_p + \|\sqrt{I \otimes \underline{A}}F\|_p,$$

Proof. By a result of Kalton and Weis [28, Theorem 6.3], applied to the sums $\underline{L}_{(1)} = L \otimes I + I \otimes \underline{A}$ and $\underline{L}_{(1)}^* = L^* \otimes I + I \otimes \underline{A}^*$, we have the estimates

$$\begin{aligned}\|(I \otimes \underline{A})F\|_p &\lesssim \|F\|_p + \|\underline{L}_{(1)}F\|_p, \quad F \in \mathsf{D}_p(\underline{L}_{(1)})\\ \|(I \otimes \underline{A}^*)F\|_p &\lesssim \|F\|_p + \|\underline{L}_{(1)}^*F\|_p, \quad F \in \mathsf{D}_p(\underline{L}_{(1)}^*).\end{aligned}$$

Since the square root domains equal the complex interpolation spaces at exponent $\frac{1}{2}$ for sectorial operators with bounded imaginary powers [23, Theorem 6.6.9], by interpolating the inclusions

$$\mathsf{D}_p(\underline{L}_{(1)}) \hookrightarrow \mathsf{D}_p(I \otimes \underline{A}), \quad \mathsf{D}_p(\underline{L}_{(1)}^*) \hookrightarrow \mathsf{D}_p(I \otimes \underline{A}^*),$$

with the identity operator, we obtain the estimates

$$\begin{aligned}\|\sqrt{I \otimes \underline{A}}F\|_p &\lesssim \|F\|_p + \big\|\sqrt{\underline{L}_{(1)}}F\big\|_p, \quad F \in \mathsf{D}_p(\underline{L}_{(1)})\\ \|\sqrt{I \otimes \underline{A}^*}F\|_p &\lesssim \|F\|_p + \big\|\sqrt{\underline{L}_{(1)}^*}F\big\|_p, \quad F \in \mathsf{D}_p(\underline{L}_{(1)}^*).\end{aligned} \tag{11.1}$$

Combining these estimates with Proposition 11.1 we obtain

$$\begin{aligned}\|F\|_p + \|\underline{D}_V F\|_p + \|\sqrt{I \otimes \underline{A}}F\|_p &\lesssim \|F\|_p + \big\|\sqrt{\underline{L}_{(1)}}F\big\|_p, \quad F \in \mathsf{D}_p(\underline{L}_{(1)})\\ \|F\|_p + \|\underline{D}_V F\|_p + \|\sqrt{I \otimes \underline{A}^*}F\|_p &\lesssim \|F\|_p + \big\|\sqrt{\underline{L}_{(1)}^*}F\big\|_p, \quad F \in \mathsf{D}_p(\underline{L}_{(1)}^*).\end{aligned}$$

Next we prove the reverse estimates. For $F \in \mathsf{D}_p(L) \otimes \mathsf{D}_p(\underline{A})$ and $G \in \mathsf{D}_q(L^*) \otimes \mathsf{D}_p(\underline{A}^*)$ ($\frac{1}{p} + \frac{1}{q} = 1$) we have $F \in \mathsf{D}_p(\underline{L}_{(1)})$, $G \in \mathsf{D}_q(\underline{L}_{(1)}^*)$, and

$$\begin{aligned}\langle\sqrt{I + \underline{L}_{(1)}}F, G\rangle &= \langle (I + \underline{L}_{(1)})F, 1/\sqrt{\underline{L}_{(1)}^* + I}G\rangle\\ &= \langle F, 1/\sqrt{I + \underline{L}_{(1)}^*}G\rangle + \langle (L \otimes I)F, 1/\sqrt{I + \underline{L}_{(1)}^*}G\rangle\\ &\quad + \langle (I \otimes \underline{A})F, 1/\sqrt{I + \underline{L}_{(1)}^*}G\rangle\\ &= \langle F, 1/\sqrt{I + \underline{L}_{(1)}^*}G\rangle + \langle B\underline{D}_V F, \underline{D}_V/\sqrt{I + \underline{L}_{(1)}^*}G\rangle\\ &\quad + \langle\sqrt{I \otimes \underline{A}}F, \sqrt{I \otimes \underline{A}^*}/\sqrt{I + \underline{L}_{(1)}^*}G\rangle.\end{aligned}$$

Using the boundedness of the three operators $1/\sqrt{I+\underline{L}^*_{(1)}}$, $\underline{D}_V/\sqrt{I+\underline{L}^*_{(1)}}$ (by Proposition 11.1(iii)), and $\sqrt{I\otimes\underline{A}^*}/\sqrt{I+\underline{L}^*_{(1)}}$ (by the second estimate in (11.1)), we find

$$\begin{aligned}\big\|\sqrt{I+\underline{L}_{(1)}}F\big\|_p &= \sup_{\|G\|_q\leqslant 1} |\langle\sqrt{I+\underline{L}_{(1)}}F, G\rangle| \\ &\leqslant \sup_{\|G\|_q\leqslant 1} \|F\|_p\,\|1/\sqrt{I+\underline{L}^*_{(1)}}G\|_q \\ &\qquad + \|B\|\,\|\underline{D}_VF\|_p\,\|\underline{D}_V/\sqrt{I+\underline{L}^*_{(1)}}G\|_q \\ &\qquad + \|\sqrt{I\otimes\underline{A}}F\|_p\,\|\sqrt{I\otimes\underline{A}^*}/\sqrt{I+\underline{L}^*_{(1)}}G\|_q \\ &\lesssim \|F\|_p + \|\underline{D}_VF\|_p + \|\sqrt{I\otimes\underline{A}}F\|_p.\end{aligned}$$

The estimate

$$\big\|\sqrt{I+\underline{L}^*_{(1)}}F\big\|_p \lesssim \|F\|_p + \|\underline{D}_VF\|_p + \|\sqrt{I\otimes\underline{A}^*}F\|_p$$

is proved similarly and will not be needed.

It remains to prove the equality of domains. Since $\mathsf{D}_p(L)\otimes\mathsf{D}(\underline{A})$ is a core for $\mathsf{D}_p(\underline{L}_{(1)})$, it is also a core for $\mathsf{D}_p(\sqrt{\underline{L}_{(1)}})$. Using this, the domain inclusion $\mathsf{D}_p(\sqrt{\underline{L}_{(1)}})\subseteq\mathsf{D}_p(\underline{D}_V)\cap\mathsf{D}_p(\sqrt{I\otimes\underline{A}})$ follows, and the equivalence of norms extends to all $F\in\mathsf{D}_p(\sqrt{\underline{L}_{(1)}})$.

Again by the equivalence of norms, $\mathsf{D}_p(\sqrt{\underline{L}_{(1)}})$ is closed in $\mathsf{D}_p(\underline{D}_V)\cap\mathsf{D}_p(\sqrt{I\otimes\underline{A}})$. It remains to prove that the inclusion is dense. This follows from Lemma 11.2, since for $F\in\mathsf{D}_p(\underline{D}_V)\cap\mathsf{D}_p(\sqrt{I\otimes\underline{A}})$ and $t>0$ we have $\underline{Q}_{(1)}(t)F\in\mathsf{D}_p(\sqrt{\underline{L}_{(1)}})$ and $\underline{Q}_{(1)}(t)F\to F$ in the norm of $\mathsf{D}_p(\underline{D}_V)\cap\mathsf{D}_p(\sqrt{I\otimes\underline{A}})$ as $t\downarrow 0$. ☐

Recall that D denotes the Malliavin derivative. Since A is a closed operator, it follows from the results in [19] that the operator AD, initially defined on $\mathscr{F}C^1_{\rm b}(E;\mathsf{D}(A))$, is closable as an operator from L^p into $L^p(E,\mu;H)$ for $1<p<\infty$. We denote its closure by D_A.

We also consider the operator D_V^2 defined by

$$\mathsf{D}_p(D_V^2) := \{f\in\mathsf{D}_p(D_V) : D_Vf\in\mathsf{D}_p(\underline{D}_V)\},\quad D_V^2 := \underline{D}_VD_V.$$

It is easy to check that this operator is closed from $\mathsf{D}_p(D_V)$ into $L^p(E,\mu;\underline{H}^{\otimes 2})$.

Lemma 11.4. *Let $1<p<\infty$. The semigroup P restricts to a C_0-semigroup on the space $\mathsf{D}_p(D_V^2)\cap\mathsf{D}_p(D_A)$.*

Proof. An easy argument based on Theorem 5.6 shows that $P(t)\mathsf{D}_p(D_V^2)\subseteq\mathsf{D}_p(D_V^2)$ and

$$D_V^2P(t)f := \underline{P}_{(2)}(t)D_V^2f,\quad f\in\mathsf{D}_p(D_V^2).$$

Similarly, we have $P(t)\mathsf{D}_p(D_A)\subseteq\mathsf{D}_p(D_A)$ and

$$D_AP(t)f = (P(t)\otimes S(t))D_Af,\quad f\in\mathsf{D}_p(D_A).$$

These identities easily imply the result. ☐

Proof of Theorem 2.2. Using the fact that $\mathsf{D}_p(L) \subseteq \mathsf{D}_p(D_V)$, Proposition 9.1, the domain equality $\mathsf{D}_p(\sqrt{\underline{L}}) = \mathsf{D}_p(D_V^* B)$ (see (5) at the end of Section 10), Theorem 11.3, the domain equality $\mathsf{D}(\sqrt{\underline{A}}) = \mathsf{D}(V^* B)$ on $\overline{\mathsf{R}(V)}$ (see (6) at the end of Section 10), and the definition of D_A, for $f \in \mathsf{D}_p(L)$ we obtain

$$\begin{aligned}
\|f\|_p + \|Lf\|_p &\eqsim \|f\|_p + \|D_V f\|_p + \|Lf\|_p \\
&= \|f\|_p + \|D_V f\|_p + \|(D_V^* B) D_V f\|_p \\
&\eqsim \|f\|_p + \|D_V f\|_p + \|\sqrt{\underline{L}} D_V f\|_p \\
&\eqsim \|f\|_p + \|D_V f\|_p + \|D_V^2 f\|_p + \|\sqrt{\underline{A}} D_V f\|_p \\
&\eqsim \|f\|_p + \|D_V f\|_p + \|D_V^2 f\|_p + \|(V^* B) D_V f\|_p \\
&\eqsim \|f\|_p + \|D_V f\|_p + \|D_V^2 f\|_p + \|D_A f\|_p,
\end{aligned}$$

This proves the equivalence of norms and the domain inclusion $\mathsf{D}_p(L) \subseteq \mathsf{D}_p(D_V^2) \cap \mathsf{D}_p(D_A)$. To obtain equality of domains it remains to show that this inclusion is both closed and dense. Closedness follows easily from the norm estimate and density follows from Lemma 11.4 in the same way as in Theorem 11.3. $\square$

Note that Theorem 2.2 is natural in view of the expression

$$\begin{aligned}
Lf(x) &= D_V^* B D_V f(x) \\
&= -\sum_{j,k=1}^{n} [BVh_j, Vh_k] \partial_j \partial_k \varphi(\phi_{h_1}, \dots, \phi_{h_n}) + \sum_{j=1}^{n} \partial_j \varphi(\phi_{h_1}, \dots, \phi_{h_n}) \cdot \phi_{Ah_j},
\end{aligned}$$

which holds for all $f \in \mathscr{F}C_{\mathrm{b}}^{\infty}(E; \mathsf{D}(A))$ of the form $f = \varphi(\phi_{h_1}, \dots, \phi_{h_n})$.

12. Proofs of Theorems 2.5 and 2.6

The first part of Theorem 2.5 has already been proved in Proposition 9.5. We begin with some preparations for the proof of the second part.

Let us denote by $\mathscr{F}P(E; \mathsf{D}(V))$ the vector space of all functions of the form $p = \varphi(\phi_{h_1}, \dots, \phi_{h_n})$ with $h_j \in \mathsf{D}(V)$ for $j = 1, \dots, n$ and $\varphi : \mathbb{R}^n \to \mathbb{R}$ a polynomial in n variables. In the proof of the next proposition we need the following auxiliary result.

Lemma 12.1. *For $1 \leqslant p < \infty$, $\mathscr{F}P(E; \mathsf{D}(V))$ is a core for $\mathsf{D}_p(D_V)$.*

Proof. A simple approximation argument shows that $\mathscr{F}P(E; \mathsf{D}(V)) \subseteq \mathsf{D}_p(D_V)$. Thus it suffices to approximate elements of $\mathscr{F}C_{\mathrm{b}}^1(E; \mathsf{D}(V))$ in the graph norm of $\mathsf{D}_p(D_V)$ with elements of $\mathscr{F}P(E; \mathsf{D}(V))$. Let $f \in \mathscr{F}C_{\mathrm{b}}^1(E; \mathsf{D}(V))$ be of the form $f = \varphi(\phi_{h_1}, \dots, \phi_{h_n})$ with $h_j \in \mathsf{D}(V)$ for $j = 1, \dots, n$ and $\varphi \in C_{\mathrm{b}}^1(\mathbb{R}^n)$. By a Gram-Schmidt argument we may assume that the elements $h_1, \dots, h_n$ are orthonormal in H. Taking Borel versions of the functions $x \mapsto \phi_{h_j}(x)$, the image measure of μ under the transformation $x \mapsto (\phi_{h_1}(x), \dots . \phi_{h_n}(x))$ is the standard Gaussian measure γ_n on $\mathbb{R}^n$.

This reduces the problem to finding polynomials p_k in n variables such that $p_k \to \varphi$ in $L^p(\mathbb{R}^n, \gamma_n)$ and $\nabla p_k \to \nabla \varphi$ in $L^p(\mathbb{R}^n, \gamma_n; \mathbb{R}^n)$. It is a classical fact that such polynomials exist. $\square$

In the remainder of this section we interpret $D_V^* B$ as a closed densely defined operator from $\underline{L}^p$ to L^p.

Proof of Theorem 2.5, second part. We shall prove separately that

$$\overline{\mathsf{R}_p(D_V)} + \mathsf{N}_p(D_V^* B) = \underline{L}^p, \tag{12.1}$$

$$\overline{\mathsf{R}_p(D_V)} \cap \mathsf{N}_p(D_V^* B) = \{0\}. \tag{12.2}$$

The proof of (12.1) is more or less standard. The idea behind the proof of (12.2) is to note that for $p = 2$ the Hodge decomposition is obtained as a special case of the Hodge decomposition theorem of Axelsson, Keith, and M^{c}Intosh [6], and to use this fact together with the fact that the L^p-norm and L^2-norm are equivalent on each summand in the Wiener-Itô decomposition.

We begin with the proof of (12.1). By Theorem 2.1(1) the operator $R := D_V/\sqrt{L}$ is well defined on $\mathsf{R}_p(\sqrt{L})$ and bounded. In view of the decomposition $L^p = \overline{\mathsf{R}_p(\sqrt{L})} \oplus \mathsf{N}_p(\sqrt{L})$ we may extend R to L^p by putting $R|_{\mathsf{N}_p(\sqrt{L})} := 0$. A similar remark applies to the operator $R_* := D_V/\sqrt{L^*}$.

For $F \in \underline{L}^p$ we claim that $RR_*^* F \in \overline{\mathsf{R}_p(D_V)}$, where $R_*^* := (R_*)^*$. Indeed, there exists $f \in \mathsf{N}_p(\sqrt{L})$ and a sequence $f_n \in \mathsf{D}_p(\sqrt{L})$ such that $f + \sqrt{L} f_n \to R_*^* F$ in L^p. Therefore $RR_*^* F = \lim_{n\to\infty} D_V f_n \in \overline{\mathsf{R}_p(D_V)}$.

Now, for functions $\psi \in \mathsf{D}_p(\sqrt{L})$ and $\phi \in \mathsf{D}_q(\sqrt{L^*})$,

$$\langle D_V \psi, B^* D_V \phi\rangle = \langle L\psi, \phi\rangle = \langle \sqrt{L}\psi, \sqrt{L^*}\phi\rangle.$$

Furthermore, approximating a function $f \in L^p$ by a sequence $(f_0 + \sqrt{L} f_n)_{n\geqslant 1}$ with $f_0 \in \mathsf{N}_p(\sqrt{L})$ and $f_n \in \mathsf{D}_p(\sqrt{L})$ we obtain

$$\begin{aligned}
\langle Rf, B^* D_V \phi\rangle &= \lim_{n\to\infty} \langle D_V f_n, B^* D_V \phi\rangle \\
&= \lim_{n\to\infty} \langle \sqrt{L} f_n, \sqrt{L^*}\phi\rangle \\
&= \langle f - f_0, \sqrt{L^*}\phi\rangle \\
&= \langle f, \sqrt{L^*}\phi\rangle.
\end{aligned}$$

Hence for the duality between L^p and L^q we obtain

$$\langle F - RR_*^* BF, B^* D_V \phi\rangle = \langle F, B^* D_V \phi\rangle - \langle F, B^* R_* \sqrt{L^*}\phi\rangle = 0.$$

This shows that $F - RR_*^* BF \in \mathsf{N}_p(D_V^* B)$. This completes the proof of (12.1).

We continue with the proof of (12.2). Assume that $G \in \mathsf{D}_p(D_V^* B)$ satisfies $D_V^* BG = 0$. Then for all $f \in \mathsf{D}_q(D_V)$ we have $\langle B^* D_V f, G\rangle = 0$, where the duality is between $\underline{L}^q$ and $\underline{L}^p$.

Let $I_{p,m}$ and $I_{q,m}$ denote the projections in L^p and L^q onto the m-th Wiener-Itô chaoses. The ranges of $I_{p,m}$ and $I_{q,m}$ are isomorphic by the equivalence of norms on the Wiener-Itô chaoses. Note that $I_{p,m}^* = I_{q,m}$. Then $I_{p,m} \otimes I$ and $I_{q,m} \otimes I$ are bounded projections in $\underline{L}^p$ and $\underline{L}^q$. Let $j_{p,m}$ denote the induced isomorphism of the range of $I_{p,m} \otimes I$ onto the range of $I_{2,m} \otimes I$.

For cylindrical polynomials $f \in \mathscr{F}P(E; \mathsf{D}(V)) \cap H^{(m)}$ (where $H^{(m)}$ is as in Section 3) we have the identity $B^* D_V f = (I_{q,m-1} \otimes I) B^* D_V f$ and

$$\begin{aligned}
[j_{p,m-1}(I_{p,m-1} \otimes I)G, B^* D_V f] &= \langle (I_{p,m-1} \otimes I)G, B^* D_V f\rangle \\
&= \langle G, (I_{q,m-1} \otimes I) B^* D_V f\rangle \\
&= \langle G, B^* D_V f\rangle \\
&= 0.
\end{aligned} \tag{12.3}$$

In the first term, the duality is the inner product of $\underline{L}^2$.

On the other hand, if $f \in \mathscr{F}P(E;\mathsf{D}(V)) \cap H^{(n)}$ for some $n \neq m$, then $j^*_{p,m-1} = j_{q,m-1}$ implies

$$\begin{aligned}
&[j_{p,m-1}(I_{p,m-1} \otimes I)G, B^* D_V f]\\
&\quad = \langle (I_{p,m-1} \otimes I)G, B^* D_V f\rangle\\
&\quad = \langle (I_{p,m-1} \otimes I)G, (I_{q,n-1} \otimes I)B^* D_V f\rangle\\
&\quad = [j_{p,n-1}(I_{p,n-1} \otimes I)(I_{p,m-1} \otimes I)G, B^* D_V f]\\
&\quad = 0,
\end{aligned} \tag{12.4}$$

since $D_V f$ is in the $(n-1)$-th chaos; in the last step we used the L^2-orthogonality of the chaoses.

Since the cylindrical polynomials form a core for $\mathsf{D}(D_V)$ by Lemma 12.1 and B is bounded on $\underline{H}$, we conclude from (12.3) and (12.4) that $j_{p,m-1}(I_{m-1} \otimes I)G$ annihilates $\mathsf{R}(B^* D_V)$ and therefore it belongs to $\mathsf{N}(D_V^* B)$.

Next we claim that if $G \in \overline{\mathsf{R}_p(D_V)}$, then $j_{p,m-1}(I_{p,m-1} \otimes I)G \in \overline{\mathsf{R}(D_V)}$. Indeed, from $G = \lim_{k\to\infty} D_V g_k$ in $\underline{L}^p$ it follows that

$$j_{p,m-1}(I_{p,m-1} \otimes I)G = \lim_{k\to\infty} D_V j_{p,m}(I_{p,m} \otimes I) g_k \in \overline{\mathsf{R}(D_V)}.$$

Combining what we have proved, we see that if $G \in \overline{\mathsf{R}_p(D_V)} \cap \mathsf{N}_p(D_V^* B)$, then $j_{p,m-1}(I_{p,m-1} \otimes I)G \in \overline{\mathsf{R}(D_V)} \cap \mathsf{N}(D_V^* B)$. Hence, $j_{p,m-1}(I_{p,m-1} \otimes I)G = 0$ by the Hodge decomposition of $\underline{L}^2$ [6]. It follows that $(I_{p,m-1} \otimes I)G = 0$ for all $m \geqslant 1$, and therefore $G = 0$. This concludes the proof of (12.2). $\square$

The next application is included for reasons of completeness.

Corollary 12.2. *If the equivalent conditions of Theorem 2.1 hold, then*

$$\overline{\mathsf{R}_p(D_V^* B)} = \overline{\mathsf{R}_p(D_V^*)}.$$

Note that by the second part of Theorem 2.5 it is immaterial whether we view $D_V^* B$ as an unbounded operator from $\underline{L}^p$ to L^p or from $\overline{\mathsf{R}_p(D_V^* B)}$ to L^p.

Proof. By the first part of Theorem 2.5 (first applied to B and then to I) we have the decompositions

$$L^p = \mathsf{N}_p(D_V) \oplus \overline{\mathsf{R}_p(D_V^* B)} = \mathsf{N}_p(D_V) \oplus \overline{\mathsf{R}_p(D_V^*)},$$

where both $D_V^* B$ and D_V^* are viewed as closed densely defined operators from $\overline{\mathsf{R}_p(D_V)}$ to L^p. The corollary will follow if we check that $\overline{\mathsf{R}_p(D_V^* B)} \subseteq \overline{\mathsf{R}_p(D_V^*)}$. This inclusion is trivial if we may interpret $D_V^* B$ and D_V^* as unbounded operators from $\underline{L}^p$ to L^p. By the preceding remark, we may indeed do so for $D_V^* B$. The proof will be finished by checking that the conditions of Theorem 2.1 also hold with B replaced by I, since then we may do the same for D_V^*. But this follows from the fact that VV^*, being self-adjoint on $\overline{\mathsf{R}(V)}$, admits a bounded H^∞-calculus on $\overline{\mathsf{R}(V)}$. $\square$

We proceed with the proof of the second part of Theorem 2.6. The first part has been proved in Section 10.

Proof of Theorem 2.6, second part. We use the notation

$$X_1 := L^p \oplus \overline{\mathsf{R}_p(D_V)} \quad \text{and} \quad X_2 := \mathsf{N}_p(D_V^* B).$$

Fix $t \in \mathbb{R} \setminus \{0\}$. First we show that $it - \Pi$ is injective on $L^p \oplus \underline{L}^p$. Theorem 2.5 implies the decomposition

$$L^p \oplus \underline{L}^p = X_1 \oplus X_2. \tag{12.5}$$

Take $x = x^{(1)} + x^{(2)} \in X_1 \oplus X_2$, and suppose that $(it - \Pi)x = 0$. Then $(it - \Pi)x^{(1)} = 0$ and $itx^{(2)} = 0$. Thus $x^{(1)} = x^{(2)} = 0$, since $\Pi|_{X_1}$ in X_1 is bisectorial.

Next we show that $it - \Pi$ is surjective on $L^p \oplus \underline{L}^p$. Let $y^{(1)} \in X_1$ and $y^{(2)} \in X_2$. The equation $(it - \Pi)(x^{(1)} + x^{(2)}) = y^{(1)} + y^{(2)}$ is solved by

$$x^{(1)} = (it - \Pi|_{X_1})^{-1} y^{(1)} \quad \text{and} \quad x^{(2)} = (it)^{-1} y^{(2)}.$$

This implies that $it - \Pi$ is surjective.

Using (12.5) and the sectoriality of Π on X_1 it follows that

$$\begin{aligned} \|x^{(1)} + x^{(2)}\| &\leqslant \|(it - \Pi|_{X_1})^{-1}\| \|y^{(1)}\| + |t|^{-1} \|y^{(2)}\| \\ &\lesssim t^{-1} \big(\|y^{(1)}\| + \|y^{(2)}\| \big) \\ &\lesssim t^{-1} \|y^{(1)} + y^{(2)}\|, \end{aligned}$$

which is the desired resolvent estimate that shows that Π is bisectorial on $X_1 \oplus X_2$.

To show R-bisectoriality of Π on $L^p \oplus \underline{L}^p$ we take $y_j = y_j^{(1)} + y_j^{(2)} \in X_1 \oplus X_2$. Let $(r_j)_{j \geqslant 1}$ be a Rademacher sequence. Using the R-bisectoriality of $\Pi|_{X_1}$ we obtain

$$\begin{aligned} \mathbb{E} \Big\| \sum_{j=1}^k r_j t_j (it_j - \Pi)^{-1} y_j \Big\|_p &\leqslant \mathbb{E} \Big\| \sum_{j=1}^k r_j t_j (it_j - \Pi|_{X_1})^{-1} y_j^{(1)} \Big\|_p \\ &\qquad + \mathbb{E} \Big\| \sum_{j=1}^k r_j t_j (it_j - \Pi|_{X_2})^{-1} y_j^{(2)} \Big\|_p \\ &\lesssim \mathbb{E} \Big\| \sum_{j=1}^N r_j y_j^{(1)} \Big\|_p + \mathbb{E} \Big\| \sum_{j=1}^k r_j t_j (t_j^{-1} y_j^{(2)}) \Big\|_p \\ &\lesssim \mathbb{E} \Big\| \sum_{j=1}^k r_j y_j \Big\|_p. \end{aligned}$$

By an application of the Kahane-Khintchine inequalities we conclude that $\{t(it - \Pi)^{-1} \ : t \in \mathbb{R} \setminus \{0\}\}$ is R-bounded on $L^p \oplus \underline{L}^p$. This completes the proof. $\square$

We finish by showing how the first part of Theorem 2.6 can be used to prove the implication (2)⇒(1) of Theorem 2.1. We need the following lemma, which is an extension of the corresponding Hilbert space result, cf. [4, Section (H)].

Proposition 12.3. *Let $1 < p < \infty$ and suppose $\mathscr{A}$ is an R-bisectorial operator on a closed subspace U of $L^p(\mu; \mathscr{H})$, where $\mathscr{H}$ is a Hilbert space. Then $\mathscr{A}^2$ is R-sectorial and for each $\omega \in (0, \frac{1}{2}\pi)$ the following assertions are equivalent:*

(1) *$\mathscr{A}$ admits a bounded $H^\infty(\Sigma_\omega)$-functional calculus;*
(2) *$\mathscr{A}^2$ admits a bounded $H^\infty(\Sigma_{2\omega}^+)$-functional calculus.*

Proof. We prove the implication (2) $\Rightarrow$ (1), the other assertions being well known. Let $\widetilde{\psi} \in H_0^\infty(\Sigma_{2\omega}^+)$ and define $\psi \in H_0^\infty(\Sigma_\omega)$ by $\psi(z) := \widetilde{\psi}(z^2)$.

Since $\mathscr{A}^2$ has a bounded $H^\infty(\Sigma_{2\omega}^+)$-functional calculus, by Proposition 7.2 $\mathscr{A}^2$ satisfies a square function estimate

$$c\|f - P_{\mathsf{N}(\mathscr{A}^2)}f\|_p \leqslant \Big\| \Big(\int_0^\infty \|\widetilde{\psi}(t\mathscr{A}^2)f\|^2 \, \frac{dt}{t} \Big)^{1/2} \Big\|_p \leqslant C\|f\|_p,$$

where $P_{\mathsf{N}(\mathscr{A}^2)}$ denotes the projection on $\mathsf{N}(\mathscr{A}) = \mathsf{N}(\mathscr{A}^2)$ with range $\overline{\mathsf{R}(\mathscr{A})} = \overline{\mathsf{R}(\mathscr{A}^2)}$.

But, $\widetilde{\psi}(t\mathscr{A}^2) = \psi(\sqrt{t}\mathscr{A})$, and therefore by the substitution $\sqrt{t} = s$ the above estimate is equivalent to

$$\frac{c}{\sqrt{2}}\|f - P_{\mathsf{N}(\mathscr{A})}f\|_p \leqslant \Big\| \Big(\int_0^\infty \|\psi(s\mathscr{A})f\|^2 \, \frac{ds}{s} \Big)^{1/2} \Big\|_p \leqslant \frac{C}{\sqrt{2}}\|f\|_p.$$

By the bisectorial version of Proposition 7.2, this estimate implies that $\mathscr{A}$ has a bounded $H^\infty(\Sigma_\omega)$-functional calculus. $\square$

Alternative proof of Theorem 2.1 (2) $\Rightarrow$ (1). We adapt an argument of [6], where more details can be found.

Consider the function sgn $\in H^\infty(\Sigma_\omega)$ given by $\mathrm{sgn}(z) = \mathbf{1}_{\Sigma_\omega^+}(z) - \mathbf{1}_{\Sigma_\omega^-}(z) = z/\sqrt{z^2}$. Since Π has a bounded H^∞-calculus on $L^p \oplus \overline{\mathsf{R}_p(D_V)}$ by Proposition 12.3, the operator $\mathrm{sgn}(\Pi)$ is bounded. By the results already proved, this implies that $\mathsf{D}_p(\Pi) = \mathsf{D}_p(\sqrt{\Pi^2})$ with

$$\|\Pi x\|_p \eqsim \big\|\sqrt{\Pi^2}x\big\|_p, \quad x \in \mathsf{D}_p(\Pi) = \mathsf{D}_p(\sqrt{\Pi^2}).$$

Clearly, on $L^p \oplus \overline{\mathsf{R}_p(D_V)}$ we have

$$\sqrt{\Pi^2} = \begin{bmatrix} \sqrt{L} & 0 \\ 0 & \sqrt{\underline{L}} \end{bmatrix},$$

and by restricting to elements of the form $x = (f, 0)$ with $f \in \mathsf{D}_p(D_V)$ we obtain the desired result. $\square$

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