ON THE DOMAIN OF NON-SYMMETRIC ORNSTEIN-UHLENBECK OPERATORS IN BANACH SPACES

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Abstract. We consider the linear stochastic Cauchy problem

$$dX(t) = AX(t)\,dt + B\,dW_H(t), \quad t \geqslant 0,$$

where A generates a C_0-semigroup on a Banach space E, W_H is a cylindrical Brownian motion over a Hilbert space H, and $B : H \to E$ is a bounded operator. Assuming the existence of a unique minimal invariant measure μ_∞, let L_p denote the realization of the Ornstein-Uhlenbeck operator associated with this problem in $L^p(E, \mu_\infty)$. Under suitable assumptions concerning the invariance of $\mathscr{R}(B)$ under the semigroup generated by A, we prove the following domain inclusions, valid for $1 < p \leqslant 2$:

$$\mathscr{D}((-L_p)^{1/2}) \hookrightarrow W_H^{1,p}(E, \mu_\infty),$$
$$\mathscr{D}(L_p) \hookrightarrow W_H^{2,p}(E, \mu_\infty).$$

Here $W_H^{k,p}(E, \mu_\infty)$ denotes the k-th order Sobolev space of functions with Fréchet derivatives up to order k in the direction of H. No symmetry assumptions are made on L_p.

1. Introduction and statement of the results

Let L be the classical Ornstein-Uhlenbeck operator, defined for functions $f \in C_c^2(\mathbb{R}^d)$ by

$$Lf(x) = \tfrac{1}{2}\Delta f(x) - \tfrac{1}{2}\langle x, \nabla f(x)\rangle.$$

The classical Meyer inequalities [18] state that for $1 < p < \infty$ one has the equivalence of norms

$$\|(I - L)^{1/2} f\|_{L^p(\mathbb{R}^d,\gamma)} \simeq \|f\|_{W^{1,p}(\mathbb{R}^d,\gamma)},$$

where γ is the standard normal distribution on $\mathbb{R}^d$ and $W^{1,p}(\mathbb{R}^d, \gamma)$ denotes the Gaussian Sobolev space of all functions $f \in L^p(\mathbb{R}^d, \gamma)$ having a weak partial derivatives belonging to $L^p(\mathbb{R}^d, \gamma)$. For various proofs of Meyer's inequalities and related results, see e.g. [2, 10, 11, 12, 22, 23, 28]. Extensions to a more general class of Ornstein-Uhlenbeck operators were obtained by Shigekawa [24], Song [26], and Chojnowska-Michalik and Goldys [3, 4], who considered the operator

$$Lf(x) = \tfrac{1}{2}\operatorname{trace} QD^2 f(x) + \langle Ax, Df(x)\rangle.$$

Here Q is a self-adjoint positive definite operator on a separable real Hilbert space E, A generates a C_0-semigroup $S = (S(t))_{t\geqslant 0}$ of bounded operators on E, and D

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denotes the Fréchet derivative. Ornstein-Uhlenbeck operators naturally occur as the generators of the transition semigroups of stochastic linear Cauchy problems on E of the form

$$dX(t) = AX(t)\,dt + B\,dW_H(t), \quad t \geqslant 0, \tag{1.1}$$

provided one sets $Q := BB^*$. Here, H is a real Hilbert space, $W_H = (W_H(t))_{t\geqslant 0}$ is an H-cylindrical Brownian motion, and $B : H \to E$ is a bounded operator.

Let us assume that (1.1) admits an invariant measure. Then it admits a unique minimal invariant measure μ_∞, and it is shown in [3, 4] that if the closure of L in $L^2(E, \mu_\infty)$ is self-adjoint, then for all $1 < p < \infty$ one has

$$\mathscr{D}((-L)^{1/2}) = W_H^{1,p}(E, \mu_\infty), \quad \mathscr{D}(L) = W_H^{2,p}(E, \mu_\infty) \cap W_{A_\infty}^{1,p}(E, \mu_\infty).$$

Here L denotes the realization of the Ornstein-Uhlenbeck operator in $L^p(E, \mu_\infty)$. For the definition of the space $W_{A_\infty}^{1,p}(E, \mu_\infty)$ we refer to Section 4.

In view of the applications to stochastic Cauchy problems it is natural to ask for generalizations in infinite dimensions beyond the self-adjoint case. The purpose of this paper is to prove some first results in this direction. Specifically, assuming that the semigroup S restricts to a C_0-contraction semigroup on H (this assumption, which may actually be relaxed somewhat, is automatically satisfied in the self-adjoint case, cf. Remark 3.4) we shall prove, for $1 < p \leqslant 2$, that there exists a constant $C_p \geqslant 0$ such that

$$\|D_H f\|_{L^p(E,\mu_\infty;H)} \leqslant C_p \|(I - L)^{1/2} f\|_{L^p(E,\mu_\infty)} \tag{1.2}$$

for a suitable class of functions f on E. Here E is allowed to be an arbitrary real Banach space, H is the reproducing kernel Hilbert space associated with Q, and D_H denotes the Fréchet derivative in the direction of H. As a result we obtain the domain inclusions

$$\mathscr{D}((-L)^{1/2}) \hookrightarrow W_H^{1,p}(E, \mu_\infty), \quad \mathscr{D}(L) \hookrightarrow W_H^{2,p}(E, \mu_\infty), \tag{1.3}$$

where, with some abuse of notation, the closure of L in $L^p(E, \mu_\infty)$ is denoted again by L. The space $W_H^{k,p}(E, \mu_\infty)$ is defined as the Banach space of all functions $f \in L^p(E, \mu_\infty)$ whose Fréchet derivatives D_H^j belong to $L^p(E, \mu_\infty; H^{\otimes j})$ for $j = 1, \dots, k$. In this context it should be noted that in the finite dimensional case $E = \mathbb{R}^d$, the full identification of the domain

$$\mathscr{D}(L) = W^{2,p}(\mathbb{R}^d, \mu_\infty)$$

was obtained recently without symmetry assumptions on L and for all $1 < p < \infty$ by Metafune, Prüss, Rhandi and Schnaubelt [16]; these authors only need a non-degeneracy assumption ensuring that $H = \mathbb{R}^d$.

Our approach builds on the methods of [3], which in turn are based on the square function approach of [24]. The main novelty of the present work is the use of H^∞-calculus, which enables us to get around the self-adjointness assumptions of [24] and [3] as far as the inclusions in (1.3) are concerned. Indeed, some of the crucial estimates in [24] and [3] can be interpreted as square function estimates, and it has been known for long that such estimates can efficiently be deduced from H^∞-calculus. In recent work on parabolic evolution equations, H^∞-calculus has become an important tool in proving maximal regularity results. In the present context, the embedding $\mathscr{D}(L) \hookrightarrow W_H^{2,p}(E, \mu_\infty)$ can be considered as a maximal regularity result.

On a technical level, instead of working with a core of cylindrical polynomials as in [3], we follow the approach of [9] and some of the references cited there, and work instead with a core of bounded cylindrical functions. This simplifies some of the technical lemmas involving approximation arguments.

The organization of the paper is as follows. In Section 2 we start with some generalities on the Cauchy semigroup associated with a bounded C_0-semigroup and prove a square function estimate and a maximal estimate. Our object of study, the non-symmetric Ornstein-Uhlenbeck operator, is introduced in Section 3. This section also contains some technical lemmas needed later on. The main results are presented in Section 4. Here we follow the ideas of [3, 24], with some simplifications due to our use of H^∞-calculus techniques. In the final Section 5 we briefly comment on the symmetric case.

2. The Cauchy semigroup

Let $T = (T(t))_{t\geqslant 0}$ be a bounded C_0-semigroup with generator G on a Banach space X. We begin our discussion with the following well-known result [1, Proposition 3.8.2].

Proposition 2.1. *There exists a unique closed densely defined operator $(-G)^{1/2}$ on X such that $((-G)^{1/2})^2 = -G$ and*

$$(-G)^{1/2}x = \lim_{\delta\downarrow 0}(\delta - G)^{1/2}x, \quad x \in \mathscr{D}(G).$$

Moreover, for all $\delta > 0$ we have

$$\mathscr{D}((-G)^{1/2}) = \mathscr{D}((\delta - G)^{1/2}),$$

and $\mathscr{D}(G)$ is a core for $\mathscr{D}((-G)^{1/2})$.

As it turns out, the operator $-(-G)^{1/2}$ is the generator of an analytic semigroup on X. This semigroup is introduced in the next definition.

Definition 2.2. *The* Cauchy semigroup *associated with T is the C_0-semigroup $T_{1/2} = (T_{1/2}(t))_{t\geqslant 0}$ on X defined by $T_{1/2}(0) = I$ and*

$$T_{1/2}(t)x = \int_0^\infty g_t(s)T(s)x\,ds, \quad x \in X,\ t > 0,$$

where $g_t : (0,\infty) \to \mathbb{R}$ is given by

$$g_t(s) := \frac{t}{2\sqrt{\pi s^3}}\exp\Big(\frac{-t^2}{4s}\Big). \tag{2.1}$$

Note that $T_{1/2}$ is a bounded semigroup and that $T_{1/2}$ is uniformly exponentially stable if T is uniformly exponentially stable.

The following result holds [1, Theorem 3.8.3]:

Proposition 2.3. *The Cauchy semigroup $T_{1/2}$ extends to a bounded analytic C_0-semigroup on X of angle $\frac{1}{4}\pi$, and its generator $G_{1/2}$ equals $-(-G)^{1/2}$.*

Let H be a real Hilbert space, let (Ω, μ) be a σ-finite measure space, and let $1 \leqslant p < \infty$. Throughout the rest of this section we assume that $P = (P(t))_{t\geqslant 0}$ and $U = (U(t))_{t\geqslant 0}$ are bounded C_0-semigroups on $L^p(\Omega,\mu)$ and H respectively. As is well known, the algebraic tensor product operators $T(t) := P(t) \otimes U(t)$ extend in a unique way to form a bounded semigroup $T = (T(t))_{t\geqslant 0}$ on $L^p(\Omega,\mu;H)$, and it is easy to check that this semigroup is a C_0-semigroup. Denoting the generators of P and U by G_P and G_U respectively, the generator G of T is given on the core $\mathscr{D}(G_P) \otimes \mathscr{D}(G_U)$ by

$$G = G_P \otimes I + I \otimes G_U.$$

The following maximal estimate is a simple vector-valued extension of a result in [17, p. 4]. For the convenience of the reader we include the proof.

Theorem 2.4 (Maximal estimate for $T_{1/2}$)**.** *Let $1 < p < \infty$ and let $P = (P(t))_{t\geqslant 0}$ be a C_0-semigroup of positive operators on $L^p(\Omega, \mu)$ satisfying*

$$\|P(t)f\|_1 \leqslant \|f\|_1, \quad \|P(t)f\|_\infty \leqslant \|f\|_\infty$$

for all $f \in L^1(\Omega, \mu) \cap L^\infty(\Omega, \mu)$ and $t \geqslant 0$. Let $U = (U(t))_{t\geqslant 0}$ be a bounded C_0-semigroup on H. For all $f \in L^p(\Omega, \mu; H)$ the maximal function

$$f^\star(\omega) := \sup_{t>0} \|T_{1/2}(t)f(\omega)\|_H, \quad \omega \in \Omega,$$

belongs to $L^p(\Omega, \mu)$ and we have

$$\|f^\star\|_p \leqslant \frac{pCm_U}{p-1}\|f\|_p,$$

where C is a universal constant and $m_U = \sup_{t\geqslant 0} \|U(t)\|$.

Proof. Note that $t^2 g_t(s) = \phi_0(s/t^2)$, where g_t is given by (2.1) and

$$\phi_0(s) := \frac{1}{2\sqrt{\pi s^3}} \exp\Big(\frac{-1}{4s}\Big).$$

Take $f \in L^p(\Omega, \mu; H)$ and put

$$M(t)f := \frac{1}{t}\int_0^t T(s)f\,ds.$$

Observe that

$$\|(P \otimes U)f\|_H \leqslant \|U\|P(\|f\|_H) \ \ \mu\text{-almost surely.} \tag{2.2}$$

To see this, let $g = \sum_{n=1}^N \mathbf{1}_{\Omega_n} \otimes h_n$ be a simple function; here the measurable sets Ω_n are disjoint and the vectors h_n are taken from H. Taking norms in H pointwise and using the positivity of P we have, μ-almost surely,

$$\begin{aligned}\|(P \otimes U)g\|_H &= \Big\|\sum_{n=1}^N P\mathbf{1}_{\Omega_n} \otimes Uh_n\Big\|_H \\ &\leqslant \|U\|\Big\|\sum_{n=1}^N P\mathbf{1}_{\Omega_n} \otimes h_n\Big\|_H \leqslant \|U\|\sum_{n=1}^N (P\mathbf{1}_{\Omega_n})\|h_n\|_H \\ &= \|U\|\sum_{n=1}^N P(\mathbf{1}_{\Omega_n}\|h_n\|_H) = \|U\|P(\|g\|_H).\end{aligned}$$

From (2.2) we have, for μ-almost all $\omega \in \Omega$,

$$Mf(\omega) := \sup_{t>0} \|M(t)f(\omega)\|_H \leqslant m_U \sup_{t>0} \frac{1}{t}\int_0^t P(s)\|f\|_H(\omega)\,ds. \tag{2.3}$$

Using that $T(t)f = \frac{d}{dt}(tM(t)f)$ we write

$$\begin{aligned}
T_{1/2}(t)f &= \int_0^\infty \phi_0(\frac{s}{t^2})T(s)f\frac{ds}{t^2} \\
&= \lim_{n\to\infty}\big[\frac{s}{t^2}\phi_0(\frac{s}{t^2})M(s)f\big]_{1/n}^n - \int_0^\infty \frac{s}{t^2}\phi_0'(\frac{s}{t^2})M(s)f\frac{ds}{t^2} \\
&= -\int_0^\infty r\phi_0'(r)M(t^2r)f\,dr,
\end{aligned}$$

where we used that $s\phi_0(s)\to 0$ as $s\to 0$ or ∞. Noting that the function $r\mapsto r\phi_0'(r)$ is integrable on $\mathbb{R}_+$ we let $C := \|r\mapsto r\phi_0'(r)\|_1$. For μ-almost all $\omega\in\Omega$ we obtain the estimate

$$\|T_{1/2}(t)f(\omega)\|_H \leqslant \int_0^\infty r|\phi_0'(r)|\,\|M(t^2r)f(\omega)\|_H\,dr \leqslant CMf(\omega).$$

This gives $f^\star \leqslant CMf$ μ-almost everywhere and

$$\|f^\star\|_p \leqslant C\|Mf\|_p \leqslant \frac{pCm_U}{p-1}\|f\|_p,$$

where the last inequality follows from (2.3) and the Hopf-Dunford-Schwartz Ergodic Theorem [13, Theorem 6.12]. □

Theorem 2.5 (Square function estimate for $T_{1/2}$)**.** *Let $1<p<\infty$. Let $P=(P(t))_{t\geqslant 0}$ be a C_0-semigroup of positive contractions on $L^p(\Omega,\mu)$ and let $U=(U(t))_{t\geqslant 0}$ be a bounded C_0-semigroup on H which admits a dilation to a bounded group on a Hilbert space $\bar{H}$. Suppose that G has dense range. Then there exist constants $0<c\leqslant C<\infty$ such that for all $f\in L^p(\Omega,\mu;H)$ we have*

$$c\|f\|_{L^p(\Omega,\mu;H)} \leqslant \Big\|\Big(\int_0^\infty \big\|tG_{1/2}T_{1/2}(t)f\big\|_H^2\,\frac{dt}{t}\Big)^{1/2}\Big\|_{L^p(\Omega,\mu)} \leqslant C\|f\|_{L^p(\Omega,\mu;H)}.$$

Proof. By the Hilbert space-valued extension of [15, Corollary 2.3] it suffices to check that the operator $-G_{1/2}$ admits a bounded $H^\infty(\Sigma_\theta)$-calculus for some $\theta\in(0,\frac{1}{2}\pi)$.

By Fendler's theorem [7], see also [14, Theorem 10.13], the semigroup P admits a dilation to a C_0-group of positive isometries $\bar{P}$ on some space $L^p(\bar{E},\bar{\mu})$ containing $L^p(E,\mu)$ as a complemented subspace. By the assumption on U it follows that the semigroup $T=P\otimes U$ admits a dilation to a bounded C_0-group $\bar{T}=\bar{P}\otimes\bar{U}$ on $L^p(\bar{E},\bar{\mu};\bar{H})$. Therefore, the negative generator $-G$ of T admits a bounded $H^\infty(\Sigma_\eta)$-calculus by [14, Corollary 10.9] for all $\eta\in(\frac{\pi}{2},\pi]$. This implies that the negative generator $-G_{1/2}$ of $T_{1/2}$ admits a bounded $H^\infty(\Sigma_\theta)$-calculus for all $\theta\in(\frac{\pi}{4},\pi]$. □

3. Notations and standing assumptions

In this section we introduce the setting and notations which shall be used in the rest of the paper.

We consider the linear stochastic Cauchy problem

$$\text{(SCP)}\qquad \begin{cases} dX(t) = AX(t)\,dt + B\,dW_H(t), & t\geqslant 0,\\ X(0)=x. \end{cases}$$

Here A is the generator of a C_0-semigroup $S = (S(t))_{t\geqslant 0}$ of bounded linear operators on a real Banach space E, $W_H = (W_H(t))_{t\geqslant 0}$ is a cylindrical Brownian motion on a real Hilbert space H, and $B : H \to E$ is a bounded operator. Throughout this paper we shall assume that the problem (3) has a (necessarily unique) weak solution $X_x = (X_x(t))_{t\geqslant 0}$. For the precise definitions of these notions as well as necessary and sufficient conditions for the existence of a weak solution we refer to [20].

The range of the operator $B : H \to E$ has the structure of a Hilbert space in a natural way by endowing it with the norm of $H \ominus \ker(B)$. More precisely, for $h \in H$ we define

$$\|Bh\|_{\mathscr{R}(B)} := \inf\{\|h'\|_H :\ Bh' = Bh\}.$$

In everything that follows, we may (and shall) replace H with $H \ominus (\ker(B))^{\perp}$ and thereby assume, without any loss of generality, that B is *injective*. Furthermore it will be convenient to identify H with its image under B in E; we shall frequently do so without further notice.

On the space $C_b(E)$ of all bounded continuous functions $f : E \to \mathbb{R}$ we define a semigroup of contractions $P = (P(t))_{t\geqslant 0}$ by the formula

$$P(t)f(x) = \mathbb{E}(f(S(t)x + X_0(t))) = \mathbb{E}(f(X_x(t))), \quad t \geqslant 0,\ x \in E,\ f \in C_b(E).$$

The semigroup P is called the *Ornstein-Uhlenbeck semigroup* associated with A and H. In general it fails to be strongly continuous with respect to the uniform topology of $C_b(E)$, but it is always strongly continuous with respect to the mixed topology of $C_b(E)$ [9]. By definition, this is the finest locally convex topology of $C_b(E)$ with agrees with the compact-open topology on all norm-bounded subsets of $C_b(E)$. The infinitesimal generator of P with respect to this topology will be denoted by L. In order to describe the operator L in more detail we introduce the following terminology.

Let $\mathscr{F}C_b^{m,n}(E)$ denote the linear subspace of $C_b(E)$ consisting of all functions $f : E \to \mathbb{R}$ of the form

$$f(x) = \phi(\langle x, x_1^*\rangle,\ \dots,\ \langle x, x_k^*\rangle) \tag{3.1}$$

where $k \geqslant 1$ is an integer, $x_1^*, \dots, x_k^* \in \mathscr{D}(A^{*n})$, and ϕ belongs to the space $C_b^m(\mathbb{R}^k)$ of bounded functions on $\mathbb{R}^k$ with bounded and continuous derivatives up to order m. The elements of $\mathscr{F}C_b^{m,n}(E)$ are referred to as *cylindrical* $C_b^{m,n}$-functions. We write $\mathscr{F}C_b^m(E) = \mathscr{F}C_b^{m,0}(E)$, with the understanding that $\mathscr{D}(A^{*0}) = E^*$.

For a function $f \in \mathscr{F}C_b^1(E)$ of the form (3.1), the Fréchet derivative Df and the Fréchet derivative $D_H f$ in the direction of H are defined by

$$Df(x) = \sum_{j=1}^{k} D_j\phi(\langle x, x_1^*\rangle, \dots, \langle x, x_k^*\rangle)x_j^* \tag{3.2}$$

and

$$D_H f(x) = \sum_{j=1}^{k} D_j\phi(\langle x, x_1^*\rangle, \dots, \langle x, x_k^*\rangle)B^* x_j^*. \tag{3.3}$$

Lemma 3.1. *For all $f \in \mathscr{F}C_b^{m,n}(E)$ and $t \geqslant 0$ we have $P(t)f \in \mathscr{F}C_b^{m,n}(E)$ and*

$$D_H P(t)f(x) = B^* S^*(t)\,\mathbb{E}(Df(X_x(t))).$$

Proof. Let f be as in (3.1). Let μ_t denote the distribution of the E-valued random variable $X_0(t)$. Then μ_t is a Radon probability measure μ_t on E and we have

$$\begin{aligned} P(t)f(x) &= \mathbb{E}(f(S(t)x + X_0(t))) \\ &= \int_E f(S(t)x + y)\, d\mu_t(y) \\ &= \int_E \phi(\langle S(t)x + y, x_1^*\rangle, \dots, \langle S(t)x + y, x_k^*\rangle)\, d\mu_t(y) \\ &= \psi_t(\langle x, S^*(t)x_1^*\rangle, \dots, \langle x, S^*(t)x_k^*\rangle), \end{aligned}$$

where

$$\psi_t(\xi_1, \dots, \xi_n) = \int_E \phi(\xi_1 + \langle y, x_1^*\rangle, \dots, \xi_k + \langle y, x_k^*\rangle)\, d\mu_t(y).$$

Hence $P(t)f \in \mathscr{F}C_b^{m,n}(E)$ and, by differentiation under the integral we obtain

$$\begin{aligned} D_H P(t)f(x) &= \sum_{j=1}^k \int_E D_j\phi(\langle S(t)x + y, x_1^*\rangle, \dots, \langle S(t)x + y, x_n^*\rangle) B^* S^*(t) x_j^*\, d\mu_t(y) \\ &= \sum_{j=1}^k \mathbb{E} D_j\phi(\langle X_x(t), x_1^*\rangle, \dots, \langle X_x(t), x_n^*\rangle) B^* S^*(t) x_j^* \\ &= B^* S^*(t)\, \mathbb{E}(Df(X_x(t))). \end{aligned}$$

□

In the same way as in [9] one can show that

$$\mathscr{C} := \{f \in \mathscr{F}C_b^{\infty,1}(E) :\ x \mapsto \langle x, A^* Df(x)\rangle \ \text{ belongs to } C_b(E)\}. \tag{3.4}$$

is a core for $\mathscr{D}(L)$ and that on this core, L is given explicitly by

$$\begin{aligned} Lf(x) :=& \tfrac{1}{2}\,\mathrm{trace}\, D_H^2 f(x) + \langle x, A^* Df(x)\rangle \\ =& \tfrac{1}{2}\,\mathrm{trace}\, Q D^2 f(x) + \langle x, A^* Df(x)\rangle, \qquad x \in E,\ f \in \mathscr{C}. \end{aligned} \tag{3.5}$$

In order to be able to discuss the properties of the Ornstein-Uhlenbeck semigroup P in an L^p-setting, for the rest of the paper we shall assume that the problem (SCP) admits an *invariant measure*, i.e., a Radon probability measure μ on E such that for all $f \in C_b(E)$ and $t \geqslant 0$ we have

$$\int_E P(t)f\, d\mu = \int_E f\, d\mu. \tag{3.6}$$

This measure is centred Gaussian but needs not be unique. However, the existence of an invariant measure implies the existence of a unique *minimal* invariant measure μ_∞, whose covariance operator $Q_\infty \in \mathscr{L}(E^*, E)$ is given by

$$\langle Q_\infty x^*, y^*\rangle = \int_0^\infty \langle S(t) Q S^*(t) x^*, y^*\rangle\, dt, \quad x^*, y^* \in E^*,$$

where $Q := B \circ B^*$. The minimality of μ_∞ is expressed by the fact that we have

$$\langle Q_\infty x^*, x^*\rangle \leqslant \langle C x^*, x^*\rangle, \quad x^* \in E^*,$$

whenever C is the covariance operator of an invariant measure μ. For proofs and more information on this topic we refer to [5, 21].

In what follows, H_∞ denotes the reproducing kernel Hilbert space associated with Q_∞ and $i_\infty : H_\infty \hookrightarrow E$ the inclusion operator.

By standard arguments, for $1 \leqslant p < \infty$ the semigroup P extends in a unique way to a C_0-semigroup of contractions, also denoted by P, on $L^p(E, \mu_\infty)$. Its generator will be denoted by L. Since $C_b(E)$ is dense in $L^p(E, \mu_\infty)$, the identity (3.6) extends to $L^p(E, \mu_\infty)$:

$$\int_E P(t)f \, d\mu_\infty = \int_E f \, d\mu_\infty, \quad f \in L^p(E, \mu_\infty).$$

From this it follows that

$$\int_E Lf \, d\mu_\infty = 0, \quad f \in \mathscr{D}(L). \tag{3.7}$$

It is shown in [8] that D_H is closable as an operator from $L^p(E, \mu_\infty)$ into $L^p(E, \mu_\infty; H)$ if and only if the mapping $i_\infty^* x^* \mapsto B^* x^*$ is closable from H_∞ into H. This condition is independent of $p \in [1, \infty)$, and if it is satisfied we will simply say that D_H is *closable*. If D_H is closable, then by abuse of notation, the closure of D_H as an operator from $L^p(E, \mu_\infty)$ into $L^p(E, \mu_\infty; H)$ will be denoted by D_H as well and we define

$$W_H^{1,p}(E, \mu_\infty)$$

to be its domain. The space $W_H^{1,p}(E, \mu_\infty; H)$ is defined similarly, noting that D_H is also closable as an operator from $L^p(E, \mu_\infty; H)$ to $L^p(E, \mu_\infty; H^{\otimes 2})$. Denoting its closure again by D_H we define

$$W_H^{2,p}(E, \mu_\infty) := \{f \in W_H^{1,p}(E, \mu_\infty) : \ D_H f \in W_H^{1,p}(E, \mu_\infty; H)\}.$$

The proofs of the following two lemmas are left to the reader.

Lemma 3.2. *Let $1 \leqslant p < \infty$. Then $\mathscr{F}C_b^\infty(E)$ is dense in $W_H^{1,p}(E, \mu_\infty)$.*

Lemma 3.3. *Let $1 \leqslant p, q, r < \infty$ satisfy $1/p + 1/q = 1/r$. If D_H is closable, then for all $f \in W_H^{1,p}(E, \mu_\infty)$ and $g \in W_H^{1,q}(E, \mu_\infty)$ we have $fg \in W_H^{1,r}(E, \mu_\infty)$ and*

$$D_H(fg) = f D_H g + g D_H f.$$

From this point on we make the following standing assumption. Recall our convention to identify H with its image under B in E.

Assumption (A1). *The space H is invariant under S and the restricted semigroup $S_H := S|_H$ admits a dilation to a bounded C_0-group on H.*

Note that if $\tilde{S}_H$ is a dilation of S_H, then the adjoint group $\tilde{S}_H^*$ is a dilation of S_H^*; this fact will be used in the proof of Theorem 4.5 below.

By the Sz.-Nagy Unitary Dilation Theorem, Assumption (A1) is fulfilled if S_H is contractive. By [9, Theorem 4.5], this includes the case where the semigroup P_2 is symmetric on $L^2(E, \mu_\infty)$.

Remark 3.4. In the setting considered in [3], assumption (A1) is automatically satisfied in the Ornstein-Uhlenbeck case. To explain this more carefully, let us recall that the authors start from an arbitrary self-adjoint contraction semigroup $\mathscr{S}$ on a Hilbert space $\mathscr{H}$, on which a non-degenerate centred Gaussian measure μ is given. Denoting by C the covariance operator of μ, they define a self-adjoint C_0-contraction semigroup $\mathscr{S}_C$ on the reproducing kernel Hilbert space $\mathscr{H}_C$ associated with C by the formula

$$\mathscr{S}_C(t) := C^{1/2} \mathscr{S}(t) C^{-1/2}, \quad t \geqslant 0.$$

This semigroup is well defined because the non-degeneracy of μ implies that $C^{1/2}$ is a unitary operator from $\mathscr{H}$ onto $\mathscr{H}_C$. The object of study in [3] is the second quantization (in the sense of [25]) of $\mathscr{S}_C(t)$ on $L^2(\mathscr{H},\mu)$.

In our setting, the spaces $\mathscr{H}$ and $\mathscr{H}_C$ correspond to E and H_∞, the semigroup $\mathscr{S}$ corresponds to S, the measure μ corresponds to μ_∞, and the operator C to Q_∞. It is important to observe, however, that the semigroup $\mathscr{S}_C$ corresponds to our semigroup S_∞ if and only if for all $t \geqslant 0$ one has $\mathscr{S}(t)C^{1/2} = C^{1/2}\mathscr{S}(t)$, or equivalently, if and only if $\mathscr{S}(t)C = C\mathscr{S}(t)$. This is the 'Ornstein-Uhlenbeck case' referred to above, which is also considered in [4]. In that paper, the results of [3] are applied to the semigroup $\mathscr{S}_0(t) := C^{-1/2}\mathscr{S}(t)C^{1/2}$ on $\mathscr{H}$.

Let us now assume that we are in the Ornstein-Uhlenbeck case. To verify that Assumption (A1) is satisfied, note that by what has been said, in the setting of [3] the semigroup $S_\infty = \mathscr{S}_C$ is self-adjoint on H_∞. Therefore by [9, Theorems 4.5 and 7.4], (A1) holds.

It should be noted that the second quantization as defined in [25], which is also used in the papers [9, 24], is different from the second quantization as it is used in [3, 4]. Indeed, the second quantizations of the operators $\mathscr{S}(t)$ in the sense of [3, 4] are equal to the second quantizations of the operators $C^{1/2}\mathscr{S}(t)C^{-1/2}(t)$ in the sense of [25]. This difference in terminology accounts for the frequent occurrence of square roots in [3, 4].

It is proved in [8] that (A1) implies that D_H is closable. Furthermore, (A1) enables us to define a bounded C_0-semigroup on $L^p(E,\mu_\infty;H)$ by

$$T(t) = P(t) \otimes S_H^*(t), \quad t \geqslant 0.$$

We denote the generator of T by G.

Proposition 3.5. *Let $1 \leqslant p < \infty$. For all $f \in W_H^{1,p}(E,\mu_\infty)$ and $t \geqslant 0$ we have $P(t)f \in W_H^{1,p}(E,\mu_\infty)$ and*

$$D_H P(t) f = T(t) D_H f. \tag{3.8}$$

Proof. For functions $f \in \mathscr{F}C_b^1(E)$, (3.8) is an immediate consequence of (3.2), (3.3), Lemma 3.1, and the identity $B^*S^*(t)x^* = S_H^*(t)B^*x^*$. The general case follows from the closability of D_H via an approximation argument. □

Our next aim is to prove a product rule and chain rule for L. Since we will deal with different exponents, we shall write L_p, rather than L, to denote the generator of the semigroup P in $L^p(E,\mu_\infty)$. We need the following basic result.

Lemma 3.6. *For all $1 \leqslant p \leqslant q < \infty$, $\mathscr{C}$ is dense in $\mathscr{D}(L_p) \cap W_H^{1,q}(E,\mu_\infty)$.*

Proof. The proof proceeds in several steps.

Step 1 – $\mathscr{C}$ is dense in $W_H^{1,q}(E,\mu_\infty)$. For this it suffices to prove that $\mathscr{C}_0$ is dense in $W_H^{1,q}(E,\mu_\infty)$, where $\mathscr{C}_0 \subseteq \mathscr{C}$ is defined as

$$\mathscr{C}_0 := \{f \in \mathscr{F}C_b^{\infty,1}(E) : \ \nabla\phi \in C_c(\mathbb{R}^k;\mathbb{R}^k)\}.$$

Here we use the notation of (3.1) and the discussion following it.

To prove that $\mathscr{C}_0$ is dense in $W_H^{1,q}(E,\mu_\infty)$, let $f \in \mathscr{F}C_b^\infty(E)$ be a given function of the form (3.1), i.e.,

$$f(x) = \phi(\langle x, x_1^*\rangle, \dots, \langle x, x_k^*\rangle)$$

with $\phi \in C_b^\infty(\mathbb{R}^k)$ and $x_1^*, \dots, x_k^* \in E^*$, and put

$$R_n := nR(n, A).$$

Choose smooth functions $\psi_n : \mathbb{R}^k \to [0,1]$ satisfying $\psi_n(\xi) = 1$ for $|\xi| \leqslant n$, $\psi_n(\xi) = 0$ for $|\xi| \geqslant n+1$, and $|\nabla \psi_n(\xi)| \leqslant 2$ for all $\xi \in \mathbb{R}^k$. The functions

$$f_n(x) := (\psi_n \phi)(\langle x, R_n^* x_1^* \rangle, \dots, \langle x, R_n^* x_k^* \rangle)$$

belong to $\mathscr{C}_0$ and satisfy $f_n \to f$ in $L^q(E, \mu_\infty)$ and $D_H f_n \to D_H f$ in $L^q(E, \mu_\infty; H)$; the second assertion follows by observing that

$$\lim_{n\to\infty} B^* R_n^* x^* = \lim_{n\to\infty} R(n, A_H^*) B^* x^* = B^* x^*$$

strongly in H. Since $\mathscr{F}C_b^\infty(E)$ is dense in $W_H^{1,q}(E, \mu_\infty)$ by Lemma 3.2, this proves the claim.

Step 2 – Fix $f \in \mathscr{D}(L_p) \cap W_H^{1,q}(E, \mu_\infty)$. By Step 1 we can find a sequence $(f_n)_{n\geqslant 1}$ in $\mathscr{C}$ such that $f_n \to f$ in $W_H^{1,q}(E, \mu_\infty)$.

Fix $\lambda_k > 0$ so large that

$$\begin{aligned}
\|\lambda_k R(\lambda_k, L_q) f - f\|_q &< 1/k, \\
\|\lambda_k R(\lambda_k, G_q) D_H f - D_H f\|_q &< 1/k, \\
\|\lambda_k R(\lambda_k, L_p) L_p f - L_p f\|_p &< 1/k.
\end{aligned}$$

For each k choose n_k so large that

$$\begin{aligned}
\|\lambda_k R(\lambda_k, L_q)(f_{n_k} - f)\|_q &< 1/k, \\
\|\lambda_k R(\lambda_k, G_q)(D_H f_{n_k} - D_H f)\|_q &< 1/k, \\
\|\lambda_k R(\lambda_k, L_p)(L_p f_{n_k} - L_p f)\|_p &< 1/k.
\end{aligned}$$

The second inequality can be achieved since $D_H f_n \to D_H f$ in $L^q(E, \mu_\infty; H)$ and the third since $\lambda_k R(\lambda_k, L_p) L_p$ is a bounded operator. With these choices we have

$$\begin{aligned}
\|\lambda_k R(\lambda_k, L_q) f_{n_k} - f\|_q &< 2/k, \\
\|\lambda_k R(\lambda_k, G_q) D_H f_{n_k} - D_H f\|_q &< 2/k, \\
\|\lambda_k R(\lambda_k, L_p) L_p f_{n_k} - L_p f\|_p &< 2/k.
\end{aligned}$$

In view of the identity $R(\lambda_k, G_q) D_H f_{n_k} = D_H R(\lambda_k, L_q) f_{n_k}$, cf. Proposition 3.5, this can be restated as saying that

$$g_k := \lambda_k R(\lambda_k, L) f_{n_k} \to f \quad \text{in} \quad \mathscr{D}(L_p) \cap W_H^{1,q}(E, \mu_\infty).$$

Step 3 – Writing $g_k = \int_0^\infty \lambda_k e^{-\lambda_k t} P(t) f_{n_k} \, dt$, each g_k can be approximated in $L^q(E, \mu_\infty)$ by Riemann sums of the form

$$g_k^{(l)} := \sum_{j=1}^{N^{(l)}} (t_j^{(l)} - t_{j-1}^{(l)}) \cdot \lambda_k e^{-\lambda_k t_j^{(l)}} P(t_j^{(l)}) f_{n_k}.$$

Letting $l \to \infty$ we obtain

$$\begin{aligned}
\lim_{l\to\infty} L_p g_k^{(l)} &= \lim_{l\to\infty} \sum_{j=1}^{N^{(l)}} (t_j^{(l)} - t_{j-1}^{(l)}) \cdot \lambda_k e^{-\lambda_k t_j^{(l)}} P(t_j^{(l)}) L_p f_{n_k} \\
&= \int_0^\infty \lambda_k e^{-\lambda_k t} P(t) L_p f_{n_k} \, dt = L_p g_k
\end{aligned}$$

and, using the closedness of D_H,

$$\begin{aligned}\lim_{l\to\infty} D_H g_k^{(l)} &= \lim_{l\to\infty} \sum_{j=1}^{N^{(l)}} (t_j^{(l)} - t_{j-1}^{(l)}) \cdot \lambda_k e^{-\lambda_k t_j^{(l)}} T(t_j^{(l)}) D_H f_{n_k} \\ &= \int_0^\infty \lambda_k e^{-\lambda_k t} T(t) D_H f_{n_k}\,dt \\ &= \int_0^\infty \lambda_k e^{-\lambda_k t} D_H P(t) f_{n_k}\,dt = D_H g_k.\end{aligned}$$

Hence,

$$\lim_{l\to\infty} g_k^{(l)} = g_k \quad\text{in}\quad \mathscr{D}(L_p) \cap W_H^{1,q}(E,\mu_\infty).$$

Step 4 – Combining Steps 2 and 3 we find

$$\lim_{k\to\infty}\lim_{l\to\infty} g_k^{(l)} = f \quad\text{in}\quad \mathscr{D}(L_p) \cap W_H^{1,q}(E,\mu_\infty).$$

Since each $g_k^{(l)}$ belongs to $\mathscr{C}$, the lemma is proved. □

Remark 3.7. We know that $\mathscr{C}$ is dense in $\mathscr{D}(L_p)$, and Step 1 in the above proof shows that $\mathscr{C}$ is also dense in $W_H^{1,q}(E,\mu_\infty)$. However, this by itself does not permit us to conclude that $\mathscr{C}$ is dense in $\mathscr{D}(L_p) \cap W_H^{1,q}(E,\mu_\infty)$.

In fact, if X_1 and X_2 are Banach spaces which are continuously embedded in a Banach space $\mathscr{X}$ and Y is a linear subspace of $\mathscr{X}$ which is dense in both X_1 and X_2, it may happen that Y fails to be dense in $X_1 \cap X_2$. An example is obtained by taking $\mathscr{X} = L^2(-1,1)$, $X_1 = L^2(-1,1) \cap C[0,1]$, $X_2 = L^2(-1,1) \cap C[-1,0]$, and $Y = \{f \in C[-1,1] : f(-1) = f(1)\}$. Clearly, Y is dense in X_1 and in X_2, but not in $X_1 \cap X_2 = C[-1,1]$.

Lemma 3.8 (Product rule)**.** *Let $1 \leqslant p,q,r < \infty$ satisfy ${}^1\!/_p + {}^1\!/_q = {}^1\!/_r$. For all $f \in \mathscr{D}(L_p) \cap W_H^{1,p}(E,\mu_\infty)$ and $g \in \mathscr{D}(L_q) \cap W_H^{1,q}(E,\mu_\infty)$ we have $fg \in \mathscr{D}(L_r)$ and*

$$L_r(fg) = g \cdot L_p f + f \cdot L_q g + [D_H f, D_H g]_H.$$

Proof. If $f, g \in \mathscr{C}$, then $f \in \mathscr{D}(L_p) \cap W_H^{1,p}(E,\mu_\infty)$ and $g \in \mathscr{D}(L_q) \cap W_H^{1,q}(E,\mu_\infty)$, and the identity follows by a direct computation based on (3.5). The general case follows by approximation via Lemma 3.6 and the closedness of the operators involved. □

Lemma 3.9 (Chain rule)**.** *Let $1 \leqslant p < \infty$. For $f \in \mathscr{D}(L_p) \cap W_H^{1,2p}(E,\mu_\infty)$ and $\varphi \in C_b^2(\mathbb{R})$ we have $\varphi \circ f \in \mathscr{D}(L_p)$ and*

$$L_p(\varphi \circ f) = (\varphi' \circ f) L_p f + \frac{1}{2}(\varphi'' \circ f)\|D_H f\|_H^2.$$

Proof. For $f \in \mathscr{C}$ this follows by a direct computation based on (3.5). The general case follows again by approximation via Lemma 3.6 and the closedness of the operators involved. □

4. Main results

In this section we shall work with the rescaled semigroup

$$\tilde{P}(t) := e^{-t} P(t)$$

in $L^p(E, \mu_\infty)$. The associated Cauchy semigroup generated by

$$\tilde{L}_{1/2} := (L - I)_{1/2}$$

is denoted by $(\tilde{P}_{1/2}(t))_{t \geqslant 0}$. We start with a technical lemma which will be needed in the proof of Theorem 4.5.

Lemma 4.1. *Let $1 < p < \infty$. For all $f \in \mathscr{C}$ we have $\tilde{L}_{1/2} f \in W_H^{1,p}(E, \mu_\infty)$.*

Proof. First we use (3.5) to show that $Lf \in W_H^{1,p}(E, \mu_\infty)$. Using this fact together with Proposition 3.5, which implies that P acts as a bounded C_0-semigroup in $W_H^{1,p}(E, \mu_\infty)$, we represent $\tilde{L}_{1/2} f$ as a Bochner integral in $W_H^{1,p}(E, \mu_\infty)$. Indeed, in $L^p(E, \mu_\infty)$ we have the standard identity

$$\tilde{L}_{1/2} f = \frac{1}{\sqrt{\pi}} \int_0^\infty t^{-1/2} e^{-t} P(t)(L - I) f \, dt, \quad f \in \mathscr{D}(L),$$

and by the observations just made this integral converges as a Bochner integral in $W_H^{1,p}(E, \mu_\infty)$. □

In the results below we perform 'pointwise' computations, which can be justified by the fact that analytic functions with values in vector-valued L^p-spaces admit pointwise analytic versions. Results of this type go back to Stein [27] and have been investigated in detail in [6]. The properties of the pointwise analytic versions $t \mapsto \tilde{P}_{1/2}(t) f(x)$ needed here are discussed in detail in [3].

Lemma 4.2. *Let $1 < p < \infty$. For $f \in \mathscr{C}$ and $\varepsilon > 0$ define*

$$F_\varepsilon(t, x) := \big(|\tilde{P}_{1/2}(t) f(x)|^2 + \varepsilon^2\big)^{1/2}, \quad t \geqslant 0, \ x \in E.$$

Then $F_\varepsilon^p(t, \cdot) \in \mathscr{D}(L)$ and

$$(D_t^2 + L) F_\varepsilon^p(t, x) \geqslant 0 \ \text{ for } \mu_\infty\text{-almost all } x \in E. \tag{4.1}$$

Moreover, for μ_∞-almost all $x \in E$, the following estimate holds for all $t > 0$:

$$\begin{aligned} &2|\tilde{L}_{1/2} \tilde{P}_{1/2}(t) f(x)|^2 + \|(D_H \tilde{P}_{1/2}(t) f)(x)\|_H^2 \\ &\qquad \leqslant \alpha_p^{-1} |\tilde{P}_{1/2}(t) f(x)|^{2-p} \liminf_{\varepsilon \downarrow 0} (D_t^2 + L) F_\varepsilon^p(t, x), \end{aligned} \tag{4.2}$$

where $\alpha_p := \frac{1}{2} p(p-1)$.

For $p > 2$, the inequality (4.1) should be interpreted by multiplying both sides with $|\tilde{P}_{1/2}(t) f(x)|^{p-2}$.

Proof. For μ_∞-almost all $x \in E$ we have

$$\begin{aligned} D_t^2 F_\varepsilon^2(t, x) &= D_t^2 |\tilde{P}_{1/2}(t) f(x)|^2 \\ &= 2(I - L)\tilde{P}_{1/2}(t) f(x) \cdot \tilde{P}_{1/2}(t) f(x) + 2|\tilde{L}_{1/2} \tilde{P}_{1/2}(t) f(x)|^2. \end{aligned}$$

Also, by Lemma 3.8, $F_\varepsilon^2 = (\tilde{P}_{1/2}(t) f)^2 + \varepsilon^2 \in \mathscr{D}(L)$ and

$$L F_\varepsilon^2(t, x) = 2L\tilde{P}_{1/2}(t) f \cdot \tilde{P}_{1/2}(t) f(x) + \|D_H \tilde{P}_{1/2}(t) f(x)\|_H^2.$$

It follows that

$$(D_t^2 + L)F_\varepsilon^2(t,x) \geqslant 2|\tilde{L}_{1/2}\tilde{P}_{1/2}(t)f(x)|^2 + \|D_H\tilde{P}_{1/2}(t)f(x)\|_H^2 =: g(t,x), \tag{4.3}$$

which gives the result for $p = 2$.

We continue with the case $p \neq 2$. For μ_∞-a.a. $x \in E$ we have, for all $t > 0$,

$$\begin{aligned} D_t^2 F_\varepsilon^p(t,x) &= D_t^2 (F_\varepsilon^2)^{p/2}(t,x) \\ &= \frac{p}{2}(F_\varepsilon^2)^{(p-2)/2}(t,x) D_t^2 F_\varepsilon^2(t,x) + \frac{p}{2}\frac{p-2}{2}(F_\varepsilon^2)^{(p-4)/2}(t,x)\big(D_t F_\varepsilon^2(t,x)\big)^2. \end{aligned}$$

To compute LF_ε^p we choose a non-negative function $\varphi_\varepsilon \in C_b^2(\mathbb{R})$ such that $\varphi_\varepsilon(t) = \sqrt{t+\varepsilon^2}$ for $t \in [0, \|f\|_\infty^2]$. Noting that $\frac{d}{dt}\varphi_\varepsilon^p(t) = \frac{p}{2}\varphi_\varepsilon^{p-2}(t)$ and $D_t^2\varphi_\varepsilon^p(t) = \frac{p}{2}\frac{p-2}{2}\varphi_\varepsilon^{p-4}(t)$ on the interval $[0, \|f\|_\infty^2]$, from Lemma 3.9 we obtain $F_\varepsilon^p \in \mathscr{D}(L)$ and

$$\begin{aligned} LF_\varepsilon^p(t,x) &= L(\varphi_\varepsilon^p \circ |\tilde{P}_{1/2}(t)f(x)|^2) \\ &= \frac{p}{2}F_\varepsilon^{p-2}(t,x) LF_\varepsilon^2(t,x) + \frac{1}{2}\frac{p}{2}\frac{p-2}{2}F_\varepsilon^{p-4}(t,x)\|D_H F_\varepsilon^2(t,x)\|_H^2. \end{aligned}$$

Hence,

$$\begin{aligned} (D_t^2 + L)F_\varepsilon^p(t,x) &= \frac{p}{2}F_\varepsilon^{p-2}(t,x)(D_t^2 + L)F_\varepsilon^2(t,x) \\ &\quad + \frac{p}{2}\frac{p-2}{2}F_\varepsilon^{p-4}(t,x)\Big(\big(D_t F_\varepsilon^2(t,x)\big)^2 + \frac{1}{2}\|D_H F_\varepsilon^2(t,x)\|_H^2\Big). \end{aligned} \tag{4.4}$$

Observe that

$$|D_t F_\varepsilon^2(t,x)| = 2|\tilde{L}_{1/2}\tilde{P}_{1/2}(t)f(x)|\,|\tilde{P}_{1/2}(t)f(x)|$$

and, by Lemma 3.3,

$$\|D_H F_\varepsilon^2(t,\cdot)\|_H = 2\|D_H\tilde{P}_{1/2}(t)f(\cdot)\|_H|\tilde{P}_{1/2}(t)f(\cdot)|.$$

Inserting these identities into (4.4) and using (4.3), we obtain

$$\begin{aligned} &(D_t^2 + L)F_\varepsilon^p(t,x) \\ &\quad= \frac{p}{2}F_\varepsilon^{p-2}(t,x)(D_t^2 + L)F_\varepsilon^2(t,x) + \frac{p}{2}(p-2)F_\varepsilon^{p-4}(t,x)\cdot|\tilde{P}_{1/2}(t)f(x)|^2 g(t,x) \\ &\quad\geqslant \frac{p}{2}F_\varepsilon^{p-4}(t,x)g(t,x)\Big(F_\varepsilon^2(t,x) + (p-2)|\tilde{P}_{1/2}(t)f(x)|^2\Big) \\ &\quad= \frac{p}{2}F_\varepsilon^{p-4}(t,x)g(t,x)\big((p-1)|\tilde{P}_{1/2}(t)f(x)|^2 + \varepsilon^2\big). \end{aligned}$$

Since the right hand side is non-negative for all $1 < p < \infty$, this implies (4.1). The estimate (4.2) also follows, since

$$\begin{aligned} g(t,x) &\leqslant \frac{2}{p}\liminf_{\varepsilon\downarrow 0}\Big(F_\varepsilon^{4-p}(t,x)\big((p-1)|\tilde{P}_{1/2}(t)f(x)|^2 + \varepsilon^2\big)^{-1}(D_t^2 + L)F_\varepsilon^p(t,x)\Big) \\ &\leqslant \frac{2}{p(p-1)}|\tilde{P}_{1/2}(t)f(x)|^{2-p}\liminf_{\varepsilon\downarrow 0}(D_t^2 + L)F_\varepsilon^p(t,x). \end{aligned}$$

$\square$

For $f \in \mathscr{C}$ we define the *Littlewood-Paley-Stein function* $\mathfrak{g}_H(f) : E \to [0,\infty]$ by

$$(\mathfrak{g}_H(f))(x) = \Big(\int_0^\infty t \big\| D_H \tilde{P}_{1/2}(t) f(x) \big\|_H^2 \, dt \Big)^{1/2}. \tag{4.5}$$

Theorem 4.3. *Let $1 < p \leqslant 2$. For all $f \in \mathscr{C}$ we have $\mathfrak{g}_H(f) \in L^p(E,\mu_\infty)$ and*

$$\|\mathfrak{g}_H(f)\|_{L^p(E,\mu_\infty)} \leqslant C_p \|f\|_{L^p(E,\mu_\infty)},$$

where $C_p := \big(\frac{pC}{p-1}\big)^{(2-p)/2} \alpha_p^{-1/2}$ with C the universal constant from Theorem 2.4.

Proof. For $\tau > 0$ put

$$\mathfrak{g}_H^\tau(f)(x) := \Big(\int_0^\tau t \| D_H \tilde{P}_{1/2}(t) f(x) \|_H^2 \, dt \Big)^{1/2}.$$

It is our aim to prove that

$$\|\mathfrak{g}_H^\tau(f)\|_p \leqslant C_p \|f\|_p. \tag{4.6}$$

The lemma is then obtained by letting $\tau \to \infty$.

Fix $f \in \mathscr{C}$. By integrating inequality (4.2) over $[0,\tau]$ and applying Fatou's Lemma we obtain

$$(\mathfrak{g}_H^\tau(f)(x))^2 \leqslant \alpha_p^{-1} (f^\star(x))^{2-p} \liminf_{\varepsilon \downarrow 0} h_\varepsilon(\tau, x), \tag{4.7}$$

where $f^\star$ is the maximal function from the scalar-valued case of Theorem 2.4 and

$$h_\varepsilon(\tau,x) := \int_0^\tau t(D_t^2 + L) F_\varepsilon^p(t,x) \, dt.$$

Note that by (3.7) we have

$$\int_E L F_\varepsilon^p(t,\cdot) \, d\mu_\infty = 0.$$

Using Fubini's theorem and an integration by parts, we obtain

$$\begin{aligned} \int_E h_\varepsilon(\tau,x) \, d\mu_\infty(x) &= \int_0^\tau \int_E t(D_t^2 + L) F_\varepsilon^p(t,x) \, d\mu_\infty(x) \, dt \\ &= \int_0^\tau \int_E t D_t^2 F_\varepsilon^p(t,x) \, d\mu_\infty(x) \, dt \\ &= \int_E \Big(\tau D_t F_\varepsilon^p(\tau,x) - \int_0^\tau D_t F_\varepsilon^p(t,x) \, dt \Big) \, d\mu_\infty(x) \\ &= \int_E \tau D_t F_\varepsilon^p(\tau,x) + F_\varepsilon^p(0,x) - F_\varepsilon^p(\tau,x) \, d\mu_\infty(x) \\ &\leqslant \tau \int_E D_t F_\varepsilon^p(\tau,x) \, d\mu_\infty(x) + \|F_\varepsilon(0,\cdot)\|_p^p. \end{aligned} \tag{4.8}$$

Since

$$\begin{aligned} D_t F_\varepsilon^p(t,x) &= \frac{p}{2} \big(F_\varepsilon^2(t,x)\big)^{(p-2)/2} D_t F_\varepsilon^2(t,x) \\ &= p F_\varepsilon^{p-2}(t,x) \cdot \tilde{L}_{1/2} \tilde{P}_{1/2}(t) f(x) \cdot \tilde{P}_{1/2}(t) f(x), \end{aligned}$$

we have

$$\begin{aligned}\int_E D_t F_\varepsilon^p(\tau,x)\,d\mu_\infty(x) &\leqslant p\int_E F_\varepsilon^{p-2}(\tau,x)\cdot|\tilde L_{1/2}\tilde P_{1/2}(\tau)f(x)|\cdot|\tilde P_{1/2}(\tau)f(x)|\,d\mu_\infty(x)\\ &\leqslant p\int_E F_\varepsilon^{p-1}(\tau,x)|\tilde L_{1/2}\tilde P_{1/2}(\tau)f(x)|\,d\mu_\infty(x)\\ &\leqslant p\|F_\varepsilon^{p-1}(\tau,\cdot)\|_{\frac{p}{p-1}}\|\tilde L_{1/2}\tilde P_{1/2}(\tau)f\|_p.\end{aligned}$$

For a suitable constant k_p,

$$\begin{aligned}\|F_\varepsilon^{p-1}(\tau,\cdot)\|_{\frac{p}{p-1}} &= \big\|\big(|\tilde P_{1/2}(\tau)f|^2+\varepsilon^2\big)^{1/2}\big\|_p^{p-1}\\ &\leqslant \big\||\tilde P_{1/2}(\tau)f|+\varepsilon\big\|_p^{p-1} \leqslant \big(\|\tilde P_{1/2}(\tau)f\|_p+\varepsilon\big)^{p-1}\\ &\leqslant k_p\big(\|\tilde P_{1/2}(\tau)f\|_p^{p-1}+\varepsilon^{p-1}\big) \leqslant k_p\big(\|f\|_p^{p-1}+\varepsilon^{p-1}\big).\end{aligned}$$

By putting these estimates together we obtain

$$\tau\int_E D_t F_\varepsilon^p(\tau,x)\,d\mu_\infty(x) \leqslant \tau p k_p\big(\|f\|_p^{p-1}+\varepsilon^{p-1}\big)\|\tilde P_{1/2}(\tau)\tilde L_{1/2}f\|_p.$$

Since the semigroup $(\tilde P_{1/2}(t))_{t\geqslant 0}$ is uniformly exponentially stable in $L^p(E,\mu_\infty)$ we conclude that

$$\lim_{\tau\to\infty}\tau\int_E D_t F_\varepsilon^p(\tau,x)\,d\mu_\infty(x)=0. \tag{4.9}$$

By (4.1) and the first identity in (4.8) it follows that

$$t\mapsto \int_E h_\varepsilon(t,x)\,d\mu_\infty(x)$$

is non-decreasing as a function of τ. Therefore it follows from (4.8) and (4.9) that

$$\|h_\varepsilon(\tau,\cdot)\|_1 = \int_E h_\varepsilon(\tau,x)\,d\mu_\infty(x) \leqslant \lim_{t\to\infty}\int_E h_\varepsilon(t,x)\,d\mu_\infty(x) \leqslant \|F_\varepsilon(0,\cdot)\|_p^p. \tag{4.10}$$

By (4.7) and Fatou's Lemma we obtain

$$\begin{aligned}\|\mathfrak{g}_H^\tau(f)\|_p^p &= \int_E (|\mathfrak{g}_H^\tau(f)|^2)^{p/2}\,d\mu_\infty\\ &\leqslant \alpha_p^{-p/2}\int_E (f^\star)^{p(2-p)/2}\liminf_{\varepsilon\downarrow 0} h_\varepsilon^{p/2}(\tau,\cdot)\,d\mu_\infty\\ &\leqslant \alpha_p^{-p/2}\liminf_{\varepsilon\downarrow 0}\int_E (f^\star)^{p(2-p)/2} h_\varepsilon^{p/2}(\tau,\cdot)\,d\mu_\infty.\end{aligned}$$

By Hölder's inequality with the dual exponents $2/(2-p)$ and $2/p$,

$$\begin{aligned}\|\mathfrak{g}_H^\tau(f)\|_p^p &\leqslant \alpha_p^{-p/2}\|(f^\star)^{p(2-p)/2}\|_{2/(2-p)}\liminf_{\varepsilon\downarrow 0}\|h_\varepsilon^{p/2}(\tau,\cdot)\|_{2/p}\\ &= \alpha_p^{-p/2}\|f^\star\|_p^{p(2-p)/2}\liminf_{\varepsilon\downarrow 0}\|h_\varepsilon(\tau,\cdot)\|_1^{p/2}.\end{aligned}$$

Using (4.10) and the maximal inequality of Theorem 2.4 we obtain

$$\begin{aligned}\|\mathfrak{g}_H^\tau(f)\|_p^p &\leqslant \alpha_p^{-p/2}\|f^\star\|_p^{p(2-p)/2}\liminf_{\varepsilon\downarrow 0}\|F_\varepsilon(0,\cdot)\|_p^{p^2/2}\\ &= \alpha_p^{-p/2}\|f^\star\|_p^{p(2-p)/2}\|f\|_p^{p^2/2} \leqslant \alpha_p^{-p/2}\Big(\frac{pC}{p-1}\Big)^{p(2-p)/2}\|f\|_p^p.\end{aligned}$$

$\square$

Corollary 4.4. *For $1 < p \leqslant 2$ the non-linear operator $f \mapsto \mathfrak{g}_H(f)$ admits a unique continuous extension from $L^p(E,\mu_\infty)$ into $L^p(E,\mu_\infty)$ satisfying*

$$\|\mathfrak{g}_H(f)\|_{L^p(E,\mu_\infty)} \leqslant C_p\|f\|_{L^p(E,\mu_\infty)}, \tag{4.11}$$

For $f \in W^{1,p}_H(E,\mu_\infty)$, $\mathfrak{g}_H(f)$ is given by the right-hand side of (4.5).

Proof. Let $f_n \to f$ in $L^p(E,\mu_\infty)$ with each $f_n \in \mathscr{C}$, then from the inverse triangle inequality in $L^2(\mathbb{R}_+, t\,dt; H)$ we obtain

$$\|\mathfrak{g}_H(f_n) - \mathfrak{g}_H(f_m)\|_p \leqslant C_p\|f_n - f_m\|_p.$$

Hence the sequence $(\mathfrak{g}_H(f_n))_{n\geqslant 1}$ is a Cauchy sequence in $L^p(E,\mu_\infty)$ and $\mathfrak{g}_H(f) := \lim_{n\to\infty}\mathfrak{g}_H(f_n)$ defines the unique continuous extension.

To prove the second statement, we note that by Proposition 3.5 and (4.11), the mapping $f \mapsto D_H\tilde{P}_{1/2}(\cdot)f$ is well-defined on $\mathscr{C}$ and admits a unique extension to a bounded linear operator from $L^p(E,\mu_\infty)$ to $L^p(E,\mu_\infty; L^2(\mathbb{R}_+, t\,dt; H))$. We claim that on $W^{1,p}_H(E,\mu_\infty)$ this extension is again given by $f \mapsto D_H\tilde{P}_{1/2}(\cdot)f$. To see this, fix $f \in W^{1,p}_H(E,\mu_\infty)$ and choose $f_n \to f$ in $W^{1,p}_H(E,\mu_\infty)$ with $f_n \in \mathscr{C}$. Then $D_H f_n \to D_H f$ in $L^p(E,\mu_\infty; H)$ and therefore for all $t > 0$ we have

$$D_H\tilde{P}_{1/2}(t)f_n = \tilde{T}_{1/2}(t)D_H f_n \to \tilde{T}_{1/2}(t)D_H f = D_H\tilde{P}_{1/2}(t)f.$$

Fixing $t > 0$, we may pass to a subsequence such that

$$\lim_{n\to\infty} D_H\tilde{P}_{1/2}(t)f_n(x) = D_H\tilde{P}_{1/2}(t)f(x) \quad \text{for } \mu_\infty\text{-almost all } x \in E. \tag{4.12}$$

On the other hand, the sequence of functions $D_H\tilde{P}_{1/2}(\cdot)f_n$ defines a Cauchy sequence in $L^p(E,\mu_\infty; L^2(\mathbb{R}_+, t\,dt; H))$. Let $\Phi \in L^p(E,\mu_\infty; L^2(\mathbb{R}_+, t\,dt; H))$ be its limit. Then we also have $D_H\tilde{P}_{1/2}(\cdot)f_n \to \Phi$ in $L^p(E,\mu_\infty; L^p(\mathbb{R}_+, t\,dt; H)) = L^p(E \times \mathbb{R}_+, \mu_\infty \times t\,dt; H)$, and by passing to a subsequence we may assume that

$$\lim_{n\to\infty} D_H\tilde{P}_{1/2}(t)f_n(x) = \Phi(t,x) \quad \text{for } (\mu_\infty \times t\,dt)\text{-almost all } (x,t) \in E \times \mathbb{R}_+. \tag{4.13}$$

Since both $(t,x) \mapsto D_H\tilde{P}_{1/2}(t)f(x)$ and $(t,x) \mapsto \Phi(t,x)$ are jointly measurable, for almost all $t > 0$ the identity (4.13) holds for μ_∞-almost all $x \in E$. Combining this with (4.12) we conclude that

$$D_H\tilde{P}_{1/2}(\cdot)f = \Phi \quad \text{in } L^p(E,\mu_\infty; L^2(\mathbb{R}_+, t\,dt; H)).$$

This proves the claim.

Now let $f \in W^{1,p}_H(E,\mu_\infty)$ and choose functions $f_n \in \mathscr{C}$ satisfying $f_n \to f$ in $W^{1,p}_H(E,\mu_\infty)$. Then, by the claim, in $L^p(E,\mu_\infty)$ we have

$$\begin{aligned}\mathfrak{g}_H(f) = \lim_{n\to\infty}\mathfrak{g}_H(f_n) &= \lim_{n\to\infty}\Big(\int_0^\infty t\big\|D_H\tilde{P}_{1/2}(t)f_n\big\|_H^2\,dt\Big)^{1/2} \\ &= \Big(\int_0^\infty t\big\|D_H\tilde{P}_{1/2}(t)f\big\|_H^2\,dt\Big)^{1/2}.\end{aligned}$$

∎

By combining the above results we obtain the main result of this paper.

Theorem 4.5. *Assume* (A1) *and let* $1 < p \leqslant 2$. *Then,*

$$\mathscr{D}((-L)^{1/2}) \hookrightarrow W_H^{1,p}(E, \mu_\infty).$$

Moreover, there exists a constant $K > 0$ *such that for all* $f \in \mathscr{D}((-L)^{1/2})$ *we have*

$$\|D_H f\|_{L^p(E,\mu_\infty;H)} \leqslant K \|(I-L)^{1/2} f\|_{L^p(E,\mu_\infty)}.$$

Proof. By Lemma 4.1, for $f \in \mathscr{C}$ we have $\tilde{L}_{1/2} f \in W_H^{1,p}(E, \mu_\infty)$, and therefore Proposition 3.5 and the second assertion in Corollary 4.4 imply that

$$\mathfrak{g}(D_H f) = \mathfrak{g}_H(\tilde{L}_{1/2} f),$$

where

$$\mathfrak{g}(g) := \Big(\int_0^\infty \big\| t \tilde{G}_{1/2} \tilde{T}_{1/2}(t) g \big\|^2 \frac{dt}{t} \Big)^{1/2}.$$

By Theorem 2.5 (which can be applied by the remark after Assumption (A1)) and Corollary 4.4 we obtain for all $f \in \mathscr{C}$,

$$\|D_H f\|_p \leqslant c^{-1} \|\mathfrak{g}(D_H f)\|_p = c^{-1} \|\mathfrak{g}_H(\tilde{L}_{1/2} f)\|_p \leqslant c^{-1} C_p \|\tilde{L}_{1/2} f\|_p.$$

Since $\mathscr{C}$ is a core for $\mathscr{D}(L_{1/2}) = \mathscr{D}(\tilde{L}_{1/2})$ and D_H is closed, the result follows from this. □

Remark 4.6. If $f \in L^p(E, \mu_\infty)$, then $\tilde{P}_{1/2}(t) f \in \mathscr{D}(L_{1/2})$ by analyticity, and therefore $\tilde{P}_{1/2}(t) f \in W_H^{1,p}(E, \mu_\infty)$ by the theorem. This shows that the right-hand side of equation (4.5) makes sense for all $f \in L^p(E, \mu_\infty)$, and by an approximation argument we see that it equals $\mathfrak{g}_H(f)$ μ_∞-almost everywhere.

We obtain the following inclusion for the domain of the Ornstein-Uhlenbeck operator.

Theorem 4.7. *Assume* (A1) *and let* $1 < p \leqslant 2$. *Then,*

$$\mathscr{D}(L) \hookrightarrow W_H^{2,p}(E, \mu_\infty).$$

Moreover, there exists a constant $K > 0$ *such that for all* $f \in \mathscr{D}(L)$ *we have*

$$\|D_H^2 f\|_{L^p(E,\mu_\infty;H)} \leqslant K \|(I-L) f\|_{L^p(E,\mu_\infty)}. \tag{4.14}$$

Proof. Using the same methods as above, cf. [3], one can show that for $1 < p \leqslant 2$ the following extension of the Littlewood-Paley-Stein inequality for H-valued functions holds:

$$\|\mathfrak{g}_{H^{\otimes 2}}(g)\|_p \leqslant C'_p \|g\|_p, \quad g \in L^p(E, \mu_\infty; H)$$

where

$$\mathfrak{g}_{H^{\otimes 2}}(g) := \left(\int_0^\infty t \big\| D_H \tilde{T}_{1/2}(t) g(x) \big\|_{H^{\otimes 2}}^2 \, dt \right)^{1/2}, \quad g \in \mathscr{C} \otimes H.$$

As in Theorem 4.5 it follows that

$$\|D_H g\|_{L^p(E,\mu_\infty;H^{\otimes 2})} \leqslant K' \|\tilde{G}_{1/2} g\|_{L^p(E,\mu_\infty)},$$

Using this we obtain for $f \in \mathscr{C}$

$$\begin{aligned} \|D_H^2 f\|_p &\leqslant K' \|\tilde{G}_{1/2} D_H f\|_p = K' \|D_H \tilde{L}_{1/2} f\|_p \\ &\leqslant K' K'' \|\tilde{L}_{1/2} \tilde{L}_{1/2} f\|_p = K' K'' \|(I-L) f\|_p. \end{aligned}$$

This proves (4.14). Since also

$$\|D_H f\|_p \leqslant K\|\tilde{L}_{1/2} f\|_p \leqslant K\|(I-L)^{-1/2}\|_{p\to p}\|(I-L)f\|_p$$

we obtain the desired domain inclusion. $\square$

The estimate (4.14) can be improved if we make an additional spectral gap assumption. Define the projection $\pi_0 \in \mathscr{L}(L^p(E,\mu_\infty))$ by $\pi_0 f := \mathbb{E}f \cdot \mathbf{1}$, where $\mathbb{E}f := \int_E f\, d\mu_\infty$, and put

$$L_0^p(E,\mu_\infty) := \mathscr{R}(I-\pi_0) = \{f \in L^p(E,\mu_\infty) :\ \mathbb{E}f = 0\}.$$

We denote the parts of P and L in $L_0^p(E,\mu_\infty)$ by P_0 and L_0.

Lemma 4.8. *Let $1 \leqslant p < \infty$. Then for all $f \in \mathscr{D}((-L)^{1/2})$ we have $(-L)^{1/2} f \in L_0^p(E,\mu_\infty)$.*

Proof. Since μ_∞ is an invariant measure it follows that $P(t)L_0^p(E,\mu_\infty) \subseteq L_0^p(E,\mu_\infty)$ and consequently $(-L)^{1/2}L_0^p(E,\mu_\infty) \subseteq L_0^p(E,\mu_\infty)$. Using this and the fact that $(-L)^{1/2}\mathbf{1} = 0$ we obtain $(-L)^{1/2} f = (-L)^{1/2}(I-\pi_0)f \in L_0^p(E,\mu_\infty)$. $\square$

It is shown in [9, Theorem 7.5] that if (A1) holds, then one has a continuous inclusion

$$H_\infty \hookrightarrow H$$

if and only if the semigroup S_∞ is uniformly exponentially stable on H_∞. By standard arguments, cf. [19, Lemma 4.2], this implies that P is uniformly exponentially stable on $L_0^p(E,\mu_\infty)$.

Theorem 4.9. *Assume* (A1) *and let $1 < p \leqslant 2$. If $H_\infty \hookrightarrow H$, then there exists a constant $C > 0$ such that for $f \in \mathscr{D}((-L)^{1/2})$ the following estimate holds:*

$$\|D_H f\|_{L^p(E,\mu_\infty;H)} \leqslant C\|(-L)^{1/2} f\|_{L^p(E,\mu_\infty)}.$$

Proof. Since the semigroup P_0 is uniformly exponentially stable on $L_0^p(E,\mu_\infty)$ we have $0 \in \varrho(L_0)$ and $(-L_0)^{-1/2}$ is well defined as a bounded operator on $L_0^p(E,\mu_\infty)$. Since $\mathscr{R}((-L_0)^{-1/2}) = \mathscr{D}((-L_0)^{1/2}) \subseteq \mathscr{D}((-L)^{1/2})$ it follows from Theorem 4.5 and the closed graph theorem that $D_H(-L_0)^{-1/2}$ is well defined and bounded as an operator from $L_0^p(E,\mu_\infty)$ into $L^p(E,\mu_\infty;H)$. This implies that for $f \in \mathscr{D}((-L)^{1/2})$ we have, with $f_0 = (I-\pi_0)f$,

$$\begin{aligned}\|D_H f\|_p = \|D_H f_0\|_p &\leqslant \|D_H(-L_0)^{-1/2}\|_{p\to p}\|(-L_0)^{1/2} f_0\|_p \\ &= \|D_H(-L_0)^{-1/2}\|_{p\to p}\|(-L)^{1/2} f\|_p.\end{aligned}$$

$\square$

5. The symmetric case

In the case that the Ornstein-Uhlenbeck semigroup P is symmetric on $L^2(E,\mu_\infty)$ we can characterize the exact domain of L in $L^p(E,\mu_\infty)$ for $1 < p < \infty$ using the methods of [3]. For this purpose we define, for $f \in \mathscr{F}C_b^{1,1}(E)$ of the form (3.1),

$$D_{A_\infty} f(x) := A_\infty^* D_{H_\infty} f(x) = \sum_{j=1}^k D_j\phi(\langle x, x_1^*\rangle, \dots, \langle x, x_k^*\rangle) A_\infty^* i_\infty^* x_j^*.$$

The operator D_{A_∞} is closable from $L^p(E,\mu_\infty)$ into $L^p(E,\mu_\infty;H_\infty)$; the domain of its closure is denoted by $W^{1,p}_{A_\infty}(E,\mu_\infty)$.

Theorem 5.1. *Assume that P is symmetric on $L^2(E,\mu_\infty)$ and let $1<p<\infty$. Then,*

$$\mathscr{D}((-L)^{1/2}) = W^{1,p}_H(E,\mu_\infty), \quad \mathscr{D}(L) = W^{2,p}_H(E,\mu_\infty)\cap W^{1,p}_{A_\infty}(E,\mu_\infty)$$

with equivalence of norms.

Sketch of the proof. First, by repeating the arguments of [3, Lemma 4.2], one establishes Theorem 4.2 for all $1<p<\infty$. From this, the first identification follows as in the proof of [3, Theorem 4.3, Lemma 5.1 and Theorem 5.2]. The second identification follows from the first by similar arguments as in [3, Theorem 5.3]. To see that the norm on $\mathscr{D}(L)$ thus obtained is equivalent to the norm of $W^{2,p}_H(E,\mu_\infty)\cap W^{1,p}_{A_\infty}(E,\mu_\infty)$ we note that for $f\in\mathscr{C}$ the self-adjointness of A_H implies

$$\begin{aligned}\|(-A_H)^{1/2}D_Hf(x)\|_H^2 &= [(-A_H)^{1/2}B^*Df(x),(-A_H)^{1/2}B^*Df(x)]_H\\ &= -\langle BA_HB^*Df(x),Df(x)\rangle\\ &= -\langle QA^*Df(x),Df(x)\rangle.\end{aligned}$$

Using the identity $Q=-2AQ_\infty$ this gives

$$\begin{aligned}\|(-A_H)^{1/2}D_Hf(x)\|_H^2 &= 2\langle AQ_\infty A^*Df(x),Df(x)\rangle\\ &= 2[i_\infty^*A^*Df(x),i_\infty^*A^*Df(x)]_{H_\infty}\\ &= 2\|A_\infty^*i_\infty^*Df(x)\|_{H_\infty}^2\\ &= 2\|A_\infty^*D_{H_\infty}f(x)\|_{H_\infty}^2\\ &= 2\|D_{A_\infty}f(x)\|_{H_\infty}^2.\end{aligned}$$

The remaining details are left to the reader. □

Proceeding as in [3], the domains $\mathscr{D}((-L)^{m/2})$, $m=1,2,\dots$, can be characterized in a similar way.

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