Analysis of
Infinite Dimensional Diffusions

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Prof. dr. J.M.A.M. van Neerven

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Introduction

Stochastic partial differential equations (SPDEs) are used to model a wide variety of phenomena in physics, population biology, finance, and other fields of science. Mathematically, SPDEs are often formulated as stochastic ordinary differential equations (SDEs) in an infinite dimensional Banach space E. The solution to such an equation is a Banach space-valued stochastic process which can be studied by means of *probabilistic* methods, in particular the theory of stochastic integration in Banach spaces, and by *analytic* methods. This thesis focusses on the latter, in particular on two specific approaches.

Firstly, associated with an SPDE is an evolution equation for functions defined on E, known as the Kolmogorov backward equation, typically a linear second order parabolic partial differential equation in infinitely many variables. Secondly, an SPDE induces a flow on the space of probability measures on E, representing the time-evolution of the law of the underlying process, and evolving according to a Kolmogorov forward (or Fokker-Planck) equation. Properties of solutions to both types of equations will be studied in this thesis.

The above-mentioned equations describe various physical phenomena, such as the erratic behaviour of particles immersed in a fluid and the statistics of laser light. Furthermore, these equations arise in the analysis of interacting particle systems, in the study of crystals in solid state theory, in models for neuronal activity, and in the kinematic approach to turbulence.

Part I: Elliptic operators on Wiener spaces

In this part of the thesis we analyse the generators of transition semigroups associated with a class of stochastic processes in a Banach space E. We start with an informal discussion of the probabilistic interpretation of the transition semigroup.

For $x \in E$ we consider the linear stochastic Cauchy problem in E given by

$$\begin{cases} dX(t) = -\mathcal{A}X(t)\,dt + i\,dW_{\mathcal{H}}(t), \qquad t \geq 0, \\ X(0) = x. \end{cases}$$

Such an equation represents an autonomous dynamical system perturbed by additive noise. The first term on the right hand side is a deterministic drift term involving the generator $-\mathcal{A}$ of a C_0-semigroup of bounded linear operators on E. The second term is a noise term containing a so-called cylindrical Wiener process $(W_{\mathcal{H}}(t))_{t\geq 0}$ associated with a Hilbert space $\mathcal{H}$, and a bounded operator $i : \mathcal{H} \to E$ injecting the noise into the system. Under mild assumptions, the Cauchy problem admits a unique solution $(X^x(t))_{t\geq 0}$, being an E-valued stochastic process starting from $X(0) = x$ and evolving in time under the joint influence of the drift and the noise.

Given a time $t \geq 0$, and a Borel set $B \subseteq E$, we are interested in the probability that the process is contained in B at time t. In other words, we would like to determine $\mathbb{P}(X^x(t) \in B) = \mathbb{E}\mathbf{1}_B(X^x(t))$. The latter expression suggests that it is natural to take a more general functional approach and compute

$$u(x,t) := (P(t)f)(x) := \mathbb{E}f(X^x(t)), \qquad x \in E,\ t \geq 0,$$

for, say, $f \in \mathscr{B}_{\mathrm{b}}(E)$, the space of bounded Borel functions on E. In this way we obtain a collection of linear operators $(P(t))_{t\geq 0}$ acting on $\mathscr{B}_{\mathrm{b}}(E)$. Since the solution to the SDE is a Markov process, these operators form a semigroup, whose generator $-L$ turns out to be a second order elliptic differential operator acting on functions defined on E. Therefore u satisfies $\partial_t u(x,t) = -Lu(x,t)$, a second order parabolic partial differential equation in infinitely many variables. Equations arising in this way are known as *Kolmogorov (backward) equations.*

Part I of this thesis is concerned with the analysis of such operators L in suitable $L^p(\mu)$-spaces, where μ is an invariant measure for the underlying stochastic process on E. Although it is relatively easy to give an explicit formula for L on a suitable core of smooth functions, it is not easy to give a precise description of the operator, since it is difficult to determine its domain as an operator on $L^p(\mu)$. Solving this problem boils down to obtaining two-sided L^p-estimates for the generator. The knowledge of the domain of the operator is useful in the study of nonlinear perturbations of the process.

The simplest infinite dimensional example contained in this framework is the classical Ornstein-Uhlenbeck operator, also known as the number operator in quantum field theory, which is obtained by taking $\mathcal{A} = I$ in the stochastic equation stated above. In this case the domain identification of L on $L^p(\mu)$ is provided by the celebrated P.-A. Meyer inequalities, which are of fundamental importance in the theory of Malliavin calculus. These inequalities allow to define fractional Gaussian Sobolev spaces in an infinite dimensional setting, and imply the boundedness of the Skorokhod stochastic integral.

The problem described above has been treated by various mathematicians who obtained several results using very different methods. In particular the problem has been solved in the finite dimensional case by Metafune, Prüss, Rhandi, and Schnaubelt [125] using Dori-Venni theorems for sums of operators. The domain identification in the infinite-dimensional case under a symmetry assumption is due to Chojnowska-Michalik and Goldys [32] (see also Shigekawa [152]), who employed the Littlewood-Paley-Stein theory for symmetric diffusion semigroups. However, both methods fail to extend to the infinite dimensional non-symmetric case.

In the first part of this thesis we present a simultaneous generalisation of both results. We prove the domain identification in the infinite dimensional non-symmetric case under the sole assumption of analyticity of the semigroup.

Part II: Wasserstein Theory for Infinite Dimensional Diffusions

Suppose that the above-mentioned SDE is considered with a random initial condition $X(0) = X_0$, where X_0 is an E-valued random variable. For $t \geq 0$, let μ_t be the law of the solution $X(t)$. In this way the stochastic process $(X(t))_{t\geq 0}$ induces an evolution $[0,\infty) \ni t \mapsto P^*(t)\mu := \mu_t$ of probability measures on E. The suggestive notation P^* is motivated by the fact that P^* is formally the adjoint of the semigroup P considered in Part I, in the sense that for all bounded continuous functions $f : E \to \mathbb{R}$,

$$\int_E P(t)f \, d\mu = \int_E f \, d(P^*(t)\mu).$$

The measures $(\mu_t)_{t\geq 0}$ solve the *Kolmogorov forward equation*, also known as the *Fokker-Planck equation*.

At the end of the 1990s a very appealing interpretation for Fokker-Planck equations in a finite dimensional space has been given by Jordan, Kinderlehrer, and Otto [85]. These authors demonstrated that certain Fokker-Planck equations can be regarded as gradient flows for entropy functionals on the space of probability measures endowed with the L^2-Wasserstein metric.

Although such an analysis is mathematically complicated due to the lack of a linear structure on the L^2-Wasserstein space, the gradient flow formulation has many advantages. First of all, it provides new schemes for numerical approximation. Furthermore, singular (Dirac) measures are naturally included in the theory, and the ideas are applicable in very general, possibly non-smooth metric spaces. Moreover, new proofs for several probabilistic and functional inequalities (such as concentration of measure, logarithmic Sobolev, and isoperimetric inequalities) can be obtained, often with sharp constants. Finally, geometric properties of the underlying metric spaces can be effectively studied using Wasserstein gradient flows.

In recent years, various works have been devoted to the implementation of these ideas in an infinite dimensional setting, in particular by Ambrosio, Gigli, Savaré, and Zambotti [4, 5] in Hilbert spaces, and by Fang, Shao, and Sturm [59, 150] in the setting of an abstract Wiener space. In Part II of this thesis we follow this line of argumentation, and present a Wasserstein framework which is suitable for the study of Fokker-Planck equations associated with the diffusions treated in Part I.

Part III: Malliavin Calculus in Banach Spaces

The main results obtained in Part I are generalisations of results from the theory of Malliavin calculus, a differential calculus for functions defined on an infinite dimensional Banach space E endowed with a Gaussian measure. Malliavin calculus can be applied to establish regularity results for the laws associated with solutions to stochastic (partial) differential equations. In several applications it is useful to develop a theory of Malliavin calculus for functions with values in an infinite dimensional space X. For instance, in [25] the theory where X is a Hilbert space has been applied to problems in mathematical finance. Unfortunately, the straightforward extension to Banach spaces breaks down.

Nevertheless, in Part III we demonstrate that a natural Banach space-valued generalisation of Malliavin calculus can be obtained by a systematic use of so-called γ-radonifying operators. This approach is inspired by recent advances in vector-valued stochastic and harmonic analysis. As an application we obtain a Clark-Ocone representation formula for random variables taking values in a UMD Banach space.

We continue with a more detailed introduction to each of the three parts.

Part I

Ornstein-Uhlenbeck operators

Ornstein-Uhlenbeck operators appear as generators of transition semigroups associated with linear stochastic differential equations on a Banach space E. There exists a vast literature on these operators. Of particular relevance for the following discussion are the papers by Chojnowska-Michalik and Goldys [29, 31, 33], and Goldys and van Neerven [70].

Let $-\mathcal{A}$ be the generator of a C_0-semigroup $(\mathcal{S}(t))_{t\geq 0}$ of bounded linear operators on a Banach space E. Let $\mathcal{H}$ be a Hilbert space, let $i \in \mathcal{L}(\mathcal{H}, E)$ be a bounded linear operator, set $Q := ii^*$, and let $W_\mathcal{H}$ be an $\mathcal{H}$-cylindrical Wiener process. For $x \in E$ we consider the linear stochastic Cauchy problem

$$\begin{cases} dX(t) = -\mathcal{A}X(t)\,dt + i\,dW_\mathcal{H}(t), & t \geq 0, \\ X(0) = x. \end{cases} \tag{0.1}$$

Under mild assumptions this problem admits a unique solution, and the transition semigroup is given by

$$P(t)f(x) := \int_E f(\mathcal{S}(t)x + y)\,d\mu_t(y), \qquad x \in E,\ t \geq 0, \tag{0.2}$$

for all bounded Borel functions $f : E \to \mathbb{R}$. In this expression, μ_t denotes the law of the solution to (0.1) with initial value $x = 0$. Assuming the existence of a weak limit $\mu_\infty := \lim_{t\to\infty} \mu_t$, the semigroup P extends to a C_0-semigroup of positive contractions on $L^p(\mu_\infty)$ for $1 \le p < \infty$. On a suitable core of functions the generator $-L$ is given by

$$Lf(x) = -\frac{1}{2}\operatorname{trace} D_V^2 f(x) + \langle x, \mathcal{A}^* \nabla f(x)\rangle, \qquad x \in E, \tag{0.3}$$

where ∇ denotes the Fréchet derivative and D_V can be regarded as the derivative in the direction of $\mathcal{H}$. Under an analyticity assumption, the operator L can be written as a divergence form elliptic differential operator on the space $L^2(\mu_\infty)$. In fact,

$$L = D_V^* B D_V,$$

where B is a bounded operator on $\mathcal{H}$ satisfying $B + B^* = I$. We will study these operators L in a more general framework.

Elliptic operators on Wiener spaces

Motivated by the considerations above we will set up an abstract framework for the study of elliptic operators on Wiener spaces. We start with the following data:

- (E, H, μ) is an (abstract) Wiener space, i.e., H is the reproducing kernel Hilbert space of a Gaussian measure μ on a real separable Banach space E;
- $\underline{H}$ is another real Hilbert space and $V : \mathsf{D}(V) \subseteq H \to \underline{H}$ is a closed and densely defined operator;
- $B \in \mathcal{L}(\underline{H})$ is a bounded operator which is coercive on $\mathsf{R}(V)$, i.e., there exists $\kappa > 0$ such that

$$[Bu, u] \ge \kappa \|u\|^2, \qquad u \in \mathsf{R}(V).$$

Here we use the notation $\mathsf{D}(T)$ and $\mathsf{R}(T)$ to denote the domain and the range of an operator T. For operators acting on an L^p-space, we will write $\mathsf{D}_p(T)$ and $\mathsf{R}_p(T)$ respectively.

Associated with this abstract framework are the following operators:

$$A := V^* B V \text{ on } H, \qquad \underline{A} := V V^* B \text{ on } \underline{H}.$$

These operators can be lifted to function spaces by means of the gradient $D_V := VD : \mathsf{D}_p(D_V) \subseteq L^p(\mu) \to L^p(\mu; \underline{H})$, where D denotes the Malliavin derivative. This leads to the operators

$$L := D_V^* B D_V \text{ on } L^2(\mu), \qquad \underline{L} := D_V D_V^* B \text{ on } L^2(\mu; \underline{H}).$$

The operators $-A$ and $-L$ are the generators of analytic contraction semigroups denoted by S and P respectively. Furthermore, the operators $-\underline{A}$ and

$-\underline{L}$ generate bounded analytic semigroups $\underline{S}$ and $\underline{P}$, which are *not* necessarily contractive. It can be shown that the semigroup P extends to an analytic contraction semigroup on all spaces $L^p(\mu)$ for $1 < p < \infty$. Similarly, the semigroup $\underline{P}$ can be extended to a bounded analytic semigroup on the closure of the range of D_V in $L^p(\mu; \underline{H})$.

Our first main result characterises the (two-sided) L^p-boundedness of the Riesz transform

$$D_V/\sqrt{L} : \sqrt{L}f \mapsto D_V f$$

in terms of the four operators considered above. We write $A \eqsim B$ to express that there exists constants $c, C > 0$, not depending on A and B, such that $cA \le B \le CA$. Similar conventions are used for the notation $A \lesssim B$ and $A \gtrsim B$.

Theorem 0.1 (Domain of $\sqrt{L}$). *Let $1 < p < \infty$. The following assertions are equivalent:*

(1) $\mathsf{D}_p(\sqrt{L}) = \mathsf{D}_p(D_V)$ *with* $\|\sqrt{L}f\|_p \eqsim \|D_V f\|_p$ *for* $f \in \mathsf{D}_p(\sqrt{L})$;
(2) $\underline{L}$ *admits a bounded H^∞-functional calculus on* $\overline{\mathsf{R}_p(D_V)}$;
(3) $\mathsf{D}(\sqrt{A}) = \mathsf{D}(V)$ *with* $\|\sqrt{A}h\| \eqsim \|Vh\|$ *for* $h \in \mathsf{D}(\sqrt{A})$;
(4) $\underline{A}$ *admits a bounded H^∞-functional calculus on* $\overline{\mathsf{R}(V)}$.

We actually prove a more refined version of this result involving necessary and sufficient conditions for one-sided estimates for the Riesz transform. An immediate consequence of the theorem is that (1) and (2) are actually independent of p.

Theorem 0.1 contains the classical Meyer inequalities from Malliavin calculus as a trivial consequence. In this case we have $H = \underline{H}$ and $V = B = I$, and (3) is trivially satisfied.

Let us briefly comment on the equivalence of (1) and (3). It turns out that the operator A can be identified with the restriction of L to the first chaos in the Wiener-Itô decomposition, on which all L^p-norms are equivalent by the Khintchine inequalities. In view of this observation, the implication (1) $\Rightarrow$ (3) is trivial. In the opposite direction, as (3) is not automatically satisfied [121], the same observation gives an obvious obstruction for the validity of (1). However, the theorem asserts that this is in fact the only obstruction.

Using Theorem 0.1 we can also characterise the domain of L. Our second main result reads as follows:

Theorem 0.2 (Domain of L). *Let $1 < p < \infty$, and let the equivalent conditions of Theorem 0.1 be satisfied. Then we have equality of domains*

$$\mathsf{D}_p(L) = \mathsf{D}_p(D_V^2) \cap \mathsf{D}_p(D_A)$$

with equivalence of norms

$$\|f\|_p + \|Lf\|_p \eqsim \|f\|_p + \|D_V f\|_p + \|D_V^2 f\|_p + \|D_A f\|_p.$$

In this result, D_V^2 denotes the second derivative associated with the operator V, and D_A denotes the first derivative associated with A. Note that the domain is the intersection of a "noise part" $\mathsf{D}(D_V^2)$ and a "drift part" $\mathsf{D}(D_A)$, which is natural in view of the expression (0.3).

The conditions of Theorem 0.1 are automatically satisfied in each of the following two cases:

(1) B is selfadjoint. Under this assumption the results had already been proved by Shigekawa [152] and Chojnowska-Michalik and Goldys [32].
(2) V has finite dimensional range. In this situation we recover the results by Metafune, Prüss, Rhandi, and Schnaubelt [125].

The Ornstein-Uhlenbeck semigroup setting considered above fits naturally into the abstract framework. In this case μ is the invariant measure μ_∞, H is its reproducing kernel Hilbert space, and $\underline{H}$ corresponds to the Hilbert space $\mathcal{H}$ associated with the noise term in (0.1).

Each of the four semigroups appearing in the discussion above has a natural interpretation:

- S corresponds to the adjoint of the restriction of the drift semigroup $\mathcal{S}$ to the reproducing kernel Hilbert space of μ_∞.
- Similarly, $\underline{S}$ can be viewed as the adjoint of the restriction of $\mathcal{S}$ to the noise Hilbert space $\mathcal{H}$.
- P is the Ornstein-Uhlenbeck transition semigroup associated with (0.1).
- $\underline{P}$ can be interpreted as the corresponding Ornstein-Uhlenbeck semigroup acting on vector fields.

The equivalence of (1) and (4) is of particular interest in this setting, since it characterises the boundedness of the Riesz transform directly in terms of the interplay between the drift $\mathcal{S}$ and the noise $\mathcal{H}$, without referring to the invariant measure.

The strategy of the proof

To prove Theorem 0.1 we will work in the setting of perturbed Hodge-Dirac operators. This is an abstract framework constructed by M^c^Intosh in the study of the Kato square root problem.

Consider the perturbed Hodge-Dirac operators defined by

$$T = \begin{bmatrix} 0 & V^*B \\ V & 0 \end{bmatrix} \text{ on } H \oplus \underline{H}, \qquad \mathit{\Pi} = \begin{bmatrix} 0 & D_V^*B \\ D_V & 0 \end{bmatrix} \text{ on } L^p(\mu) \oplus L^p(\mu; \underline{H}).$$

The operators A, $\underline{A}$, L, and $\underline{L}$ can be reconstructed by taking squares:

$$T^2 = \begin{bmatrix} V^*BV & 0 \\ 0 & VV^*B \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & \underline{A} \end{bmatrix}, \qquad \mathit{\Pi}^2 = \begin{bmatrix} D_V^*BD_V & 0 \\ 0 & D_V D_V^*B \end{bmatrix} = \begin{bmatrix} L & 0 \\ 0 & \underline{L} \end{bmatrix}.$$

To prove the boundedness of the Riesz transform $D_V/\sqrt{L}$, we would like to apply the following formal argument for $f \in \mathsf{D}_p(L)$:

$$\begin{aligned}\|D_V f\|_p &= \|\Pi(f,0)\|_p \\ &\lesssim \|\Pi/\sqrt{\Pi^2}\|_p \,\|\sqrt{\Pi^2}(f,0)\|_p = \|\Pi/\sqrt{\Pi^2}\|_p \,\|\sqrt{L}f\|_p.\end{aligned}$$

Therefore, still arguing formally, we obtain the desired estimate as soon as we can prove the boundedness of the operator $\Pi/\sqrt{\Pi^2}$. Applying the formal rules of functional calculus we may write $\Pi/\sqrt{\Pi^2} = (\frac{z}{\sqrt{z^2}})(\Pi) = \operatorname{sgn}(\Pi)$, where $\operatorname{sgn} : \mathbb{C} \setminus i\mathbb{R} \to \mathbb{C}$ is the bounded *holomorphic* function defined by

$$\operatorname{sgn}(z) := \begin{cases} 1, & \operatorname{Re} z > 0, \\ -1, & \operatorname{Re} z < 0. \end{cases} \tag{0.4}$$

Therefore, we conclude from this formal argument that in order to prove the boundedness of the Riesz transform $D_V/\sqrt{L}$, it suffices to show that the operator $\operatorname{sgn}(\Pi)$ is bounded. In fact, we shall prove that Π has a bounded H^∞-functional calculus in the sense that $\psi(\Pi)$ defines a bounded operator for all bounded holomorphic functions ψ defined on a certain bisector in the complex plane.

Let us emphasise that proving the boundedness of operators of the form $\operatorname{sgn}(\Pi)$ is not an easy task in general, even if $p = 2$. In fact, in the related setting where D_V is replaced by the Euclidean gradient, and B by a matrix-valued multiplication operator with L^∞-coefficients, the boundedness of $\operatorname{sgn}(\Pi)$ is equivalent to the famous Kato square root conjecture in harmonic analysis, which has been proved in [8].

It turns out that with some work the formal argument above can be made rigorous, provided we prove that the operator Π is randomised bisectorial. Loosely speaking, this means that the operator Π should satisfy appropriate resolvent estimates which remain stable under randomisation. These so-called $\mathcal{R}$-boundedness results form the core of the proof of Theorem 0.1; they are essentially equivalent to various Littlewood-Paley-Stein type inequalities. We will prove the following result:

Theorem 0.3 (Gradient bounds). *For $1 < p < \infty$ the following statements hold.*

(1) *For all smooth cylindrical functions $f : E \to \mathbb{R}$ and $t > 0$ we have, for μ-almost all $x \in E$,*

$$\sqrt{t}\|D_V P(t) f(x)\| \lesssim (P(t)|f|^2(x))^{1/2}.$$

(2) *The set $\{\sqrt{t} D_V P(t) : \ t \geq 0\}$ is $\mathcal{R}$-bounded in $\mathcal{L}(L^p(\mu), L^p(\mu; \underline{H}))$.*

(3) *For all $f \in L^p(\mu)$ we have*

$$\Big\| \Big(\int_0^\infty \|\sqrt{t} D_V P(t) f\|^2 \, \frac{dt}{t} \Big)^{1/2} \Big\|_p \lesssim \|f\|_p.$$

The proof is based on Gaussian techniques such as the Mehler formula and reproducing kernel Hilbert spaces. Once we have obtained this result, the boundedness of the H^∞-calculus can be proved in a relatively straightforward manner.

Let us finally remark that we could have taken a different but equivalent point of departure. Instead of starting with the space $\underline{H}$ and the operators V and B, we could have taken an abstract Wiener space (E, H, μ) and an analytic contraction C_0-semigroup $(e^{-tA})_{t\geq 0}$ on H. Then the semigroup $(e^{-tL})_{t\geq 0}$ on $L^p(\mu)$ can be constructed by second quantisation. In this context we obtain a characterisation for the L^p-boundedness of the Riesz transform associated with L on $L^p(\mu)$ in terms of the corresponding result for A on H.

Organisation

In Chapter 1 we collect some relevant background material concerning analysis on Wiener spaces. Chapter 2 contains the basic theory of Ornstein-Uhlenbeck operators. In Chapter 3 we summarise the Hilbertian theory of Hodge-Dirac operators, which underlies the study of elliptic operators on Wiener spaces presented in Chapter 4. The latter chapter is based on [107] and contains most of the new results obtained in Part I of this thesis. In particular the proofs of the main Theorems 0.1 and 0.2 can be found there. Chapter 5 is an appendix containing a review of the main tools: randomised boundedness, radonifying operators, and H^∞-functional calculus.

Part II

This second part of the thesis is in a sense the dual of Part I. We will construct a framework for the study of Fokker-Planck equations corresponding to (0.1) as entropy gradient flows with respect to a suitable Wasserstein metric.

The idea of formulating diffusion equations as gradient flows with respect to the Wasserstein metric goes back to the seminal paper by Jordan, Kinderlehrer and Otto [85]. Since then this approach has been very popular (see, e.g., [1, 26, 27, 37, 172]). A systematic theory of gradient flows in metric spaces has been developed by Ambrosio, Gigli, and Savaré [4]. This theory has been applied to Fokker-Planck equations in Hilbert spaces (see [4] and the paper by Ambrosio, Savaré, and Zambotti [5]).

Closely related to our work are the papers by Fang, Shao and Sturm [59, 150]. These authors construct a theory of gradient flows in Wasserstein spaces over abstract Wiener spaces. In particular, they prove that the entropy gradient flow of the Wiener measure is a solution to the Fokker-Planck equation associated with the classical Ornstein-Uhlenbeck operator. The philosophy of these works is similar to [4, 5], but additional technical complications arise due to the fact that the Wasserstein metric is defined in terms of the Cameron-Martin distance, which behaves very irregularly on E.

Adapting the approach of these works, we establish a framework which also covers the more general Ornstein-Uhlenbeck setting considered in Part I.

Geometry of the Wasserstein space

We consider a Hilbert space H, a separable Banach space E and a continuous embedding $i : H \hookrightarrow E$. Let $\mathscr{P}(E)$ denote the collection of Borel probability measures on E. For $\mu, \nu \in \mathscr{P}(E)$ we define the Wasserstein distance by

$$W_H(\mu,\nu) := \inf \left\{ \left(\int_{E\times E} |x-y|_H^2 \, d\Sigma(x,y) \right)^{1/2} : \Sigma \in \Gamma(\mu,\nu) \right\} \in [0,\infty],$$

where $\Gamma(\mu,\nu)$ denotes the collection of all Borel probability measures on the product $E \times E$ having marginals μ and ν. Endowed with the Wasserstein metric W_H, $\mathscr{P}(E)$ is a complete separable pseudo-metric space (in the sense that the W_H attains the value $+\infty$). The non-continuity of $|\cdot|_H$ as a function on E makes the analysis technically more involved than in the Euclidean or Hilbertian case. Contrary to [59], we do *not* assume that H is the reproducing kernel Hilbert space associated with a Gaussian measure on E.

We investigate the geometric structure of $\mathscr{P}(E)$. For $\mu \in \mathscr{P}(E)$ we set

$$T_\mu^H := \overline{\{\nabla_H f : f \in \mathcal{C}\}} \subseteq L^2(\mu; H),$$

where ∇_H denotes the gradient in the direction of H, $\mathcal{C}$ is a class of smooth cylindrical real-valued functions defined on E, and the closure is taken in $L^2(\mu;H)$. The space T_μ^H is interpreted as the tangent space to $\mathscr{P}(E)$ at μ. We show that to each W_H-smooth curve $(\mu_t)_{t\in[0,1]} \subseteq \mathscr{P}(E)$ corresponds a unique measurable function $Z : [0,1] \times E \to \mathbb{R}$ having the following properties:

(1) $Z_t \in T_{\mu_t}^H$ for a.e. $t \geq 0$ and $\int_0^1 \int_E |Z_t(x)|_H^2 \, d\mu_t(x) \, dt < \infty$;
(2) The continuity equation

$$\partial_t \mu_t + \nabla_H \cdot (Z_t \mu_t) = 0$$

holds in the following distributional sense: for all $\alpha \in C_c^\infty(0,1)$ and all smooth cylindrical functions $f \in \mathcal{C}$,

$$\int_0^1 \int_E \Big(\alpha'(t) f(x) + \alpha(t) [\nabla_H f(x), Z_t(x)]_H \Big) \, d\mu_t(x) \, dt = 0.$$

We will view $(Z_t)_{t\in[0,1]}$ as the velocity field along the curve $(\mu_t)_{t\in[0,1]}$. These objects allow to perform Riemannian-like computations ("Otto calculus") in the space $\mathscr{P}(E)$. Such a Riemannian structure on $\mathscr{P}(\mathbb{R}^n)$ has been formally introduced in [140] and rigorously implemented in [4] in a Hilbert space setting. Our work follows the approach of [59], where the Wiener space case has been considered.

Gradient flows

A gradient flow associated with a smooth convex functional $\phi \in C^2(\mathbb{R}^n)$ is a solution $u \in C^1(0,\infty;\mathbb{R}^n)$ to the equation $\partial_t u(t) = -\nabla\phi(u(t))$. Loosely speaking, at each moment in time the solution moves in the direction of steepest descent of the functional ϕ.

Clearly, the definition of a gradient flow refers to the differentiable structure of $\mathbb{R}^n$. However, a theory of gradient flows in metric spaces (X,d) without any differentiable structure has been developed by Ambrosio, Gigli, and Savaré in [4]. These authors consider functionals $\phi : X \to \mathbb{R}\cup\{\infty\}$ which are convex along suitable curves in X, and consider a notion of a gradient flow which only involves the metric d and the functional ϕ. More precisely, a locally absolutely continuous mapping $u : (0,\infty) \to X$ is said to be a gradient flow for a proper lower semicontinuous functional ϕ if there exists $\lambda \in \mathbb{R}$ such that for any $y \in \mathsf{D}(\phi)$ the evolution variational inequality (EVI)

$$\frac{1}{2}\partial_t d^2(u(t),y) + \frac{\lambda}{2} d^2(u(t),y) \le \phi(y) - \phi(u(t)) \tag{0.5}$$

holds for almost every $t \in (0,\infty)$. Under suitable convexity assumptions, existence and uniqueness of gradient flows has been proved in [4]. Moreover, the theory has been applied to entropy functionals on the Wasserstein space over a Hilbert space. We will study functionals and gradient flows in $(\mathscr{P}(E), W_H)$ based on this line of argumentation.

The Riemannian structure on the Wasserstein space can be used to introduce the subdifferential $\partial\phi$ associated with a functional $\phi : \mathscr{P}(E) \to \mathbb{R}\cup\{\infty\}$. Combining this with the velocity fields along curves in $\mathscr{P}(E)$, which we already discussed above, we give a rigorous meaning to the Wasserstein gradient flow equation

$$\partial_t \mu_t \in -\partial\phi(\mu_t), \qquad t \ge 0. \tag{0.6}$$

One of the main results asserts that under appropriate convexity assumptions on ϕ, this *differential geometric* approach is equivalent to the *metric* approach to gradient flows via the EVI (0.5).

As an example we let γ be a Gaussian measure on E and consider the relative entropy functional defined by

$$\mathcal{H}_\gamma : \mathscr{P}(E) \to [0,+\infty], \qquad \mathcal{H}_\gamma(\mu) := \begin{cases} \int_E \rho \log \rho \, d\gamma, & \mu \ll \gamma, \mu = \rho\gamma \\ +\infty, & \text{otherwise.} \end{cases}$$

We show that these functionals are convex along generalised geodesics in the sense of [4] if the reproducing kernel Hilbert space of γ is contained in H.

Application to infinite dimensional diffusions

In this section we connect the above-mentioned theory to the first part of the thesis and apply the results to infinite dimensional diffusions.

Under suitable assumptions we show that the relative entropy functional $\mathcal{H}_{\mu_\infty}$ associated with the invariant measure μ_∞ of the SDE (0.1) is generalised displacement convex with respect to the Wasserstein distance induced by the noise Hilbert space $\mathcal{H}$. This result generalises the Wiener space result from [59] and allows us to apply the abstract metric theory from [4], which guarantees for each $\sigma \in \overline{\mathsf{D}(\mathcal{H}_{\mu_\infty})}$ the existence of a unique gradient flow $(\sigma_t)_{t\geq 0}$ in $\mathscr{P}(E)$ associated with the metric $W_\mathcal{H}$ and the functional $\mathcal{H}_{\mu_\infty}$, satisfying the initial condition $\sigma_0 = \sigma$.

Using the approach of [85] (see also [59] for the Wiener space setting) we show in Theorem 10.17 that the measures $(\sigma_t)_{t\geq 0}$ satisfy the Fokker-Planck equation associated with the Ornstein-Uhlenbeck operator L in the following distributional sense: for any test function $\alpha \in C_\mathrm{c}^\infty[0,\infty)$ and any sufficiently smooth function $f \in \mathsf{D}(L)$ we have

$$\begin{aligned}-\int_0^\infty \alpha'(t) \int_E f(x)\, d\sigma_t(x)\, dt \\ + 2\int_0^\infty \alpha(t) \int_E Lf(x)\, d\sigma_t(x)\, dt = \alpha(0) \int_E f(x)\, d\sigma(x).\end{aligned}$$

This generalises the result from [59], where the classical Ornstein-Uhlenbeck operator corresponding to $\mathcal{A} = I$ in (0.1) has been considered.

We believe that the theory described in this chapter can be used as a framework for more general evolution equations in the space of probability measures over infinite dimensional spaces.

Organisation

Part II of the thesis is based on [106]. In Chapter 6 we collect some background results and study the basic properties of the Wasserstein space $(\mathscr{P}(E), W_H)$. Chapter 7 is concerned with the analysis of smooth paths in $\mathscr{P}(E)$ and their velocity fields. Functionals and their subdifferentials are investigated in Chapter 8, and in Chapter 9 we analyse the corresponding gradient flows on $\mathscr{P}(E)$. Finally, in Chapter 10 we consider the relative entropy functionals associated with Gaussian measures on E and connect the theory to Part I of this thesis.

Part III

The theory of Malliavin calculus [82, 138] has been initiated in the seventies by Malliavin [111], who gave a probabilistic proof of Hörmander's "sums of squares"-theorem. The Malliavin calculus generalises in a natural way to Hilbert space-valued random variables. We refer to [25] for a recent account of this infinite dimensional setting with applications to mathematical finance.

In recent years many Hilbert space results in stochastic and harmonic analysis have been transferred to a Banach space setting [81, 90]. Of particular relevance for this work is the theory of stochastic integration in Banach spaces developed by van Neerven, Veraar, and Weis [134, 135]. Inspired by these developments we construct in Part III a theory of Malliavin calculus for random variables taking values in a Banach space.

The focus in Chapter 11 is on the interplay between Malliavin calculus and decoupling inequalities for vector-valued random variables. These decoupling inequalities underly the proofs of Theorems 11.5, 11.9 and 11.12. In the opposite direction, we apply the theory developed here to give a new proof of a known decoupling result in Theorem 11.24. Different aspects of vector-valued Malliavin calculus have been considered by several authors in [112, 115, 117, 153].

In Chapter 12 (see also [109]) the vector-valued Malliavin calculus is used to construct a Skorokhod integral in so-called UMD Banach spaces which extends the stochastic integral from [134]. This integral is used to obtain a Clark-Ocone representation formula in UMD spaces. This result is new (see [116]).

Banach space-valued Malliavin calculus

Let us briefly discuss some of the main results of this part of the thesis. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let H be a Hilbert space, and let E be a Banach space. We consider an isometry $W : H \to L^2(\Omega)$ onto a closed subspace consisting of Gaussian random variables, and assume that $\mathcal{F}$ is the σ-field generated by $\{W(h) : h \in H\}$. According to the classical Wiener-Itô decomposition, $L^2(\Omega, \mathcal{F}, \mathbb{P})$ admits an orthogonal decomposition into Gaussian chaoses $L^2(\Omega, \mathcal{F}, \mathbb{P}) = \bigoplus_{m\geq 0} H^{(m)}$. Moreover, there exist canonical isometries Φ_m from the symmetric Hilbert space tensor powers $H^{\circledS m}$ onto $H^{(m)}$.

We show in Theorem 11.5 that this result admits a natural Banach space-valued generalisation. For this purpose we introduce higher order versions of the space of γ-radonifying operators $\gamma(H,E) \subseteq \mathcal{L}(H,E)$. These spaces $\gamma^{\circledS m}(H,E)$ consist of so-called symmetric γ-radonifying operators, which turn out to be the natural vector-valued analogues of the symmetric Hilbert space tensor powers in this setting. We prove that Φ_m extends to an L^p-isomorphism between $\gamma^{\circledS m}(H,E)$ and the vector-valued Gaussian chaos $H^{(m)}(E)$ for $1 \leq p < \infty$,

$$\|(\Phi_m \otimes I)T\|_{L^p(\Omega;E)} \eqsim_{m,p} \|T\|_{\gamma^{\circledS m}(H,E)}, \qquad T \in \gamma^{\circledS m}(H,E).$$

In Section 11.3 we consider the particular case where $H = L^2(M,\mu)$ for some σ-finite measure space (M,μ). Theorem 11.9 shows that the Wiener-Itô isomorphism is given in this case by a multiple stochastic integral I_m for Banach space-valued functions, for which we prove two-sided estimates on the L^p-norms. In fact, for any $1 \leq p < \infty$ we obtain the estimates

$$\|I_m F\|_{L^p(\Omega;E)} \eqsim_{m,p} \|F\|_{\gamma^{\circledS m}(L^2(M),E)}, \qquad F \in \gamma^{\circledS m}(L^2(M),E),$$

which generalise the estimates for (single) Banach space-valued stochastic integrals in [135].

The proofs of these results rely on known decoupling inequalities for so-called tetrahedral and symmetric chaos. The idea to use decoupling in the study of multiple stochastic integrals already appears in the pioneering work on decoupling by McConnell and Taqqu [119, 120], Kwapień [96] and others.

In Section 11.4 we consider the Banach space-valued Malliavin derivative D, which for $1 \le p < \infty$ acts as a closed operator

$$D : \mathbb{D}^{1,p}(\Omega; E) \subset L^p(\Omega; E) \to L^p(\Omega; \gamma(H, E)),$$

where $\mathbb{D}^{1,p}(\Omega; E)$ denotes the E-valued Gaussian Sobolev space. The main result of this section (Theorem 11.12) asserts that the restriction of the Malliavin derivative to each chaos is an L^p-isomorphism for $1 \le p < \infty$,

$$\|DF\|_{L^p(\Omega;\gamma(H,E))} \eqsim_{p,m} \|F\|_{L^p(\Omega;E)}, \qquad F \in H^{(m)}(E),$$

a fact which is by no means obvious for general Banach spaces. The use of decoupling in this context appears to be new. In UMD spaces this result is an easy consequence of Pisier's extension of Meyer's inequalities to UMD Banach spaces. These inequalities are considered in more detail in Section 11.5. We discuss several of their consequences and obtain a version of Meyer's multiplier theorem in UMD spaces. We return to decoupling in Theorem 11.24 where we present a new proof of a known decoupling result for Gaussian chaoses in UMD spaces based on Meyer's inequalities.

The Clark-Ocone formula

A classical result of Clark [34] and Ocone [139] asserts that if $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ is the augmented filtration generated by a Brownian motion $(W_t)_{t\in[0,T]}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, then every $\mathcal{F}_T$-measurable Malliavin differentiable random variable $F \in \mathbb{D}^{1,p}(\Omega)$, $1 < p < \infty$, admits a representation

$$F = \mathbb{E}(F) + \int_0^T \mathbb{E}(D_t F|\mathcal{F}_t)\, dW_t,$$

where D_t is the Malliavin derivative of F at time t. An extension to $\mathcal{F}_T$-measurable random variables $F \in \mathbb{D}^{1,1}(\Omega)$ was given by Karatzas, Ocone, and Li [91]. The Clark-Ocone formula is used in mathematical finance to obtain explicit expressions for hedging strategies.

The aim of Chapter 12 is to extend this formula to the infinite-dimensional setting using the theory of stochastic integration of $\mathcal{L}(\mathscr{H}, E)$-valued processes with respect to $\mathscr{H}$-cylindrical Brownian motions, developed in [134]. Here, $\mathscr{H}$ is a separable Hilbert space and E is a UMD Banach space.

For this purpose we study the properties of the divergence operator, which is a closed operator acting from $L^p(\Omega; \gamma(H, E))$ to $L^p(\Omega; E)$. This operator is in duality with the Malliavin derivative. In the special case where $H = L^2(0,T;\mathscr{H})$ for another Hilbert space $\mathscr{H}$, the divergence operator turns out to be a UMD space-valued Skorokhod integral, which extends the stochastic integral of [134] to non-adapted processes.

The main result in Chapter 12 asserts that if E is a UMD Banach space, $1 \le p < \infty$, and $F \in \mathbb{D}^{1,p}(\Omega;E)$ is $\mathcal{F}_T$-measurable, then

$$F = \mathbb{E}(F) + \int_0^T P_{\mathbb{F}}(DF)\, dW_{\mathscr{H}},$$

where D is the Malliavin derivative of F, $W_{\mathscr{H}}$ is an $\mathscr{H}$-cylindrical Wiener process, and $P_{\mathbb{F}}$ is the projection onto the $\mathbb{F}$-adapted elements in a suitable Banach space of L^p-stochastically integrable $\mathcal{L}(\mathscr{H},E)$-valued processes.

Organisation

In Chapter 11, which is based on [105], we develop a Banach space-valued theory of Malliavin calculus and study some of the fundamental objects, in particular Wiener-Itô chaos, multiple stochastic integrals, and the Malliavin derivative. Chapter 12 is concerned with the Skorokhod stochastic integral and the Clark-Ocone representation formula. This chapter is based on [109].

Part I

Elliptic operators on Wiener spaces

1

Analysis on Wiener Spaces

In this chapter we review some parts of the theory of analysis on Wiener spaces. These are separable Banach spaces endowed with a Gaussian measure. Before discussing the infinite dimensional framework, we start by recalling some properties of Gaussian measures on $\mathbb{R}$.

Gaussian measures on the real line

A Borel probability measure γ on $\mathbb{R}$ is called *Gaussian* if either $\gamma = \delta_0$, or there exists $q > 0$ such that γ has a density $\frac{d\gamma}{d\xi}$ with respect to the Lebesgue measure given by

$$\frac{d\gamma}{d\xi} = \frac{1}{\sqrt{2\pi q}} \exp\left(-\frac{\xi^2}{2q}\right), \qquad \xi \in \mathbb{R}.$$

The number q is called the *covariance* of γ. In the case where $\gamma = \delta_0$ we say that $q = 0$. We have

$$\int_{\mathbb{R}} \xi^2 \, d\mu(\xi) = q.$$

Lemma 1.1. *A Borel probability measure μ on $\mathbb{R}$ is Gaussian with covariance $q \geq 0$ if and only if its Fourier transform is given by*

$$\widehat{\mu}(t) := \int_{\mathbb{R}} \exp(-it\xi) \, d\gamma(\xi) = \exp\left(-\frac{qt^2}{2}\right), \qquad t \in \mathbb{R}. \tag{1.1}$$

The *standard Gaussian measure* is the Gaussian measure with $q = 1$.

1.1 Gaussian measures on Banach spaces

- Throughout the rest of Chapter 1, we let E and F be real separable Banach spaces.

Let $\mathscr{B}(E)$ be the Borel σ-algebra of E. The collection of all Borel probability measures on E will be denoted by $\mathscr{P}(E)$. For $\mu \in \mathscr{P}(E)$ and a Borel mapping $T : E \to F$, the *push-forward measure* $T_{\#}\mu \in \mathscr{P}(F)$ is defined by

$$T_{\#}\mu(B) := \mu(T^{-1}(B)), \qquad B \in \mathscr{B}(F).$$

Definition 1.2. *A Borel probability measure μ on E is said to be* Gaussian *if for any $x^* \in E^*$ the measure $x^*_{\#}\mu$ is Gaussian on $\mathbb{R}$.*

The following celebrated result is known as Fernique's Theorem [60].

Theorem 1.3. *Let μ be a Gaussian measure on E. There exists a constant $\beta > 0$ such that*

$$\int_E \exp(\beta\|x\|^2)\, d\mu(x) < \infty.$$

Proof. See [14, Theorem 2.8.5]. □

We will often use the much weaker result that μ has integrable p-moments, i.e., $I_E \in L^p(\mu; E)$ for all $1 \leq p < \infty$. In particular, the following definition makes sense:

Definition 1.4. *For a Gaussian measure μ on E the* covariance operator $Q \in \mathcal{L}(E^*, E)$ *is defined by the Bochner integral*

$$Qx^* := \int_E \langle x, x^* \rangle x \, d\mu(x), \qquad x^* \in E^*.$$

Note that if $\mu \in \mathscr{P}(E)$ is Gaussian with covariance $Q \in \mathcal{L}(E^*, E)$, then for each $x^* \in E^*$ the covariance of the Gaussian measure $x^*_{\#}\mu \in \mathscr{P}(\mathbb{R})$ equals $\langle Qx^*, x^* \rangle$.

In order to study covariance operators in more detail we introduce some terminology. An operator $R \in \mathcal{L}(E^*, E)$ is said to be

- *positive*, if $\langle Rx^*, x^* \rangle \geq 0$ for any $x^* \in E^*$;
- *symmetric*, if $\langle Rx^*, y^* \rangle = \langle Ry^*, x^* \rangle$ for any $x^*, y^* \in E^*$.

It is obvious that covariances of Gaussian measures are positive and symmetric. However, not every positive symmetric operator $Q \in \mathcal{L}(E^*, E)$ is the covariance of a Gaussian measure, unless E is finite dimensional.

To prove that a positive symmetric operator $Q \in \mathcal{L}(E^*, E)$ is the covariance of a Gaussian measure, the following result is often useful. Recall that a subset $M \subseteq \mathscr{P}(E)$ is said to be *tight* if for each $\varepsilon > 0$ there exists a compact set $K \subseteq E$ such that $\mu(K) > 1 - \varepsilon$ for every $\mu \in M$.

Proposition 1.5 (Covariance domination). *Let $Q \in \mathcal{L}(E^*, E)$ be the covariance of a Gaussian measure μ on E. Suppose that $\mathscr{R} \subseteq \mathcal{L}(E^*, E)$ is a* *collection of positive symmetric operators such that for some $K \geq 0$ and all $R \in \mathscr{R}$ we have*

$$\langle Rx^*, x^* \rangle \leq K^2 \langle Qx^*, x^* \rangle, \qquad x^* \in E^*.$$

Then each $R \in \mathscr{R}$ is the covariance of a Gaussian measure μ_R on E. Moreover, the collection $\{\mu_R : R \in \mathscr{R}\}$ is tight, and for all $1 \leq p < \infty$ we have

$$\int_E |x|_E^p \, d\mu_R(x) \leq K^p \int_E |x|_E^p \, d\mu(x).$$

Proof. See [130, Theorem 4.10]. $\square$

For $\mu \in \mathscr{P}(E)$ we consider its *Fourier transform* $\widehat{\mu} : E^* \to \mathbb{C}$ defined by

$$\widehat{\mu}(x^*) := \int_E \exp(-i\langle x, x^* \rangle) \, d\mu(x), \qquad x^* \in E^*.$$

The following proposition characterises Gaussian measures in terms of their Fourier transforms.

Proposition 1.6. *For $\mu \in \mathscr{P}(E)$ the following assertions are equivalent:*

(1) μ is Gaussian with covariance operator Q;
(2) There exists a positive symmetric operator $Q \in \mathcal{L}(E^*, E)$ such that the Fourier transform of μ is given by

$$\widehat{\mu}(x^*) = e^{-\frac{1}{2}\langle Qx^*, x^* \rangle}, \qquad \forall x^* \in E^*.$$

Proof. If μ is Gaussian with covariance operator Q, then

$$\int_{\mathbb{R}} \xi^2 \, d(x^*_{\#}\mu)(\xi) = \int_E \langle x, x^* \rangle^2 \, d\mu(x) = \langle Qx^*, x^* \rangle,$$

hence $x^*_{\#}\mu$ is Gaussian with covariance $\langle Qx^*, x^* \rangle$. Therefore (2) follows from Lemma 1.1.

Conversely, (2) implies that for $t \in \mathbb{R}$ and $x^* \in E^*$,

$$\widehat{(x^*_{\#}\mu)}(t) = \int_{\mathbb{R}} e^{-it\xi} \, d(x^*_{\#}\mu)(\xi) = \int_E e^{-it\langle x, x^* \rangle} \, d\mu(x) = e^{-\frac{1}{2}t^2\langle Qx^*, x^* \rangle}.$$

Therefore Lemma 1.1 implies that $x^*_{\#}\mu$ is Gaussian with covariance $\langle Qx^*, x^* \rangle$, hence μ is Gaussian. Note that

$$\int_E \langle x, x^* \rangle^2 \, d\mu(x) = \int_{\mathbb{R}} \xi^2 \, d(x^*_{\#}\mu)(\xi) = \langle Qx^*, x^* \rangle,$$

hence by polarisation we obtain for $y^* \in E^*$,

$$\int_E \langle x, x^* \rangle \langle x, y^* \rangle \, d\mu(x) = \langle Qx^*, y^* \rangle.$$

This implies that Q is the covariance of μ. $\square$

A useful consequence is the following result.

Lemma 1.7. *Let μ be a Gaussian measure on E with covariance $Q \in \mathcal{L}(E^*, E)$. For all $T \in \mathcal{L}(E, F)$ the measure $T_{\#}\mu$ is Gaussian on F with covariance TQT^*.*

Proof. For any $x^* \in E^*$, Proposition 1.6 implies that

$$\begin{aligned}\widehat{(T_{\#}\mu)}(x^*) &= \int_E e^{-i\langle x, x^*\rangle}\, d(T_{\#}\mu)(x) = \int_E e^{-i\langle Tx, x^*\rangle}\, d\mu(x)\\ &= \int_E e^{-i\langle x, T^*x^*\rangle}\, d\mu(x) = \langle QT^*x^*, T^*x^*\rangle = \langle TQT^*x^*, x^*\rangle.\end{aligned}$$

Therefore the result follows by another application of Proposition 1.6. □

1.2 Reproducing kernel Hilbert spaces

It turns out that each Gaussian measure on E comes with a canonical Hilbert space, which plays a prominent role in the differential calculus on E. In this section we will study these Hilbert spaces in a slightly more general setting following the presentation in [130].

- Throughout this section we let $Q \in \mathcal{L}(E^*, E)$ be a positive symmetric operator.

Consider the bilinear form $[\cdot,\cdot]_{H_Q}$ on $\mathsf{R}(Q)$ defined for $x^*, y^* \in E^*$ by

$$[Qx^*, Qy^*]_{H_Q} := \langle Qx^*, y^*\rangle, \qquad x, y \in \mathsf{R}(Q).$$

Note that this expression is well-defined. Indeed, if $Qy^* = Q\tilde{y}^*$ for some $y^*, \tilde{y}^* \in E^*$, then

$$\langle Qx^*, y^*\rangle = \langle Qy^*, x^*\rangle = \langle Q\tilde{y}^*, x^*\rangle = \langle Qx^*, \tilde{y}^*\rangle.$$

Lemma 1.8. *The bilinear form $[\cdot,\cdot]_{H_Q}$ defines an inner product on $\mathsf{R}(Q)$.*

Proof. Clearly, $[\cdot,\cdot]_{H_Q}$ is positive and symmetric (in the sense of bilinear forms). It remains to show that $[x, x]_{H_Q} > 0$ for any non-zero $x \in \mathsf{R}(Q)$. To show this, let $x := Qx^*$ for some $x^* \in E^*$, and assume that $[x, x]_{H_Q} = 0$. The Cauchy-Schwarz inequality implies that for any $y^* \in E^*$,

$$\langle Qx^*, y^*\rangle^2 \le \langle Qx^*, x^*\rangle\langle Qy^*, y^*\rangle = 0,$$

thus $x = 0$. □

This leads to the following definition:

Definition 1.9. *The* reproducing kernel Hilbert space *(RKHS) associated with Q is the completion of $\mathsf{R}(Q)$ with respect to the norm induced by the inner product $[\cdot,\cdot]_{H_Q}$.*

It turns out that the RKHS is continuously embedded in E:

Proposition 1.10. *The identity operator on $\mathsf{R}(Q)$ extends to a continuous embedding $i_Q : H_Q \hookrightarrow E$. Moreover, we have the factorisation*

$$Q := i_Q i_Q^*.$$

Proof. Let $x^* \in E^*$.Taking the supremum over all y^* in the unit ball of E^*, we obtain

$$\|Qx^*\|_E = \sup_{y^*} \langle Qx^*, y^*\rangle = \sup_{y^*} [Qx^*, Qy^*]_{H_Q} \leq \|Qx^*\|_{H_Q} \sup_{y^*} \|Qy^*\|_{H_Q}.$$

Combined with the fact that for any $y^* \in E^*$,

$$\|Qy^*\|_{H_Q} = \langle Qy^*, y^*\rangle^{1/2} \leq \|Q\|_{\mathcal{L}(E^*,E)}^{1/2} \|y^*\|_{E^*},$$

it follows that

$$\|Qx^*\|_E \leq \|Q\|_{\mathcal{L}(E^*,E)}^{1/2} \|Qx^*\|_{H_Q},$$

which proves that the identity operator on $\mathsf{R}(Q)$ extends to a bounded operator $i_Q : H_Q \to E$.

For $x^* \in E^*$, let h_{x^*} be the element in H_Q corresponding to Qx^* in E, i.e., $i_Q h_{x^*} = Qx^*$. For $y^* \in E^*$ we have

$$[i_Q^* x^*, h_{y^*}]_{H_Q} = \langle i_Q h_{y^*}, x^*\rangle = \langle Qy^*, x^*\rangle = [h_{x^*}, h_{y^*}]_{H_Q}.$$

Since elements of the form h_{y^*} form a dense subset of H_Q, it follows that $i_Q^* x^* = h_{x^*}$, hence $Q = i_Q i_Q^*$.

To show that i_Q is injective, we suppose that $i_Q g = 0$ for some $g \in H_Q$. For any $y^* \in E^*$ we obtain

$$[g, h_{y^*}]_{H_Q} = [g, i_Q^* y^*] = \langle i_Q g, y^*\rangle = 0,$$

which implies that $g = 0$, thus i_Q is injective. □

From now on we will explicitly distinguish between an element $h \in H_Q$ and the corresponding element $i_Q h \in E$.

The following characterisation of H_Q will be useful.

Lemma 1.11. *We have equality of sets*

$$i_Q(H_Q) = \{x \in E : \langle x, y^*\rangle^2 \lesssim \langle Qy^*, y^*\rangle \ \forall y^* \in E^*\}.$$

Moreover, for any $h \in H_Q$,

$$\|h\|_{H_Q} := \inf\{C \geq 0 : \langle i_Q h, y^*\rangle^2 \leq C^2 \langle Qy^*, y^*\rangle \ \forall y^* \in E^*\}.$$

Proof. Take $h \in H_Q$. For any $y^* \in E^*$ we have

$$\langle i_Q h, y^* \rangle = [h, i_Q^* y^*]_{H_Q} \leq \|h\|_{H_Q} \|i_Q^* y^*\|_{H_Q} = \|h\|_{H_Q} \langle Q^* y^*, y^* \rangle^{1/2},$$

which shows that $i_Q(H_Q)$ is contained in the set under consideration, together with the desired lower bound for $\|h\|_{H_Q}$. The corresponding upper bound follows from the identity $\langle Qy^*, y^* \rangle = \|i_Q^* y^*\|_{H_Q}^2$ and the fact that $\mathsf{R}(Q)$ is dense in H_Q.

To prove that $i_Q(H_Q)$ contains the set in question, suppose that $x \in E$ satisfies $\langle x, y^* \rangle^2 \leq C^2 \langle Qy^*, y^* \rangle$ for some $C \geq 0$ and all $y^* \in E^*$. Then the mapping $i_Q^* y^* \mapsto \langle x, y^* \rangle$ extends to a bounded linear functional on H_Q. By the Riesz Representation Theorem there exists $h \in H$ with $\|h\|_{H_Q} \leq C$ such that $\langle x, y^* \rangle = [h, i_Q^* y^*]_{H_Q} = \langle i_Q h, y^* \rangle$ for every $y^* \in E^*$. It follows that $x = i_Q h$. □

1.3 The Wiener-Itô chaos decomposition

- In this section we consider an *(abstract) Wiener space*, i.e., a triple (E, H, μ), where μ is a Gaussian measure on E with covariance operator $Q \in \mathcal{L}(E^*, E)$ and reproducing kernel Hilbert space H. We let $i : H \hookrightarrow E$ denote the canonical embedding.

We will study an orthogonal decomposition of $L^2(\mu)$, which is very useful in the analysis on Wiener spaces. In the one dimensional case, a prominent role is played by the Hermite polynomials $(H_m)_{m \geq 0}$, which are recursively defined for $\xi \in \mathbb{R}$ by

$$H_0(\xi) = 1, \qquad H_1(\xi) = \xi, \qquad (m+1) H_{m+1}(\xi) = \xi H_m(\xi) - H_{m-1}(\xi).$$

Their importance in this setting is due to the fact that

$$\big(m!^{1/2} H_m\big)_{m \geq 0}$$

is an orthonormal basis of $L^2(\gamma)$, where γ denotes the standard Gaussian measure on $\mathbb{R}$.

The starting point for an infinite dimensional generalisation of this orthogonal decomposition is the following observation:

Proposition 1.12. *The mapping*

$$\phi : i^* x^* \mapsto \langle \cdot, x^* \rangle, \qquad x^* \in E^*,$$

extends to an isometry from H to $L^2(\mu)$.

Proof. Observe that for every $x^* \in E^*$,

$$\int_E \langle x, x^* \rangle^2 \, d\mu = \langle Qx^*, x^* \rangle = \|i_Q^* x^*\|_{H_Q}^2.$$

This shows at once that ϕ is well-defined and isometric. □

The isometry ϕ is known as the *Paley-Wiener map*. For $h \in H$ we will usually write $\phi_h := \phi(h)$.

Remark 1.13. For $h \in H$, it can be shown that the function $\phi_h \in L^2(\mu)$ agrees μ-a.e. with the μ-a.e. uniquely defined measurable extension of the bounded linear functional $\overline{\phi}_h : H \to \mathbb{R}$ defined by $\overline{\phi}_h(g) := [g, h]$ (see [14, Theorem 2.10.11] or [61]).

Using the Paley-Wiener map, we will construct an orthogonal decomposition of $L^2(\mu)$. Set $H^{(\leq 0)} := \mathbb{R}\mathbf{1}$, and define $H^{(\leq m)}$ inductively as the closed linear span of $H^{(\leq (m-1))}$ together with all products of the form $\phi_{h_1} \cdot \ldots \cdot \phi_{h_m}$ with $h_1, \ldots, h_m \in H$. We then define $H^{(0)} := \mathbb{R}\mathbf{1}$ and define $H^{(m)}$ as the orthogonal complement of $H^{(\leq (m-1))}$ in $H^{(\leq m)}$. The space $H^{(m)}$ is usually referred to as the m-th *Wiener-Itô chaos*. Note that $H^{(1)} = \phi(H)$. The orthogonal projection in $L^2(\mu)$ onto $H^{(m)}$ will be denoted by I_m.

The main result of this section expresses each Wiener-Itô chaos in terms of Hermite polynomials, and gives a decomposition of $L^2(\mu)$ as an infinite orthogonal direct sum of chaoses.

Theorem 1.14. *For each $m \geq 0$ we have*

$$H^{(m)} := \overline{\mathrm{lin}}\big\{H_m(\phi_h) : h \in H, \|h\| = 1\big\}. \tag{1.2}$$

Moreover, we have the orthogonal Wiener-Itô decomposition

$$L^2(\mu) = \overset{\perp}{\bigoplus_{m \geq 0}} H^{(m)}. \tag{1.3}$$

Proof. See [138, Theorem 1.1.1]. □

Some properties of Hermite polynomials in this setting are collected in the following result.

Proposition 1.15. *Let $m, n \geq 0$, and let $(e_j)_{j \geq 1}$ be an orthonormal basis of H. Then*

$$\Big\{ \prod_{j \geq 0} \alpha_j!^{1/2} H_{\alpha_j}(\phi_{e_j}) : \sum_{j \geq 0} \alpha_j = m \Big\}$$

is an orthonormal basis of $H^{(m)}$. Moreover, for $g, h \in H$ with $\|g\| = \|h\| = 1$,

$$\int_E H_m(\phi_g) H_n(\phi_h)\, d\mu = \delta_{mn} \frac{1}{m!} [g, h]^m, \tag{1.4}$$

$$I_m(\phi_h^m) = m! H_m(\phi_h). \tag{1.5}$$

Proof. See [138, Lemma 1.1.1], [138, Proposition 1.1.1], and [84, Theorem 3.19]. Note that a different normalisation for the Hermite polynomials is used in the latter reference. □

Gaussian exponentials

A distinguished role in the analysis on Wiener spaces is played by the Gaussian exponentials. These functions appear in the literature under various names (Wick exponentials, coherent states, etc.) and are often useful in performing explicit computations.

For $h \in H$, the Gaussian exponential $E_h : E \to \mathbb{R}$ is (μ-a.e.) defined by

$$E_h(x) := \exp\big(\phi_h(x) - \tfrac{1}{2}\|h\|^2\big), \qquad x \in E. \tag{1.6}$$

Proposition 1.16. *Let $1 \le p < \infty$. For $g, h \in H$ we have*

$$E_h = \sum_{m=0}^{\infty} \frac{1}{m!} I_m(\phi_h^m) \tag{1.7}$$

where the sum converges absolutely in $L^p(\mu)$. Moreover, the linear span of the functions $\{E_h : h \in H\}$ is dense in $L^p(\mu)$, and

$$\int_E E_g E_h \, d\mu = \exp\big([g,h]\big),$$

$$\|E_h\|_p = \exp\big(\tfrac{p-1}{2}\|h\|^2\big).$$

Proof. See [84, Theorem 3.33, Corollary 3.37, 3.38 & 3.40]. □

Gaussian exponentials appear as Radon-Nikodým derivatives in the following celebrated Cameron-Martin theorem:

Theorem 1.17. *For $x \in E$ consider the shift operator $T^x : E \to E$ defined by $T^x(y) := x + y$ for $y \in E$. Then $T^x_{\#}\mu \ll \mu$ if and only if $x = i_Q h$ for some $h \in H$. In this case the Radon-Nikodým derivative is given by $\frac{dT^x_{\#}\mu}{d\mu} = E_h$.*

Proof. See [14, Corollary 2.4.3]. □

Wiener-Itô chaos in L^p

It is a reformulation of the classical Khintchine inequalities that all L^p-norms are equivalent when restricted to the first Wiener-Itô chaos $H^{(1)}$. The next result expresses the remarkable fact that this equivalence holds true on every chaos:

Theorem 1.18. *For $1 \le p \le q < \infty$ the following assertions hold.*

(1) *The space $\bigcup_{n\ge 0} \bigoplus_{0\le m\le n} H^{(m)}$ is dense in $L^p(\mu)$.*
(2) *There exist constants $C_{p,q} \ge 1$ such that for any $m \ge 0$,*

$$\|f\|_p \le \|f\|_q \le C_{p,q}^m \|f\|_p, \qquad f \in H^{(m)}. \tag{1.8}$$

(3) *For $1 < p < \infty$ the operators I_m extend to bounded projections on $L^p(\mu)$.*

Proof. See [84, Theorem 5.10]. □

Let us remark that the operators $(I_m)_{m\ge 0}$ are *not* uniformly bounded on $L^p(\mu)$ for $p \neq 2$.

1.4 Second quantisation

- We continue with the notation from section 1.3.

To continue our treatment of the Wiener-Itô chaos decomposition we use the language of (Hilbert space) tensor products.

For $m \geq 1$ and $i = 1, \ldots, m$, let H_i be a real Hilbert space with orthonormal basis $(e_k^i)_{k\geq 1}$. The *Hilbert space tensor product* $H_1 \otimes \cdots \otimes H_m$ is defined as the Hilbert space of all multilinear operators $\xi : H_1 \oplus \cdots \oplus H_m \to \mathbb{R}$ satisfying

$$\sum_{i_1,\ldots,i_m=1}^{\infty} |\xi(e_{i_1}, \ldots, e_{i_m})|^2 < \infty,$$

endowed with the inner product defined for $\xi, \eta \in H_1 \otimes \cdots \otimes H_m$ by

$$[\xi, \eta] := \sum_{i_1,\ldots,i_m=1}^{\infty} \xi(e_{i_1}, \ldots, e_{i_m})\eta(e_{i_1}, \ldots, e_{i_m}).$$

In particular, we can consider the tensor powers $H^{\otimes m}$ for $m \geq 0$ with the convention that $H^{\otimes 0} := \mathbb{R}$.

For $g_i \in H_i$ we define $g_1 \otimes \cdots \otimes g_m \in H_1 \otimes \cdots \otimes H_m$ by

$$g_1 \otimes \cdots \otimes g_m(h_1, \ldots, h_m) := [g_1, h_1] \cdot \ldots \cdot [g_m, h_m], \qquad h_i \in H_i.$$

The *m-fold symmetric tensor power* $H^{\circledS m}$ is the range of the orthogonal projection $P_{\circledS} \in \mathcal{L}(H^{\otimes m})$ defined by

$$P_{\circledS}(h_1 \otimes \cdots \otimes h_m) := \frac{1}{m!} \sum_{\sigma \in S_m} h_{\sigma(1)} \otimes \cdots \otimes h_{\sigma(m)},$$

where S_m denotes the permutation group on m elements. The (symmetric) *Fock space* $\mathscr{F}(H)$ is defined as the Hilbert space direct sum

$$\mathscr{F}(H) := \bigoplus_{m\geq 0} H^{\circledS m}.$$

The following result, known as the Wiener-Itô isometry, establishes a canonical identification of $H^{\circledS m}$ and $H^{(m)}$.

Theorem 1.19. *The mapping*

$$\Phi_m : \frac{1}{\sqrt{m!}} \sum_{\sigma \in S_m} h_{\sigma(1)} \otimes \cdots \otimes h_{\sigma(m)} \mapsto I_m(\phi_{h_1} \cdot \ldots \cdot \phi_{h_m}).$$

extends to a isometry from $H^{\circledS m}$ *onto* $H^{(m)}$. *Consequently, the mapping*

$$\Phi := \bigoplus_{m=0}^{\infty} \Phi_m : \mathscr{F}(H) \to L^2(\mu)$$

is an isometric isomorphism.

Proof. See [84, Theorem 4.1]. □

The Wiener-Itô isometry gives a natural way to lift contractions in $\mathcal{L}(H)$ to $\mathcal{L}(L^2(\mu))$:

Definition 1.20. *Let $T \in \mathcal{L}(H)$.*

(1) The operator $\mathscr{F}_m(T) \in \mathcal{L}(H^{\circledS m})$ is defined by $\mathscr{F}_0(T)\mathbf{1} := \mathbf{1}$, and for $m \geq 1$ and $h_1, \dots, h_m \in H$ by

$$\mathscr{F}_m(T) \sum_{\sigma \in S_m} h_{\sigma(1)} \otimes \cdots \otimes h_{\sigma(m)} := \sum_{\sigma \in S_m} Th_{\sigma(1)} \otimes \cdots \otimes Th_{\sigma(m)}.$$

(2) The operator $\Gamma_m(T) \in \mathcal{L}(H^{(m)})$ is defined by

$$\Gamma_m(T) := \Phi_m \circ \mathscr{F}_m(T) \circ \Phi_m^{-1}.$$

It is easy to see that $\|\Gamma_m(T)\|_{\mathcal{L}(H^{(m)})} = \|T\|^m_{\mathcal{L}(H)}$. Consequently, the following definition makes sense.

Definition 1.21. *Let $T \in \mathcal{L}(H)$ be contractive, i.e., $\|T\|_{\mathcal{L}(H)} \leq 1$. The operators $\mathscr{F}(T) \in \mathcal{L}(\mathscr{F}(H))$ and $\Gamma(T) \in \mathcal{L}(L^2(\mu))$ are defined by*

$$\mathscr{F}(T) := \bigoplus_{m \geq 0} \mathscr{F}_m(T), \qquad \Gamma(T) := \bigoplus_{m \geq 0} \Gamma_m(T).$$

The operator $\Gamma(T)$ is called the second quantisation *of T.*

It is immediate from Theorem 1.19 that second quantised operators are contractions on $L^2(\mu)$. The next result shows that much more is true:

Theorem 1.22. *Let $1 \leq p \leq \infty$ and let $T, T_1, T_2 \in \mathcal{L}(H)$ be contractive.*

(1) $\Gamma(T)$ extends to a positive operator on $L^p(\mu)$ satisfying $\|\Gamma(T)\|_{\mathcal{L}(L^p(\mu))} = 1$ and

$$\int_E \Gamma(T) f \, d\mu = \int_E f \, d\mu, \qquad f \in L^p(\mu). \tag{1.9}$$

(2) As operators on $L^p(\mu)$ the following identities hold:

$$\Gamma(I) = I, \quad \Gamma(T_1 T_2) = \Gamma(T_1)\Gamma(T_2), \quad (\Gamma(T))^* = \Gamma(T^*). \tag{1.10}$$

(3) For all $h, h_1, \dots, h_m \in H$ we have

$$\Gamma(T) I_m(\phi_{h_1} \cdot \ldots \cdot \phi_{h_m}) = I_m(\phi_{Th_1} \cdot \ldots \cdot \phi_{Th_m}), \tag{1.11}$$

$$\Gamma(T) E_h = E_{Th}. \tag{1.12}$$

(4) Mehler's formula holds: if $f = \varphi(\phi_{h_1}, \dots, \phi_{h_n})$ with $\varphi \in C_{\mathrm{b}}(\mathbb{R}^n)$ and $h_1, \dots, h_n \in H$, then for μ-almost all $x \in E$ we have

$$\begin{aligned} \Gamma(T) f(x) = \int_E \varphi\big(&\phi_{Th_1}(x) + \phi_{\sqrt{I - T^*T} h_1}(y), \dots \\ &\dots, \phi_{Th_n}(x) + \phi_{\sqrt{I - T^*T} h_n}(y)\big) \, d\mu(y). \end{aligned} \tag{1.13}$$

Proof. See [84, Example 4.8, Theorem 4.12] and [155]. □

Complex second quantisation

By complexification, the Wiener-Itô decomposition extends to an isometry between the *complex* symmetric Fock space $\Gamma(H_{\mathbb{C}}) := \bigoplus_{m\geq 0} H_{\mathbb{C}}^{\circledS m}$ and $L^2_{\mathbb{C}}(\mu)$. By mimicking the real definitions, for a contraction $T \in \mathcal{L}(H_{\mathbb{C}})$ it is possible to define the complex second quantisation $\Gamma(T)$ as a contraction on $L^2_{\mathbb{C}}(\mu)$. However, in general $\Gamma(T)$ does not extend to a bounded operator on $L^p_{\mathbb{C}}(\mu)$ for $p \neq 2$. In this thesis there are only very few places where we deal with complex second quantisation.

The following lemmas will be useful in the sequel. The imaginary unit i appearing below should not be confused with the embedding from H into E.

Lemma 1.23. *For $1 \leq p < \infty$ and $h, g \in H$ consider the complex Gaussian exponential $E_{h+ig} \in L^2_{\mathbb{C}}(\mu)$ defined by*

$$E_{h+ig} := \exp\big(\phi_{h+ig} - \frac{1}{2}\big(\|h\|^2 + 2i[h,g] + \|g\|^2\big)\big).$$

The following identity holds:

$$\|E_{h+ig}\|_p = \exp\Big(\frac{p-1}{2}\|h\|^2 + \frac{1}{2}\|g\|^2\Big).$$

Proof. See [84, Corollary 3.38]. □

Lemma 1.24. *Let $1 < p < \infty$, and let $P \in \mathcal{L}(L^p_{\mathbb{C}}(\mu))$ be such that $P|_{H^{(m)}_{\mathbb{C}}} = \Gamma_m(T)$ for some $T \in \mathcal{L}(H_{\mathbb{C}})$. Then $\|T\|_{\mathcal{L}(H_{\mathbb{C}})} \leq 1$.*

Proof. Suppose that $\|P\|_{\mathcal{L}(L^p_{\mathbb{C}}(\mu))} = e^k$ for some $k \in \mathbb{R}$. As a consequence of the complexified version of (1.7), it follows that $PE_h = E_{Th}$ for each $h \in H_{\mathbb{C}}$. Therefore, Lemma 1.23 implies that

$$\frac{p-1}{2}\|\mathrm{Re}\, Th\|^2 + \frac{1}{2}\|\mathrm{Im}\, Th\|^2 \leq k + \frac{p-1}{2}\|\mathrm{Re}\, h\|^2 + \frac{1}{2}\|\mathrm{Im}\, h\|^2. \tag{1.14}$$

Replacing h by αh for $\alpha > 0$, multiplying (1.14) by $\frac{2}{\alpha^2}$, and passing to the limit $\alpha \to \infty$, we arrive at

$$(p-1)\|\mathrm{Re}\, Th\|^2 + \|\mathrm{Im}\, Th\|^2 \leq (p-1)\|\mathrm{Re}\, h\|^2 + \|\mathrm{Im}\, h\|^2.$$

Applying this estimate to ih we obtain

$$(p-1)\|\mathrm{Im}\, Th\|^2 + \|\mathrm{Re}\, Th\|^2 \leq (p-1)\|\mathrm{Im}\, h\|^2 + \|\mathrm{Re}\, h\|^2.$$

Adding the latter inequalities, we conclude that $\|T\|_{\mathcal{L}(H_C)} \leq 1$. □

Second quantisation of semigroups

We return to the real setting and turn our attention to second quantisation of semigroups of operators.

Proposition 1.25. *Let $(T(t))_{t\geq 0}$ be a C_0-semigroup of contractions on H, and set $P(t) := \Gamma(T(t))$ for $t \geq 0$. For $1 \leq p < \infty$ the operators $(P(t))_{t\geq 0}$ form a C_0-semigroup of positive contractions on $L^p(\mu)$. Moreover, we have $\|P(t)\|_{\mathcal{L}(L^\infty(\mu))} \leq 1$ and*

$$\int_E P(t)f\,d\mu = \int_E f\,d\mu, \qquad f \in L^p(\mu).$$

Proof. Everything follows from Theorem 1.22 except for the strong continuity. Note that for each $m \geq 0$, the mapping $h \to H_m(\phi_h)$ is continuous from H to $L^p(\mu)$. Therefore, for each $h \in H$ with $\|h\| = 1$, we obtain from (1.5) and (1.11),

$$P(t)H_m(\phi_h) = \|S(t)h\|^m H_m(\phi_{S(t)h/\|S(t)h\|}) \to H_m(\phi_h), \qquad \text{as } t \downarrow 0.$$

Since these elements are dense in $L^p(\mu)$ by Theorems 1.14 and 1.18, the strong continuity follows. □

Our next aim is to study analyticity of second quantised semigroups. We emphasise that analytic contraction C_0-semigroups are required to be contractive on a sector in the complex plane (see also Definition 5.44) and not only on the positive real axis. We will prove the following result from [68]:

Theorem 1.26. *Let $(T(t))_{t\geq 0}$ be a C_0-semigroup of contractions on H, and set $P(t) := \Gamma(T(t))$ for $t \geq 0$. For $1 < p < \infty$ the following assertions are equivalent:*

(1) P extends to an analytic C_0-semigroup on $L^p(\mu)$;
(2) P extends to an analytic contraction C_0-semigroup on $L^p(\mu)$;
(3) T extends to an analytic contraction C_0-semigroup on H.

If these equivalent conditions are fulfilled, we have $P(z) = \Gamma(T(z))$ for any z in the sector of analyticity of P.

The proof of this result makes use of the following sectorial version of the Stein Interpolation Theorem [156].

Lemma 1.27. *Let $1 \leq p_0, p_1 \leq \infty$ and $0 \leq \omega_0 < \omega_1 < \pi$. Set $\Sigma^+_{\omega_0,\omega_1} := \Sigma_{\omega_1} \setminus \overline{\Sigma_{\omega_0}}$, and for $0 < t < 1$,*

$$\frac{1}{p_t} := \frac{1-t}{p_0} + \frac{t}{p_1}, \qquad \omega_t := (1-t)\omega_0 + t\omega_1.$$

Consider a collection of operators

$$N(z) : L^{p_0}(\mu) \cap L^{p_1}(\mu) \to L^{p_0}(\mu) + L^{p_1}(\mu), \qquad z \in \overline{\Sigma^+_{\omega_0,\omega_1}},$$

and suppose that for each $f \in L^{p_0}(\mu) \cap L^{p_1}(\mu)$*, the mapping* $z \to N(z)f$ *is continuous on* $\overline{\Sigma^+_{\omega_0,\omega_1}}$ *and analytic on* $\Sigma^+_{\omega_0,\omega_1}$*. If, for* $i = 0, 1$*,*

$$\|N(z)\|_{\mathcal{L}(L^{p_i}(\mu))} \le C_i, \qquad |\arg z| = \omega_i,$$

then, for each $0 \le t \le 1$*,*

$$\|N(z)\|_{\mathcal{L}(L^{p_t}(\mu))} \le C_0^{1-t} C_1^t, \qquad |\arg z| = \omega_t.$$

Proof. See [94, Lemma 5.8]. □

Proof (of Theorem 1.26). (2) ⇒ (1) is trivial.

(1) ⇒ (3): Let $(P(z))_{z\in\Sigma^+_\omega}$ be an analytic extension of P for some $\omega \in (0, \frac{1}{2}\pi)$. For $f \in H^{(m)}_{\mathbb{C}}$ and $g \in H^{(n)}_{\mathbb{C}}$ we have $\int_E f P(t) g \, d\mu = 0$ for any $m \ne n$ and $t \ge 0$. By the uniqueness of the analytic extension we have $\int_E f P(z) g \, d\mu = 0$ for any $z \in \Sigma^+_\omega$. It follows that $P(z)$ maps $H^{(m)}_{\mathbb{C}}$ into itself for any $m \ge 0$. In particular, for $m = 1$ we conclude that there exists an analytic extension $(T(z))_{z\in\Sigma^+_\omega}$ of T. Using the fact that $P(t) = \Gamma(T(t))$ and the uniqueness of the analytic extension once more, we find that $P(z)|_{H^{(m)}_{\mathbb{C}}} = \Gamma_m(T)$ for every $m \ge 0$. Lemma 1.24 implies that $\|T(z)\|_{\mathcal{L}(H_{\mathbb{C}})} \le 1$, which completes the proof.

(3) ⇒ (2): Let $\omega \in (0, \frac{1}{2}\pi)$ be such that $T(te^{\pm i\omega})$ is contractive on $H_{\mathbb{C}}$ for all $t \ge 0$. It follows that $\|P(te^{\pm i\omega})\|_{\mathcal{L}(L^2_{\mathbb{C}}(\mu))} \le 1$.

Suppose that $p > 2$ and take $p' > p$, (resp. suppose that $p < 2$ and take $1 < p' < p$). Since $\|P(t)\|_{\mathcal{L}(L^{p'}(\mu))} \le 1$ for $t \ge 0$, Lemma 1.27 implies that $P(z)$ is contractive on $L^p(\mu)$ for every $z \in \Sigma^+_\omega$. This proves (2).

The final assertion has been proved in the course of the proof of (1) ⇒ (3). □

1.5 Differentiation in Wiener spaces

- We continue with the notation from section 1.3.

An abstract Wiener space comes with a nice differential structure. There is a natural gradient, the Malliavin derivative, which differentiates functions defined on E in the direction of the reproducing kernel Hilbert space H. However, in applications to stochastic differential equations one is naturally led to consider gradients in the direction of different Hilbertian subspaces $\underline{H} \hookrightarrow E$. The study of such gradients is the topic of this section. The first step is to define a gradient on a suitable core of functions.

Cylindrical functions

When working with functions on an infinite dimensional space, its is often useful to approximate them by functions which only depend on finitely many "coordinates". These functions are called cylindrical.

When H_0 is a linear subspace of H and $k \in \mathbb{N} \cup \{\infty\}$, we let $\mathcal{F}C_{\mathrm{b}}^k(E; H_0)$ denote the vector space of all (μ-almost everywhere defined) functions $f : E \to \mathbb{R}$ of the form

$$f(x) := \varphi(\phi_{h_1}(x), \ldots, \phi_{h_n}(x)) \tag{1.15}$$

with $n \geq 1$, $\varphi \in C_{\mathrm{b}}^k(\mathbb{R}^n)$, and $h_1, \ldots, h_n \in H_0$. Here $C_{\mathrm{b}}^k(\mathbb{R}^n)$ is the space consisting of all bounded continuous functions having bounded continuous derivatives up to order k. In the case that $H_0 = H$, we simply write $\mathcal{F}C_{\mathrm{b}}^k(E)$.

The space $\mathcal{F}\mathscr{P}(E; H_0)$ is defined by replacing $C_{\mathrm{b}}^k(\mathbb{R}^n)$ in the definition of $\mathcal{F}C_{\mathrm{b}}^k(E; H_0)$ by the collection of all polynomials.

Lemma 1.28. *Let $H_0 \subseteq H$ be a dense subspace. For $1 \leq p < \infty$ the spaces $\mathcal{F}\mathscr{P}(E; H_0)$ and $\mathcal{F}C_{\mathrm{b}}^\infty(E; H_0)$ are dense in $L^p(\mu)$.*

Proof. Theorem 1.18 and the density of H_0 in H imply that $\mathcal{F}\mathscr{P}(E; H_0)$ is dense in $L^p(\mu)$. The density of $\mathcal{F}C_{\mathrm{b}}^\infty(E; H_0)$ follows from a straightforward approximation argument. □

Directional gradients

Now we are in a position to study gradients of functions defined on E.

- In the remainder of this section we consider a real separable Hilbert space $\underline{H}$ and a densely defined operator $V : \mathsf{D}(V) \subseteq H \to \underline{H}$.

Definition 1.29. *For $f \in \mathcal{F}C_{\mathrm{b}}^1(E; \mathsf{D}(V))$ of the form* (1.15)*, the gradient "in the direction of V" is the function $D_V f : E \to \underline{H}$ defined by*

$$D_V f(x) := \sum_{j=1}^n \partial_j \varphi(\phi_{h_1}(x), \ldots, \phi_{h_n}(x)) \otimes V h_j,$$

for μ-a.e. x in E.

Remark 1.30. The classical Malliavin derivative corresponds to the special case $\underline{H} = H$ and $V = I$.

In order to define Sobolev spaces associated with the gradient D_V, we have to show that the operator

$$D_V : \mathcal{F}C_{\mathrm{b}}^1(E; \mathsf{D}(V)) \subseteq L^p(\mu) \to L^p(\mu; \underline{H})$$

is closable.

First we recall the definition of this concept. Let X, Y be Banach spaces. An operator $A : \mathsf{D}(A) \subseteq X \to Y$ is said to be *closable* if there exists a closed operator $\overline{A} : \mathsf{D}(\overline{A}) \subseteq X \to Y$ satisfying $A \subseteq \overline{A}$. The following useful characterisation of closability can be found in many textbooks on functional analysis.

Lemma 1.31. *For a densely defined operator $A : \mathsf{D}(A) \subseteq X \to Y$ the following assertions are equivalent:*

(1) *A is closable;*
(2) *$\mathsf{D}(A^*)$ is weak*-dense in Y^*;*
(3) *If $(x_n)_{n\geq 1} \subseteq X$ and $y \in Y$ satisfy $x_n \to 0$ and $Ax_n \to y$, then $y = 0$.*

In order to prove that D_V is closable, we will use the following lemma.

Lemma 1.32. *Let $1 < p < \infty$. For all $f \in \mathcal{F}C_{\mathrm{b}}^1(E; \mathsf{D}(V))$ and $u \in \mathsf{D}(V^*)$ we have $f \otimes u \in \mathsf{D}_{p'}(D_V^*)$ and*

$$D_V^*(f \otimes u) = f\phi_{V^*u} - [D_V f, u],$$

where D_V^* denotes the adjoint of the operator $D_V : \mathcal{F}C_{\mathrm{b}}^1(E; \mathsf{D}(V)) \subseteq L^p(\mu) \to L^p(\mu; \underline{H})$.

Proof. Let $g \in \mathcal{F}C_{\mathrm{b}}^1(E; \mathsf{D}(V))$. Using the Gram-Schmidt algorithm we can find an orthonormal basis $(h_j)_{j\geq 1}$ of H consisting of elements from $\mathsf{D}(V)$, such that f and g can be written as

$$f = \varphi(\phi_{h_1}, \dots, \phi_{h_n}), \qquad g = \psi(\phi_{h_1}, \dots, \phi_{h_n}),$$

for suitable $n \geq 1$ and $\varphi, \psi \in C_{\mathrm{b}}^1(\mathbb{R}^n)$. Let γ_n be the standard Gaussian measure on $\mathbb{R}^n$. Using integration by parts we obtain

$$\begin{aligned}
\int_E [f \otimes u, D_V g] \, d\mu &= \sum_{j=1}^n \int_E (\varphi \cdot \partial_j \psi)(\phi_{h_1}, \dots, \phi_{h_n})[u, V h_j] \, d\mu \\
&= \sum_{j=1}^n [V^* u, h_j] \int_{\mathbb{R}^n} \varphi(\xi) \partial_j \psi(\xi) \, d\gamma_n(\xi) \\
&= \sum_{j=1}^n [V^* u, h_j] \int_{\mathbb{R}^n} \big(\xi_j \varphi(\xi) - \partial_j \varphi(\xi)\big) \psi(\xi) \, d\gamma_n(\xi) \\
&= \sum_{j=1}^n [V^* u, h_j] \int_E \Big(\phi_{h_j} \varphi(\phi_{h_1}, \dots, \phi_{h_n}) \\
&\qquad\qquad - \partial_j \varphi(\phi_{h_1}, \dots, \phi_{h_n})\Big) \psi(\phi_{h_1}, \dots, \phi_{h_n}) \, d\mu,
\end{aligned}$$

and therefore

$$\begin{aligned}\int_E [f\otimes u, D_V g]\,d\mu &= \sum_{j=1}^n [V^*u,h_j]\int_E \big(\phi_{h_j} f - \partial_j\varphi(\phi_{h_1},\dots,\phi_{h_n})\big)g\,d\mu\\ &= \int_E \big(\phi_{\sum_{j=1}^n [V^*u,h_j]h_j} f - [D_V f,u]_H\big)g\,d\mu\\ &= \int_E \big(\phi_{V^*u} f - [D_V f,u]_H\big)g\,d\mu.\end{aligned}$$

In the final step we used the fact that for $k > n$,

$$\int_E \phi_{h_k} fg\,d\mu = \int_{\mathbb{R}^k} \xi_k\varphi(\xi_1,\dots,\xi_n)\psi(\xi_1,\dots,\xi_n)\,d\gamma_k(\xi) = 0.$$

This proves the desired result. □

The following result characterises L^p-closability of D_V in terms of the operator V.

Theorem 1.33. *Let $1 < p < \infty$. The operator D_V defined on $\mathcal{F}C_{\mathrm{b}}^1(E,\mathsf{D}(V))$ is closable as an operator from $L^p(\mu)$ into $L^p(\mu;\underline{H})$ if and only if V is closable.*

Proof. Suppose that V is closable. Then $\mathsf{D}(V^*)$ is weak*-dense, hence weakly in $\underline{H}$. Since weak and strong closures of convex sets coincide, $\mathsf{D}(V^*)$ is norm dense in $\underline{H}$, and therefore $\mathcal{F}C_{\mathrm{b}}^1(E;\mathsf{D}(V))\otimes\mathsf{D}(V^*)$ is dense in $L^{p'}(\mu;\underline{H})$. Consequently, Lemma 1.32 implies that $\mathsf{D}_{p'}(D_V^*)$ is dense in $L^{p'}(\mu;\underline{H})$, hence D_V is closable as an operator from $L^p(\mu)$ into $L^p(\mu;\underline{H})$ by Lemma 1.31.

Conversely, let $(h_n)_{n\geq 1}\subseteq \mathsf{D}(V)$ and $u\in\underline{H}$ be such that $h_n\to 0$ in H and $Vh_n\to u$ in $\underline{H}$. We have to show that $u=0$. Proposition 1.12 and Theorem 1.18 imply that $\phi_{h_n}\to 0$ in $L^p(\mu)$. Moreover, $D_V\phi_{h_n} = \mathbf{1}\otimes Vh_n \to \mathbf{1}\otimes u$ in $L^p(\mu;\underline{H})$. Since D_V is closable, it follows from Lemma 1.31 that $u=0$. □

If V is closable, we will denote the closure of D_V by D_V again. We will sometimes write

$$W_V^{1,p}(\mu) := \mathsf{D}_p(D_V).$$

For later use we will state two simple lemmas.

Lemma 1.34. *Let $1 < p < \infty$ and suppose that V is closable. For $h\in\mathsf{D}(V)$ we have $E_h\in\mathsf{D}_p(D_V)$ and*

$$D_V E_h = E_h\otimes Vh. \tag{1.16}$$

Proof. This follows from the representation $E_h(x) := \exp(\phi_h(x) - \frac{1}{2}\|h\|^2)$ and a routine approximation argument using the closedness of D_V. □

Lemma 1.35. *Let $1 < p < \infty$ and suppose that V is closable. Then the space $\mathcal{FP}(E;\mathsf{D}(V))$ is a core for $\mathsf{D}_p(D_V)$.*

Proof. An easy approximation argument shows that $\mathcal{FP}(E;\mathsf{D}(V)) \subseteq \mathsf{D}_p(D_V)$. Thus it suffices to approximate elements of $\mathcal{F}C_{\mathrm{b}}^1(E;\mathsf{D}(V))$ in the graph norm of $\mathsf{D}_p(D_V)$ with elements of $\mathcal{FP}(E;\mathsf{D}(V))$. Let $f \in \mathcal{F}C_{\mathrm{b}}^1(E;\mathsf{D}(V))$ be of the form $f = \varphi(\phi_{h_1},\dots,\phi_{h_n})$ with $h_j \in \mathsf{D}(V)$ for $j = 1,\dots,n$ and $\varphi \in C_{\mathrm{b}}^1(\mathbb{R}^n)$. By a Gram-Schmidt argument we may assume that the elements $h_1,\dots,h_n$ are orthonormal in H. Taking Borel versions of the functions $x \mapsto \phi_{h_j}(x)$, the image measure of μ under the transformation $x \mapsto (\phi_{h_1}(x),\dots.\phi_{h_n}(x))$ is the standard Gaussian measure γ_n on $\mathbb{R}^n$.

This reduces the problem to finding polynomials p_k in n variables such that $p_k \to \varphi$ in $L^p(\gamma_n)$ and $\nabla p_k \to \nabla\varphi$ in $L^p(\gamma_n;\mathbb{R}^n)$. It is a classical fact that such polynomials exist. $\square$

Higher order derivatives

The closability of the gradient D_V allows to define higher order Sobolev spaces recursively. For notational simplicity we restrict ourselves to derivatives of second order. Assuming that V is closed, we will first differentiate $\underline{H}$-valued functions. We consider the operator $D_V \otimes I$, initially defined on the algebraic tensor product $\mathsf{D}_p(D_V)\otimes\underline{H}$, which we regard as a dense subspace of $L^p(\mu;\underline{H})$.

Proposition 1.36. *Let $1 < p < \infty$ and suppose that V is closed. The operator $D_V \otimes I$ is closable as an operator from $L^p(\mu;\underline{H})$ to $L^p(\mu;\underline{H}^{\otimes 2})$.*

Proof. Take $(F_n)_{n\geq 1} \subseteq \mathsf{D}_p(D_V)\otimes\underline{H}$ and $G \in L^p(\mu;\underline{H}^{\otimes 2})$ such that

$$F_n \to 0 \text{ in } L^p(\mu;\underline{H}), \qquad (D_V\otimes I)F_n \to G \text{ in } L^p(\mu;\underline{H}^{\otimes 2}).$$

For all $u \in \underline{H}$ we have $[F_n,u]_{\underline{H}} \to 0$ in $L^p(\mu)$ and $D_V[F_n,u]_{\underline{H}} \to [G,u]_{\underline{H}}$. Since D_V is closed, it follows that $[G,u]_{\underline{H}} = 0$ for all $u \in \underline{H}$, hence $G = 0$. $\square$

We will denote the closure of $D_V \otimes I$ by

$$D_V^{(1)} : \mathsf{D}_p(D_V^{(1)}) \subseteq L^p(\mu;\underline{H}) \to L^p(\mu;\underline{H}^{\otimes 2}).$$

Sometimes we will write

$$W_V^{1,p}(\mu;\underline{H}) := \mathsf{D}_p(D_V^{(1)}).$$

Now we are in a position to introduce higher order derivatives. We define the operator $D_V^2 : \mathsf{D}_p(D_V^2) \subseteq W_V^{1,p}(\mu) \to L^p(\mu;H^{\otimes 2})$ by

$$\mathsf{D}_p(D_V^2) := \{f \in W_V^{1,p}(\mu) : D_V f \in W_V^{1,p}(\mu;\underline{H})\}, \qquad D_V^2 := D_V^{(1)} \circ D_V.$$

Proposition 1.37. *Let $1 < p < \infty$ and suppose that V is closed. Then the operator D_V^2 is closed as an operator from $W_V^{1,p}(\mu)$ to $L^p(\mu;\underline{H}^{\otimes 2})$.*

Proof. Let $f_n \in \mathsf{D}_p(D_V^2)$ be such that

$$f_n \to f \text{ in } W_V^{1,p}(\mu), \qquad D_V^2 f_n \to G \text{ in } L^p(\mu; \underline{H}^{\otimes 2})$$

for some $f \in \mathsf{D}_p(D_V)$ and $G \in L^p(\mu; \underline{H}^{\otimes 2})$. In particular, $D_V f_n \to D_V f$ in $L^p(\mu; \underline{H})$. Since $D_V^{(1)}$ is closed, it follows that $D_V f \in \mathsf{D}(D_V^{(1)})$ and $D_V^{(1)} f = G$, hence $f \in \mathsf{D}(D_V^2)$ and $D_V^2 f = G$. □

We will use the notation

$$W_V^{2,p}(\mu) := \mathsf{D}_p(D_V^2).$$

1.6 Notes

The standard reference on Gaussian measures on infinite dimensional spaces is the monograph by Bogachev [14]. Pioneering work on Gaussian measures in infinite dimensions is due to Segal [149]). Gross [72] introduced the notion of an abstract Wiener space. In his approach the starting point is the Hilbert space H. Abstract Wiener spaces are modeled after the classical Wiener space, where $E = C_0[0,\infty)$ is the Banach space of all continuous functions x on $[0,\infty)$ with $x(0) = 0$, and $H = W_0^{1,2}(0,\infty)$ consists of all absolutely continuous functions h on $[0,\infty)$ satisfying $\int_0^\infty |h'(t)|^2\,dt < \infty$, endowed with the inner product $[g,h]_H := \int_0^\infty g'(t)h'(t)\,dt$. The associated Gaussian measure on $C_0[0,\infty)$ is the law of a Brownian motion.

With minor changes the theory presented in this chapter carries over to a slightly more general setting without any reference to a Banach space. Here, the starting point is a Hilbert space H and a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ which are related by means of an isonormal Gaussian process, i.e., an isometry $W \in \mathcal{L}(H, L^2(\mathbb{P}))$. The abstract Wiener space framework is obtained by taking $(\Omega, \mathbb{P}) = (E, \mu)$, H is the RKHS of μ, and $W = \phi$.

The chaos decomposition of $L^2(\mu)$ is due to Wiener [171]. If H is an L^2-space, then the isomorphisms Φ_m are multiple stochastic integrals in the sense of Itô [83].

Non-commutative analogues of abstract Wiener spaces arise in fermionic analysis, see, e.g., [73, 23], and free probability theory e.g., [167]. The corresponding L^p-spaces are non-commutative. There is also a theory of complex abstract Wiener spaces [151, 159].

Second quantised operators appear in quantum field theory. See the monograph by Simon [155]. More on the mathematical aspects can be found in the book by Janson [84].

The closability of the Malliavin derivative D_I is a basic result in the Malliavin calculus. Directional gradients in the direction of arbitrary Hilbertian subspaces have been considered by Goldys, Gozzi, and van Neerven [69]. The result presented here is an easy generalisation of their result. A slightly more subtle argument shows that Theorem 1.33 remains valid for $p = 1$, but we will not use this fact in this work.

2

Ornstein-Uhlenbeck Operators

This chapter is devoted to the study of Ornstein-Uhlenbeck operators on $L^p(\mu)$, where μ denotes a suitable Gaussian measure on a Banach space E. These operators arise naturally as generators of transition semigroups associated with linear stochastic differential equations in E with additive noise.

We present the basic properties of the semigroups and investigate conditions for analyticity and symmetry. The L^p-theory for the generators will be studied in a more general setting in Chapter 4.

2.1 Ornstein-Uhlenbeck semigroups

Let $\mathcal{H}$ be a real separable Hilbert space and let E be a real separable Banach space. We consider the following operators:

- $-\mathcal{A}$ is the generator of a C_0-semigroup $(\mathcal{S}(t))_{t\geq 0}$ on E.
- i is a bounded operator from H into E.

It is immediate that the operator $Q := ii^* \in \mathcal{L}(E^*, E)$ is positive and symmetric, i.e.

$$\langle Qx^*, x^*\rangle \geq 0, \quad \langle Qx^*, y^*\rangle = \langle Q^*y^*, x^*\rangle, \qquad \forall x^*, y^* \in E^*.$$

Throughout Chapter 2 we assume that

- for $t > 0$, the operator $Q_t \in \mathcal{L}(E^*, E)$ defined by

  $$Q_t x^* := \int_0^t \mathcal{S}(s) Q \mathcal{S}^*(s) x^* \, ds \tag{2.1}$$

  is the covariance of a Gaussian measure μ_t on E.

Although the integrand in the definition of Q_t fails to be strongly continuous in general, the definition is justified by the following result:

Lemma 2.1. *For $t > 0$, the integral in (2.1) exists as a Bochner integral.*

Proof. For $y^* \in E^*$ and an orthonormal basis $(u_j)_{j\geq 1}$ of H we have

$$\begin{aligned}\langle \mathcal{S}(s)Q\mathcal{S}^*(s)x^*, y^*\rangle &= [i^*\mathcal{S}^*(s)x^*, i^*\mathcal{S}^*(s)y^*] \\ &= \sum_{j=1}^{\infty}[i^*\mathcal{S}^*(s)x^*, u_j]\,[u_j, i^*\mathcal{S}^*(s)y^*]. \\ &= \sum_{j=1}^{\infty}\langle \mathcal{S}(s)iu_j, x^*\rangle\,\langle \mathcal{S}(s)iu_j, y^*\rangle,\end{aligned}$$

thus $\langle \mathcal{S}(\cdot)Q\mathcal{S}^*(\cdot)x^*, y^*\rangle$ can be written as a countable sum of continuous functions. This proves weak measurability of the integrand. Since E is separable, strong measurability follows from the Pettis measurability theorem [53, Chapter II]. $\square$

For future use, we record that the definition of Q_t implies the algebraic identity

$$Q_{s+t} := Q_s + \mathcal{S}(s)Q_t\mathcal{S}^*(s), \qquad s,t \geq 0. \tag{2.2}$$

The *Ornstein-Uhlenbeck semigroup* $(P(t))_{t\geq 0}$ associated with $(\mathcal{A}, i)$ is defined on the space $\mathcal{B}_{\mathrm{b}}(E)$ of bounded Borel functions on E, by

$$(P(t)f)(x) := \int_E f(\mathcal{S}(t)x + y)\,d\mu_t(y), \qquad t \geq 0,\ f \in \mathcal{B}_{\mathrm{b}}(E),\ x \in E.$$

Some basic properties of P are collected in the following result. In the proof we let $\mu * \nu \in \mathscr{P}(E)$ denote the convolution, defined for $\mu, \nu \in \mathscr{P}(E)$ by $\mu * \nu := S_{\#}(\mu \otimes \nu)$, where $S : E \times E \to E$ is given by $S(x, y) := x + y$.

Proposition 2.2. *For all $s, t \geq 0$ and $f \in \mathcal{B}_{\mathrm{b}}(E)$ we have*

(1) $P(s)P(t)f := P(s+t)f$;
(2) $P(t)f \geq 0$ *whenever* $f \geq 0$;
(3) $P(t)f \in C_{\mathrm{b}}(E)$ *whenever* $f \in C_{\mathrm{b}}(E)$.

Proof. (1) Taking Fourier transforms we see that that (2.2) implies

$$\mu_{s+t} := \mu_t * (\mathcal{S}(t)_{\#}\mu_s).$$

Using this identity we obtain

$$
\begin{aligned}
P(s)P(t)f &= \int_E (P(t)f)(\mathcal{S}(s)x+y)\,d\mu_s(y) \\
&= \int_E \int_E f(\mathcal{S}(t)(\mathcal{S}(s)x+y)+z)\,d\mu_t(z)\,d\mu_s(y) \\
&= \int_E \int_E f(\mathcal{S}(s+t)x+w+z)\,d\mu_t(z)\,d(\mathcal{S}(t)_{\#}\mu_s)(w) \\
&= \int_E \int_E f(\mathcal{S}(s+t)x+y)\,d\mu_{s+t}(y) \\
&= P(s+t)f(x).
\end{aligned}
$$

(2) Trivial.

(3) Suppose that $x_n \to x$ in E. Since f is bounded and $\mathcal{S}$ is strongly continuous, we may apply the dominated convergence theorem to obtain

$$
\begin{aligned}
P(t)f(x_n) &= \int_E f(\mathcal{S}(t)x_n+y)\,d\mu_t(y) \\
&\to \int_E f(\mathcal{S}(t)x+y)\,d\mu_t(y) = P(t)f(x).
\end{aligned}
$$

This proves continuity of $P(t)f$. The boundedness is immediate. □

Remark 2.3. The semigroup $(P(t))_{t\geq 0}$ is not strongly continuous on $C_{\mathrm{b}}(E)$ whenever $\mathcal{A} \neq 0$ (see, e.g., [93]).

Remark 2.4. Let us briefly discuss the relationship between the operators considered in this section and the linear stochastic abstract Cauchy problem in E.

Let W_H be an H-cylindrical Wiener process (see Section 12.1), and consider for $x \in E$ the linear stochastic Cauchy problem

$$
\begin{cases} dX(t) = -\mathcal{A}X(t)\,dt + i\,dW_H(t), & t \geq 0, \\ X(0) = x. \end{cases} \tag{2.3}
$$

It has been shown in [135] that (2.3) admits a unique solution if and only if for any $t > 0$ the operator Q_t defined in (2.1) is the covariance of a Gaussian measure on E. In this case the solution is given by

$$
X_x(t) = \mathcal{S}(t)x + \int_0^t \mathcal{S}(t-s)i\,dW_H(s), \qquad t \geq 0,
$$

where the integral is the Banach space valued stochastic integral defined in [19, 135]. It readily follows that for any $C_{\mathrm{b}}(E)$,

$$
P(t)f(x) = \mathbb{E}f(X_x(t)), \qquad x \in E,\ t \geq 0,
$$

which means that P is the transition semigroup associated with (2.3).

We refer to Section 12.1 for an outline of the construction of the vector-valued stochastic integral.

Ornstein-Uhlenbeck semigroups with an invariant measure

In the remainder of Chapter 2 we impose the following additional assumption:

- $Q_\infty := \lim_{t\to\infty} Q_t$ exists in the weak operator topology, and Q_∞ is the covariance of a Gaussian measure μ_∞ on E.

This assumption allows us to study the semigroup P in an L^p-setting.

Passing to the weak operator limit $t \to \infty$ in (2.2) we obtain

$$Q_\infty = Q_s + \mathcal{S}(s)Q_\infty\mathcal{S}^*(s), \qquad s \geq 0. \tag{2.4}$$

A Borel probability measure $\mu \in \mathscr{P}(E)$ is said to be *invariant* for the semigroup P if

$$\int_E P(t)f\,d\mu = \int_E f\,d\mu, \qquad f \in \mathcal{B}_\mathrm{b}(E),\ t \geq 0.$$

Theorem 2.5. *The measure μ_∞ is invariant for P. Moreover, P extends to a C_0-semigroup of contractions on $L^p(\mu_\infty)$ for all $1 \leq p < \infty$.*

Proof. The identity (2.4) implies that $\mu_\infty = \mu_t * \mathcal{S}(t)_\#\mu_\infty$ for $t > 0$. Therefore, for $f \in \mathcal{B}_\mathrm{b}(E)$ we obtain

$$\begin{aligned}\int_E P(t)f(x)\,d\mu_\infty(x) &= \int_E\int_E f(\mathcal{S}(t)x + y)\,d\mu_t(y)\,d\mu_\infty(x)\\ &= \int_E f(z)\,d(\mu_t * \mathcal{S}(t)_\#\mu_\infty)(z)\\ &= \int_E f(x)\,d\mu_\infty(x),\end{aligned}$$

which shows that μ_∞ is invariant for P.

For $f \in C_\mathrm{b}(E)$ and $x \in E$ we obtain by Jensen's inequality

$$\begin{aligned}|P(t)f(x)|^p &= \Big|\int_E f(\mathcal{S}(t)x + y)\,d\mu_t(y)\Big|^p\\ &\leq \int_E |f|^p(\mathcal{S}(t)x + y)\,d\mu_t(y)\\ &= P(t)|f|^p(x).\end{aligned}$$

Integrating this inequality and using the invariance of μ_∞, we obtain

$$\|P(t)f\|_{L^p(\mu_\infty)} \leq \int_E P(t)|f|^p\,d\mu_\infty = \int_E |f|^p\,d\mu_\infty = \|f\|^p_{L^p(\mu_\infty)}.$$

To prove strong continuity we take $f \in \mathcal{F}C^1_\mathrm{b}(E)$ of the form

$$f(x) := \varphi(\langle x, x_1^*\rangle, \ldots, \langle x, x_n^*\rangle)$$

where $x_1^*, \dots, x_n^* \in E^*$ and $\varphi \in C_{\mathrm{b}}^1(\mathbb{R}^n)$. For $x \in E$ we obtain

$$\begin{aligned}
|P(t)f(x) - f(x)| &= \Big| \int_E \varphi(\langle \mathcal{S}(t)x + y, x_1^* \rangle, \dots, \langle \mathcal{S}(t)x + y, x_n^* \rangle) \\
&\qquad\qquad - \varphi(\langle x, x_1^* \rangle, \dots, \langle x, x_n^* \rangle)\, d\mu_t(y) \Big| \\
&\le \sum_{j=1}^n \|\partial_j \varphi\|_\infty \int_E \big| \langle \mathcal{S}(t)x - x + y, x_j^* \rangle \big|\, d\mu_t(y) \\
&\le \sum_{j=1}^n \|\partial_j \varphi\|_\infty \Big[\|\mathcal{S}(t)x - x\| \, \|x_j^*\| + \Big(\int_E \langle y, x_j^* \rangle^2 \, d\mu_t(y) \Big)^{1/2} \Big]
\end{aligned}$$

Since $\mathcal{S}(t)x \to x$ and

$$\int_E \langle y, x_j^* \rangle^2 \, d\mu_t(y) = \langle Q_t x_j^*, x_j^* \rangle \to 0,$$

it follows that $P(t)f(x) \to f(x)$ as $t \downarrow 0$. By the dominated convergence theorem, it follows that $\|P(t)f - f\|_{L^p(\mu_\infty)} \to 0$. Combined with the density of $\mathcal{F}C_{\mathrm{b}}^1(E)$ in $L^p(\mu_\infty)$, the result follows. □

In order to obtain a useful description of P in Theorem 2.8 below, we will study the reproducing kernel Hilbert space H_∞ associated with Q_∞. Let $i_\infty : H_\infty \hookrightarrow E$ denote the canonical embedding, and let $\phi : H_\infty \to L^2(\mu_\infty)$ be the Paley-Wiener map. The next result shows that $\mathcal{S}$ behaves remarkably nice when restricted to H_∞:

Proposition 2.6. *For $t \ge 0$ the semigroup $\mathcal{S}$ maps $i_\infty H_\infty$ into itself and restricts to a C_0-semigroup of contractions $\mathcal{S}_\infty$ on H_∞.*

The proof relies on the following general lemma.

Lemma 2.7. *Let j be a continuous embedding from a Hilbert space $\mathcal{H}$ into a Banach space E, and let $T \in \mathcal{L}(E)$. Then T restricts to a bounded operator of norm $\le M$ on $\mathcal{H}$ if and only if*

$$\|j^* T^* x^*\| \le M \|j^* x^*\|, \qquad x^* \in E^*.$$

Proof. First we observe that, as j is injective, j^* has dense range in $\mathcal{H}$. The estimate implies that the mapping $j^* x^* \mapsto j^* T^* x^*$ is well-defined on $j^*(E^*)$ and extends to a bounded operator $R \in \mathcal{L}(\mathcal{H})$ of norm $\le M$. For $h \in \mathcal{H}$ and $x^* \in E^*$ we have

$$\langle j R^* h, x^* \rangle = [h, R j^* x^*] = [h, j^* T^* x^*] = \langle T j h, x^* \rangle,$$

hence $Tjh = jR^*h$, which proves one implication.

Conversely, if $Tj = jS$ for some $S \in \mathcal{L}(\mathcal{H})$ with $\|S\|_{\mathcal{L}(\mathcal{H})} \le M$, then

$$[h, j^*T^*x^*] = \langle Tjh, x^*\rangle = \langle jSh, x^*\rangle = [Sh, j^*x^*].$$

Taking the supremum over all h in the unit ball of $\mathcal{H}$, we obtain

$$\|j^*Tx^*\| = \sup_h [h, j^*T^*x^*] = \sup_h [Sh, j^*x^*] \le M\|j^*x^*\|.$$

□

Proof (of Proposition 2.6). Invariance and contractivity follows from Lemma 2.7 combined with the estimate

$$\begin{aligned}
\|i_\infty^* \mathcal{S}^*(t)x^*\|^2 &= \langle Q_\infty \mathcal{S}^*(t)x^*, \mathcal{S}^*(t)x^*\rangle \\
&= \int_0^\infty \langle Q\mathcal{S}^*(s+t)x^*, \mathcal{S}^*(s+t)x^*\rangle\, ds \\
&= \int_t^\infty \langle Q\mathcal{S}^*(s)x^*, \mathcal{S}^*(s)x^*\rangle\, ds \\
&\le \int_0^\infty \langle Q\mathcal{S}^*(s)x^*, \mathcal{S}^*(s)x^*\rangle\, ds \\
&= \langle Q_\infty x^*, x^*\rangle \\
&= \|i_\infty^* x^*\|^2.
\end{aligned}$$

By Proposition 1.12, strong continuity of $\mathcal{S}_\infty^*$ is equivalent to continuity of $t \mapsto \phi_{\mathcal{S}_\infty^*(t)h}$ in $L^2(\mu_\infty)$. Since $i_\infty^* E^*$ is dense in H_∞, it suffices to prove this for any element of the form $h := i_\infty^* x^*$ with $x^* \in E^*$. Using the dominated convergence theorem we obtain

$$\begin{aligned}
\|\phi_{\mathcal{S}_\infty^*(t)h} - \phi_h\|_2^2 &= \int_E |\langle x, \mathcal{S}^*(t)x^* - x^*\rangle|^2\, d\mu_\infty \\
&= \int_E |\langle \mathcal{S}(t)x - x, x^*\rangle|^2\, d\mu_\infty \to 0.
\end{aligned}$$

This shows that $\mathcal{S}_\infty^*$ (hence $\mathcal{S}_\infty$) is strongly continuous. □

Now we are in a position to prove that Ornstein-Uhlenbeck semigroups can be obtained by second quantisation of $\mathcal{S}_\infty^*$. We refer to Section 1.4 for more information on second quantisation.

Theorem 2.8. *For $t \ge 0$ and $f \in L^p(\mu_\infty)$ we have $P(t) = \Gamma(\mathcal{S}_\infty^*(t))$.*

Proof. In view of Proposition 1.16, Theorem 1.22, and Theorem 2.5, it suffices to prove that both semigroups agree on all elements of the form E_h for $h \in H_\infty$. Since $i_\infty^* E^*$ is dense in H_∞, we may take $h = i_\infty^* x^*$ for some $x^* \in E^*$. By (1.12) we have

$$\begin{aligned}
\Gamma(\mathcal{S}_\infty^*(t))E_{i_\infty^* x^*}(x) &= E_{\mathcal{S}_\infty^*(t) i_\infty^* x^*}(x) \\
&= E_{i_\infty^* \mathcal{S}^*(t) x^*}(x) \\
&= \exp(\langle x, \mathcal{S}^*(t)x^*\rangle - \frac{1}{2}\|i_\infty^* \mathcal{S}^*(t)x^*\|^2). \qquad (2.5)\\
&= \exp(\langle \mathcal{S}(t)x, x^*\rangle - \frac{1}{2}\langle Q_\infty \mathcal{S}^*(t)x^*, \mathcal{S}^*(t)x^*\rangle).
\end{aligned}$$

On the other hand, using that $(x^*)_{\#}\mu_t$ is centered Gaussian with covariance $\langle Q_t x^*, x^*\rangle$ we obtain

$$\begin{aligned}
P(t)E_h(x) &= \int_E E_h(\mathcal{S}(t)x + y)\, d\mu_t(y) \\
&= \int_E \exp\left(\langle \mathcal{S}(t)x + y, x^*\rangle - \frac{1}{2}\langle Q_\infty x^*, x^*\rangle\right) d\mu_t(y) \\
&= \exp\left(\langle \mathcal{S}(t)x, x^*\rangle - \frac{1}{2}\langle Q_\infty x^*, x^*\rangle\right)\cdot \\
&\quad \cdot \frac{1}{\sqrt{2\pi\langle Q_t x^*, x^*\rangle}} \int_{\mathbb{R}} \exp\left(\xi - \frac{\xi^2}{2\langle Q_t x^*, x^*\rangle}\right) d\xi \\
&= \exp\left(\langle \mathcal{S}(t)x, x^*\rangle - \frac{1}{2}\langle Q_\infty x^*, x^*\rangle\right) \exp\left(\frac{1}{2}\langle Q_t x^*, x^*\rangle\right).
\end{aligned}$$

Using (2.4) we see that the last expression coincides with (2.5). □

The Lyapunov equation

The following result concerning the generator $-\mathcal{A}_\infty^*$ of $\mathcal{S}_\infty^*$ will be used frequently.

Lemma 2.9. *For $x^* \in \mathsf{D}(\mathcal{A}^*)$ we have $i_\infty^* x^* \in \mathsf{D}(\mathcal{A}_\infty^*)$ and $\mathcal{A}_\infty^* i_\infty^* x^* = i_\infty^* \mathcal{A}^* x^*$. Moreover, the space $i_\infty^*(\mathsf{D}(\mathcal{A}^*))$ is a core for $\mathsf{D}(\mathcal{A}_\infty^*)$.*

Proof. For $h \in H_\infty$ we have

$$[h, (\mathcal{S}_\infty^*(t) - I)i_\infty^* x^*] = [h, i_\infty^*(\mathcal{S}^*(t) - I)x^*] = \langle i_\infty h, (\mathcal{S}^*(t) - I)x^*\rangle,$$

which implies that

$$\begin{aligned}
\lim_{t\to 0}[h, \frac{1}{t}(\mathcal{S}_\infty^*(t) - I)i_\infty^* x^*] &= \lim_{t\to 0}\langle i_\infty h, \frac{1}{t}(\mathcal{S}^*(t) - I)x^*\rangle \\
&= -\langle i_\infty h, \mathcal{A}^* x^*\rangle = -[h, i_\infty^* \mathcal{A}^* x^*].
\end{aligned}$$

By a standard result in semigroup theory [57], the weak generator of a C_0-semigroup equals the strong generator. Therefore we obtain that $i_\infty^* x^* \in \mathsf{D}(\mathcal{A}_\infty^*)$ and $\mathcal{A}_\infty^* i_\infty x^* = i_\infty^* \mathcal{A}^* x^*$.

To prove the second claim, according to another well-known result from semigroup theory [57], it suffices to show that $i_\infty^*(\mathsf{D}(\mathcal{A}^*))$ is dense in H_∞ and invariant under $\mathcal{S}_\infty^*$.

Since $\mathsf{D}(\mathcal{A}^*)$ is weak*-dense in E^* and i_∞^* is weak*-to-weakly continuous, $i_\infty^*(\mathsf{D}(\mathcal{A}^*))$ is weakly dense, hence dense in H.

For $t \geq 0$ and $x^* \in \mathsf{D}(\mathcal{A}^*)$ we have $\mathcal{S}_\infty^*(t) i_\infty^* x^* = i_\infty^* \mathcal{S}^*(t) x^*$. Since $\mathcal{S}^*(t)$ maps $\mathsf{D}(\mathcal{A}^*)$ into itself, it follows that $i_\infty^*(\mathsf{D}(\mathcal{A}^*))$ is invariant under $\mathcal{S}_\infty^*$, which completes the proof. □

The next result is an algebraic identity which be crucial in the investigation of analyticity of the semigroups $\mathcal{S}_\infty$ and P.

Proposition 2.10 (Lyapunov equation). *For any $x^* \in \mathsf{D}(\mathcal{A}^*)$ we have $Q_\infty x^* \in \mathsf{D}(\mathcal{A})$ and*

$$Qx^* = Q_\infty \mathcal{A}^* x^* + \mathcal{A} Q_\infty x^*.$$

Proof. For $y^* \in \mathsf{D}(\mathcal{A}^*)$, (2.4) implies that

$$\langle Q_\infty x^*, y^* \rangle = \langle Q_t x^*, y^* \rangle + \langle Q_\infty \mathcal{S}^*(t) x^*, \mathcal{S}^*(t) y^* \rangle, \qquad t > 0.$$

Note that $t \mapsto \mathcal{S}^*(t)x^*$ is weak*-continuous. Being a symmetric operator, Q is weak*-to-weakly continuous. Since $\mathcal{S}$ is a C_0-semigroup, $t \mapsto \mathcal{S}(t)x$ is weakly continuous. Putting these observations together, we obtain that $t \mapsto \langle \mathcal{S}(t) Q \mathcal{S}^*(t) x^*, y^* \rangle$ is continuous. Therefore, by the fundamental theorem of calculus, $t \mapsto \langle Q_t x^*, y^* \rangle$ is differentiable at 0, and

$$\partial_t\big|_{t=0} \langle Q_t x^*, y^* \rangle = \partial_t\big|_{t=0} \int_0^t \langle \mathcal{S}(s) Q \mathcal{S}^*(s) x^*, y^* \rangle \, ds = \langle Q x^*, y^* \rangle.$$

On the other hand, we have

$$\begin{aligned}
&\langle Q_\infty \mathcal{S}^*(t) x^*, \mathcal{S}^*(t) y^* \rangle - \langle Q_\infty x^*, y^* \rangle \\
&\quad = \langle Q_\infty (\mathcal{S}^*(t) - I) x^*, \mathcal{S}^*(t) y^* \rangle + \langle Q_\infty x^*, (\mathcal{S}^*(t) - I) y^* \rangle \\
&\quad = [(\mathcal{S}_\infty^*(t) - I) i_\infty^* x^*, \mathcal{S}_\infty^*(t) i_\infty^* y^*] + \langle Q_\infty x^*, (\mathcal{S}^*(t) - I) y^* \rangle.
\end{aligned}$$

Dividing by t and passing to the limit $t \downarrow 0$ we obtain in view of Proposition 2.6 and Lemma 2.9,

$$\begin{aligned}
\partial_t\big|_{t=0} \langle Q_\infty \mathcal{S}^*(t) x^*, \mathcal{S}^*(t) y^* \rangle &= -[\mathcal{A}_\infty^* i_\infty^* x^*, i_\infty^* y^*] - \langle Q_\infty x^*, \mathcal{A}^* y^* \rangle \\
&= -\langle Q_\infty \mathcal{A}^* x^*, y^* \rangle - \langle Q_\infty x^*, \mathcal{A}^* y^* \rangle.
\end{aligned}$$

Putting these identities together we find

$$\langle Q_\infty \mathcal{A}^* x^*, y^* \rangle + \langle Q_\infty x^*, \mathcal{A}^* y^* \rangle = \langle Q x^*, y^* \rangle.$$

Since $\mathcal{A}$ is closed and densely defined, it follows from the Hahn-Banach theorem that $Q_\infty x^* \in \mathsf{D}(\mathcal{A})$ and $Qx^* = Q_\infty \mathcal{A}^* x^* + \mathcal{A} Q_\infty x^*$. □

The following simple lemma allows the construction of a useful operator.

Lemma 2.11. *We have* $\mathsf{N}(i_\infty^*) \subseteq \mathsf{N}(i^*)$.

Proof. Suppose that $i_\infty^* x^* = 0$, hence $\int_0^\infty \|i^*\mathcal{S}^*(t)x^*\|^2\,dt = 0$. This implies that $i^*\mathcal{S}^*(t)x^* = 0$ for a.e. $t \ge 0$. In particular, we can find a sequence t_k converging to 0, with $i^*\mathcal{S}^*(t_k)x^* = 0$ for all $k \ge 1$. The weak*-continuity of $t \mapsto \mathcal{S}^*(t)x^*$ implies that for all $u \in H$,

$$[u, i^*x^*] = \langle iu, x^*\rangle = \lim_{k\to\infty} \langle iu, \mathcal{S}^*(t_k)x^*\rangle = \lim_{k\to\infty} \langle u, i^*\mathcal{S}^*(t_k)x^*\rangle = 0,$$

hence $i^*x^* = 0$. □

Thanks to Lemma 2.11 it makes sense to define the following operator:

$$V : i_\infty^*(E^*) \subseteq H_\infty \to H, \qquad V(i_\infty^* x^*) := i^*x^*. \tag{2.6}$$

Using this operator we can formulate the following useful consequence of the Lyapunov equation.

Corollary 2.12. *For all* $g, h \in i_\infty^*(\mathsf{D}(\mathcal{A}^*))$ *we have*

$$[\mathcal{A}_\infty^* g, h] + [g, \mathcal{A}_\infty^* h] = [Vg, Vh].$$

Proof. In view of Lemma 2.9 and Proposition 2.10 we obtain, for $x^*, y^* \in \mathsf{D}(\mathcal{A}^*)$,

$$\begin{aligned}
[\mathcal{A}_\infty^* i_\infty^* x^*, i_\infty^* y^*] &= [i_\infty^* \mathcal{A}^* x^*, i_\infty^* y^*] \\
&= \langle Q_\infty \mathcal{A}^* x^*, y^*\rangle \\
&= -\langle \mathcal{A} Q_\infty x^*, y^*\rangle + \langle Q x^*, y^*\rangle \\
&= -[i_\infty^* x^*, i_\infty^* \mathcal{A}^* y^*] + [i^* x^*, i^* y^*] \\
&= -[i_\infty^* x^*, \mathcal{A}_\infty^* i_\infty^* y^*] + [V i_\infty^* x^*, V i_\infty^* y^*].
\end{aligned}$$

□

2.2 Analyticity

The following result provides a well-known criterion for analyticity of contraction semigroups on Hilbert spaces, which is sometimes called the strong sector condition. We refer to Definition 5.44 for the definition of an analytic contraction C_0-semigroup.

Proposition 2.13. *Let* $-G$ *be the generator of a* C_0*-semigroup* T *of contractions on a real Hilbert space* $\mathcal{H}$*. The following assertions are equivalent:*

(i) T *extends to an analytic contraction* C_0*-semigroup;*
(ii) $|[Gg, h]| \lesssim [Gg, g]^{1/2}[Gh, h]^{1/2}$ *for all* $g, h \in \mathsf{D}(G)$.

Proof. The result follows from [141, Theorems 1.53 and 1.58] combined with [103, Proposition 2.17]. □

From now on we shall make the additional assumption that i is injective. Note that there is no loss of generality, since we may replace i by the canonical embedding $H_Q \hookrightarrow E$, where H_Q denotes the RKHS associated with $Q = ii^*$, without affecting the semigroups P and $\mathcal{S}_\infty$.

The next theorem is taken from [68, 70] and provides various characterisations of analyticity for Ornstein-Uhlenbeck semigroups. In view of (vi) we recall that Q_∞ maps $\mathsf{D}(\mathcal{A}^*)$ into $\mathsf{D}(A)$ by Proposition 2.10.

Theorem 2.14. *For $1 < p < \infty$ the following assertions are equivalent:*

(i) P extends to an analytic C_0-semigroup on $L^p(\mu)$;
(ii) P extends to an analytic contraction C_0-semigroup on $L^p(\mu)$;
(iii) $\mathcal{S}_\infty$ extends to an analytic contraction C_0-semigroup on H_∞;
(iv) For $g, h \in \mathsf{D}(\mathcal{A}_\infty^*)$ we have

$$|[\mathcal{A}_\infty^* g, h]| \lesssim [\mathcal{A}_\infty^* g, g]^{1/2} [\mathcal{A}_\infty^* h, h]^{1/2}.$$

(v) There exists a bounded operator $B \in \mathcal{L}(H)$ satisfying

$$iBi^* x^* = Q_\infty \mathcal{A}^* x^* \qquad x^* \in \mathsf{D}(\mathcal{A}^*);$$

(vi) There exists a bounded operator $C \in \mathcal{L}(H)$ satisfying

$$iCi^* x^* = \mathcal{A} Q_\infty x^* \qquad x^* \in \mathsf{D}(\mathcal{A}^*).$$

In this case we have

$$B^* = C, \qquad B + B^* = C + C^* = I. \tag{2.7}$$

If the equivalent conditions of the theorem are fulfilled, we will simply say that P is *analytic*.

Proof. $(i) \Leftrightarrow (ii) \Leftrightarrow (iii)$ follows from Theorems 1.26 and 2.8.

$(iii) \Leftrightarrow (iv)$ follows from Proposition 2.13.

$(iv) \Rightarrow (v)$: For $x^*, y^* \in \mathsf{D}(\mathcal{A}^*)$ we have by Lemma 2.9,

$$[\mathcal{A}_\infty^* i_\infty^* x^*, i_\infty^* y^*] = [i_\infty^* \mathcal{A}^* x^*, i_\infty^* y^*] = \langle Q_\infty \mathcal{A}^* x^*, y^* \rangle, \tag{2.8}$$

and by Proposition 2.10,

$$[\mathcal{A}_\infty^* i_\infty^* x^*, i_\infty^* x^*] = \langle Q_\infty \mathcal{A}^* x^*, x^* \rangle = \frac{1}{2} \langle Q x^*, x^* \rangle = \frac{1}{2} \|i^* x^*\|^2. \tag{2.9}$$

Therefore, (iv) implies that, for some $k \geq 0$,

$$\langle Q_\infty \mathcal{A}^* x^*, y^* \rangle \leq k \|i^* x^*\| \, \|i^* y^*\|.$$

Since i^* is weak*-to-weakly continuous, $i^*(\mathsf{D}(\mathcal{A}^*))$ is dense in H (here we use the injectivity of i). Therefore the Riesz Representation Theorem guarantees for each $x^* \in E^*$ the existence of $h_{x^*} \in H$ satisfying $\|h_x^*\| \le k\|i^*x^*\|$ and

$$\langle Q_\infty \mathcal{A}^* x^*, y^*\rangle = [h_{x^*}, i^*y^*] = \langle i h_{x^*}, y^*\rangle.$$

Since this holds for all y^* in the weak*-dense subspace $\mathsf{D}(\mathcal{A}^*) \subseteq E^*$, we infer that $Q_\infty \mathcal{A}^* x^* = i h_{x^*}$. The result follows with $Bi^*x^* := h_{x^*}$.

$(v) \Rightarrow (iv)$: For $x^*, y^* \in \mathsf{D}(\mathcal{A}^*)$ we obtain in view of (2.8) and (2.9),

$$\begin{aligned}
[\mathcal{A}_\infty^* i_\infty^* x^*, i_\infty^* y^*] &= \langle Q_\infty \mathcal{A}^* x^*, y^*\rangle = \langle iBi^*x^*, y^*\rangle \\
&= [Bi^*x^*, i^*y^*] \le \|B\|\,\|i^*x^*\|\,\|i^*y^*\| \\
&= \|B\|\,[\mathcal{A}_\infty^* i_\infty^* x^*, i_\infty^* x^*]^{1/2}\,[\mathcal{A}_\infty^* i_\infty^* y^*, i_\infty^* y^*]^{1/2}.
\end{aligned}$$

Since $i_\infty^*(\mathsf{D}(\mathcal{A}^*))$ is a core for $\mathsf{D}(\mathcal{A}_\infty^*)$ by Lemma 2.9, the result follows.

$(iv) \Leftrightarrow (vi)$: This is proved in the same way as $(iv) \Leftrightarrow (v)$.

The final identities follow from the fact that for all $x^*, y^* \in \mathsf{D}(\mathcal{A}^*)$,

$$[Bi^*x^*, i^*y^*] = \langle Q_\infty \mathcal{A}^* x^*, y^*\rangle = \langle x^*, \mathcal{A} Q_\infty y^*\rangle = [i^*x^*, Ci^*y^*],$$

and, in view of Proposition 2.10,

$$\begin{aligned}
[Bi^*x^*, i^*y^*] &= \langle Q_\infty \mathcal{A}^* x^*, y^*\rangle \\
&= -\langle \mathcal{A} Q_\infty x^*, y^*\rangle + \langle Qx^*, y^*\rangle \\
&= -[Ci^*x^*, i^*y^*] + [i^*x^*, i^*y^*].
\end{aligned}$$

□

A consequence of analyticity is the following result.

Proposition 2.15. *If P is analytic, then the operator V defined in* (2.6) *is closable from H_∞ into H.*

Proof. According to Lemma 1.31, it suffices to show that $Bi^*(\mathsf{D}(\mathcal{A}^*))$ is a dense subspace of H, which is contained in $\mathsf{D}(V^*)$.

For all $x^* \in \mathsf{D}(\mathcal{A}^*)$ and $y^* \in E^*$ we have

$$[Bi^*x^*, V i_\infty^* y^*] = \langle Q_\infty \mathcal{A}^* x^*, y^*\rangle = [i_\infty^* \mathcal{A}^* x^*, i_\infty^* y^*] = [\mathcal{A}_\infty^* i_\infty^* x^*, i_\infty^* y^*].$$

Consequently, $Bi^*x^* \in \mathsf{D}(V^*)$ and

$$V^* B i^* x^* = \mathcal{A}_\infty^* x^*.$$

To prove the density of $Bi^*(\mathsf{D}(\mathcal{A}^*))$ in H, we note (as in the proof of Theorem 2.14) that $i^*\mathsf{D}(\mathcal{A}^*)$ is dense in H, since $\mathsf{D}(\mathcal{A}^*)$ is weak*-dense in E^* and i^* is weak*-to-weakly continuous. Moreover, B is an isomorphism on H, since (2.7) implies that

$$\|u\|^2 = [(B+B^*)u, u] = 2[Bu, u] \le 2\|Bu\|\,\|u\|, \qquad u \in H,$$

hence $\|u\| \le 2\|Bu\|$. We infer that $Bi^*(\mathsf{D}(\mathcal{A}^*))$ is dense in H, hence V is closable. □

As a consequence we obtain the following result which motivates the study of elliptic operators on Wiener spaces in Chapter 4.

Theorem 2.16. *Suppose that P is analytic. Then we have the factorisation*

$$\mathcal{A}_\infty^* = V^*BV.$$

Proof. "$\subseteq$": Take $h \in \mathsf{D}(\mathcal{A}_\infty^*)$. Using Lemma 2.9 we find $(x_n^*)_n \subseteq \mathsf{D}(\mathcal{A}^*)$ such that $i_\infty^* x_n^* \to h$ and $\mathcal{A}_\infty^* i_\infty^* x_n^* = i_\infty^* \mathcal{A}^* x_n^* \to \mathcal{A}_\infty^* h$. The desired inclusion follows from the following two claims:

- $h \in \mathsf{D}(V)$ and $Vh = \lim_{n\to\infty} i^* x_n^*$.

Indeed, by Corollary 2.12,

$$\|V i_\infty^*(x_m^* - x_n^*)\| = 2[\mathcal{A}_\infty^* i_\infty^*(x_m^* - x_n^*), i_\infty^*(x_m^* - x_n^*)] \to 0.$$

Since V is closed, the claim follows.

- $BVh \in \mathsf{D}(V^*)$ and $V^*BVh = \mathcal{A}_\infty^* h$.

For $y^* \in E^*$ we have

$$\begin{aligned}[]
[\mathcal{A}_\infty^* i_\infty^* x_n^*, i_\infty^* y^*] &= [i_\infty^* \mathcal{A}^* x_n^*, i_\infty^* y^*] = \langle Q_\infty \mathcal{A}^* x_n^*, y^* \rangle \\
&= \langle iBi^* x_n^*, y^* \rangle = [Bi^* x_n^*, i^* y^*] = [Bi^* x_n^*, V i_\infty^* y^*].
\end{aligned}$$

Since $\mathcal{A}_\infty^* i_\infty^* x_n^* \to \mathcal{A}_\infty^* h$ and $i^* x_n^* \to Vh$, we infer that

$$[\mathcal{A}_\infty^* h, i_\infty^* y^*] = [BVh, V i_\infty^* y^*].$$

Since $i_\infty^*(E^*)$ is a core for $\mathsf{D}(V)$, it follows that $BVh \in \mathsf{D}(V^*)$ and $V^*BVh = \mathcal{A}_\infty^* h$.

"$\supseteq$": Since V is closed and $[Bu, u] = \frac{1}{2}\|u\|^2$ for $u \in H$, it follows that the (complexification of the) bilinear form

$$a : \mathsf{D}(V) \times \mathsf{D}(V) \subseteq H_\infty \times H_\infty \to \mathbb{R}, \qquad a(g,h) := [BVg, Vh],$$

is closed, densely defined and sectorial (see Section 5.5 for the definitions of these notions). Hence, by the theory of sesquilinear forms (see [141, Chapter 1]), the operator $-G$ defined by

$$\mathsf{D}(G) := \{h \in \mathsf{D}(V) : \exists f \in H \ \forall g \in \mathsf{D}(V) \ \ a(h,g) = [f,g]\}, \qquad Gh := f,$$

generates a C_0-semigroup on H. By definition, $G = V^*BV$. Since $-\mathcal{A}_\infty^* \subseteq -V^*BV$ and both operators generate a C_0-semigroup, it follows that the operators are equal. □

For selfadjoint Ornstein-Uhlenbeck semigroups P it has been shown in [33, 70] that that H is invariant under the dirft semigroup $\mathcal{S}$. The following theorem extends this result to analytic P.

Theorem 2.17. *If P is analytic, then $\mathcal{S}$ restricts to a bounded analytic C_0-semigroup $\mathcal{S}_H$ on H. The generators of $\mathcal{S}_H$ and $\mathcal{S}_H^*$ are the operators $-B^*VV^*$ and $-VV^*B$ respectively.*

Proof. We will use the fact, proved in Section 3.4 below, that the operator $-G := -VV^*B$ generates a bounded analytic C_0-semigroup $(T(t))_{t\geq 0}$ on H. To prove the first claim, it suffices to show that $iT^*(t) = \mathcal{S}(t)i$ for all $t \geq 0$.

Take $y^* \in \mathsf{D}(\mathcal{A}^*)$. By Lemma 2.9 and Theorem 2.16 we have $BVi_\infty^* y^* \in \mathsf{D}(V^*)$ and $V^*BVi_\infty^* y^* = i_\infty^* \mathcal{A}^* y^*$. This implies that $V^*BVi_\infty^* y^* \in \mathsf{D}(V)$, hence $Vi_\infty^* y^* \in \mathsf{D}(G)$ and

$$GVi_\infty^* y^* = Vi_\infty^* \mathcal{A}^* y^*.$$

For $\lambda > 0$ it follows that $(I + \lambda G)Vi_\infty^* y^* = Vi_\infty^*(I + \lambda\mathcal{A}^*)y^*$. Applying this to $y^* = (I + \lambda\mathcal{A}^*)^{-1}x^*$ for $x^* \in E^*$, we obtain

$$V(I + \lambda\mathcal{A}_\infty^*)^{-1} i_\infty^* x^* = Vi_\infty^*(I + \lambda\mathcal{A}^*)^{-1}x^* = (I + \lambda G)^{-1} Vi_\infty^* x^*.$$

Taking $\lambda = \frac{t}{n}$ and repeating this argument n times we obtain

$$V(I + \tfrac{t}{n}\mathcal{A}_\infty^*)^{-n} i_\infty^* x^* = (I + \tfrac{t}{n}G)^{-n} Vi_\infty^* x^*.$$

Passing to the limit $n \to \infty$ and using the closedness of V, it follows that $\mathcal{S}_\infty^*(t)i_\infty^* x^* \in \mathsf{D}(V)$ and

$$V\mathcal{S}_\infty^*(t)i_\infty^* x^* = T(t)Vi_\infty^* x^*.$$

In view of the identities

$$V\mathcal{S}_\infty^*(t)i_\infty^* x^* = Vi_\infty^*\mathcal{S}^*(t)x^* = i^*\mathcal{S}^*(t)x^*, \qquad T(t)Vi_\infty^* x^* = T(t)i^*x^*,$$

we infer that $T(t)i^* = i^*\mathcal{S}^*(t)$, thus by duality $iT^*(t) = \mathcal{S}(t)i$.

To complete the proof, it remains to show that $(VV^*B)^* = B^*VV^*$. This follows by combining Proposition 3.4 below with the fact that VV^* is selfadjoint. □

2.3 Symmetry

After the investigation of analyticity of Ornstein-Uhlenbeck semigroups in the previous section, we now turn to the stronger property of symmetry.

The following result gives a characterisation of selfadjointness of the Ornstein-Uhlenbeck semigroup in terms of the noise Q and the drift semigroup $\mathcal{S}$. The result has been proved in [33, Theorem 2.4] a Hilbert space setting and extended to Banach spaces in [70, Theorem 4.5]. Here we present a partly different proof which employs Theorem 2.17. As before we assume that the operator $i : H \to E$ is injective.

Proposition 2.18 (Characterisation of symmetry). *The following assertions are equivalent:*

(1) *For any $x^* \in E^*$ and $t \geq 0$ we have $Q\mathcal{S}^*(t) = \mathcal{S}(t)Q$;*
(2) *The semigroup $\mathcal{S}$ maps $i(H)$ into itself and the restricted semigroup $\mathcal{S}_H$ is selfadjoint on H;*
(3) *For any $x^* \in E^*$ and $t \geq 0$ we have $Q_\infty\mathcal{S}^*(t) = \mathcal{S}(t)Q_\infty$;*
(4) *The semigroup $\mathcal{S}_\infty$ is selfadjoint on H_∞;*
(5) *The semigroup P is selfadjoint on $L^2(\mu_\infty)$.*

Proof. (1) $\Rightarrow$ (3): For $t \geq 0$ and $x^*, y^* \in E^*$ we obtain

$$\begin{aligned}\langle Q_\infty\mathcal{S}^*(t)x^*, y^*\rangle &= \int_0^\infty \langle Q\mathcal{S}^*(s+t)x^*, \mathcal{S}^*(s)y^*\rangle\, ds \\ &= \int_0^\infty \langle Q\mathcal{S}^*(s)x^*, \mathcal{S}^*(s+t)y^*\rangle\, ds \\ &= \langle Q_\infty x^*, \mathcal{S}^*(t)y^*\rangle \\ &= \langle \mathcal{S}(t)Q_\infty x^*, y^*\rangle,\end{aligned}$$

from which we infer that $Q_\infty\mathcal{S}^*(t) = \mathcal{S}(t)Q_\infty$.

(3) $\Rightarrow$ (4): For $t \geq 0$ and $x^*, y^* \in E^*$ we have

$$\begin{aligned}[\mathcal{S}_\infty^*(t)i_\infty^*x^*, i_\infty^*y^*] &= \langle Q_\infty\mathcal{S}^*(t)x^*, y^*\rangle = \langle \mathcal{S}(t)Q_\infty x^*, y^*\rangle = \langle Q_\infty x^*, \mathcal{S}^*(t)y^*\rangle \\ &= [i_\infty^*x^*, \mathcal{S}_\infty^*(t)i_\infty^*y^*] = [\mathcal{S}_\infty(t)i_\infty^*x^*, i_\infty^*y^*].\end{aligned}$$

Since $i_\infty^*(E^*)$ is dense in H_∞, the result follows.

(4) $\Rightarrow$ (2): Since $\mathcal{S}_\infty$ is selfadjoint and therefore analytic, the invariance of H under $\mathcal{S}$ follows from Theorem 2.17. Moreover, for $x^*, y^* \in \mathsf{D}(\mathcal{A}^*)$, Lemma 2.9 implies that

$$\begin{aligned}2[Bi^*x^*, i^*y^*] &= 2[\mathcal{A}_\infty^* i_\infty^*x^*, i_\infty^*y^*] \\ &= [\mathcal{A}_\infty^* i_\infty^*x^*, i_\infty^*y^*] + [i_\infty^*x^*, \mathcal{A}_\infty^* i_\infty^*y^*] \\ &= [Vi_\infty^*x^*, Vi_\infty^*y^*] \\ &= [i^*x^*, i^*y^*].\end{aligned}$$

The argument in the proof of Lemma 2.9 shows that $i^*(\mathsf{D}(\mathcal{A}^*))$ is dense in H. Consequently, the computation above implies that $B = \frac{1}{2}I$, hence $\mathcal{A}_H = \mathcal{A}_H^* = \frac{1}{2}VV^*$ by Theorem 2.17, and therefore $\mathcal{S}_H$ is selfadjoint.

(2) $\Rightarrow$ (1): For $x^*, y^* \in E^*$ we have

$$\begin{aligned}\langle Q\mathcal{S}^*(t)x^*, y^*\rangle &= [i^*\mathcal{S}^*(t)x^*, i^*y^*] = [\mathcal{S}_H^*(t)i^*x^*, i^*y^*] \\ &= [\mathcal{S}_H^*(t)i^*y^*, i^*x^*] = \langle Q\mathcal{S}^*(t)y^*, x^*\rangle \\ &= \langle Qx^*, \mathcal{S}^*(t)y^*\rangle = \langle \mathcal{S}(t)Qx^*, y^*\rangle,\end{aligned}$$

which gives the desired result.

(4) $\Leftrightarrow$ (5): This follows immediately from the identification of P as the second quantisation of $\mathcal{S}_\infty^*$ in Theorem 2.8. □

2.4 Notes

A MathSciNet search on "Ornstein-Uhlenbeck" gives 1647 hits (on 11th March 2009), which makes a complete overview of the literature impossible. We present a subjective and incomplete selection.

Finite dimensional Ornstein-Uhlenbeck processes have been introduced by the physicists Ornstein and Uhlenbeck [162] in their study of the kinetic theory of gases.

In infinite dimensions, the Ornstein-Uhlenbeck semigroup $(\Gamma(e^{-t}I))_{t\geq 0}$ seems to have first appeared in the PhD-thesis of Piech [143]. It plays an important role in the mathematical physics literature, such as [67, 137, 155] and many other works where the generator is known as the number operator.

With the advent of Malliavin calculus [111], the Ornstein-Uhlenbeck semigroup became one of the central objects in stochastic analysis (see also Stroock [157], P.A. Meyer [126, 127] and many other papers).

Later Ornstein-Uhlenbeck semigroups appeared in a wide range of applications: in the work of Holley and Stroock [80]) on interacting particle systems; Walsh used them as a model for neuronal activity in [168]; they also appear in Kolmogorov's kinematic approach to turbulence (see Carmona [24] and Avellaneda and Majda [11]).

Non-symmetric Ornstein-Uhlenbeck semigroups have been studied extensively during the last 15 years, in particular by the Polish and Italian schools. We refer to the books by Da Prato and Zabczyk [45, 46, 47], the sequence of papers by Chojnowska-Michalik and Goldys [29, 30, 31, 32, 33], and among many possible references we quote [15, 16, 44, 65, 70, 102, 124, 125, 132, 133].

The identification of non-symmetric Ornstein-Uhlenbeck semigroups as second quantised operators has been proved in [29] (see also [15]).

There exists examples of Ornstein-Uhlenbeck semigroups which fail to be analytic, although $\mathcal{S}_\infty$ is an analytic semigroup satisfying $\|\mathcal{S}_\infty(t)\|_{\mathcal{L}(H_\infty)} \leq 1$ for all $t \geq 0$. Of course, Theorem 2.14 implies that in this situation $\mathcal{S}_\infty$ fails to be contractive on any sector of strictly positive angle. An example of this phenomenon in $E = \mathbb{R}^2$ has been constructed in [64]. It follows from Theorem 2.14 that Ornstein-Uhlenbeck semigroups are analytic if E is finite dimensional and the noise is non-degenerate, in the sense $\mathsf{N}(Q) = \{0\}$.

All results that we presented in this chapter are known, with the exception of the H-invariance of analytic Ornstein-Uhlenbeck semigroups (Theorem 2.17). This generalises a result from [70]. A different proof of Proposition 2.15 can be found in the same paper.

3

Perturbed Hodge-Dirac Operators on Hilbert Spaces

In this chapter we discuss an operator theoretic framework which underlies many results in harmonic analysis. It has been developed and used by Axelsson, Keith, and M$^{\text{c}}$Intosh [12] as a new approach to the famous Kato square root problem. The basic philosophy is as follows.

Suppose that one is interested in the boundedness of a (singular integral) operator, say, the Riesz transform $R = \nabla L^{-1/2}$ associated with a second order differential operator L on $L^2(\mathbb{R}^n)$. We would like to apply the theory of H^∞-calculus to this problem, but unfortunately R does not belong to the functional calculi of ∇ and L. To circumvent this difficulty, we consider the Hodge-Dirac operator associated with ∇ and L. This is a bisectorial operator T with the property that R belongs to its functional calculus: $R = \psi(T)$ for some bounded analytic function ψ defined on a bisector. In many examples ψ is the sgn-function. By the theory of H^∞-calculus, to prove the boundedness of R, it therefore suffices to prove square function estimates for T.

At this point, one usually needs harmonic analysis to prove the square function estimates in each particular case, but the abstract idea works in a very general setting.

In Chapter 4 we will apply the Hilbert space theory from this chapter to the first Wiener-Itô chaos in the study of L^p-estimates for elliptic operators on Wiener spaces.

3.1 Hodge-Dirac operators on Hilbert spaces

In this section we will present some operator theoretic aspects of a class of abstract Hodge-Dirac operators on Hilbert spaces following [12]. Before turning to the general setup, we collect some elementary facts on coercive operators and compositions of closed operators.

Coercive operators

Let $\mathcal{U}$ be a subspace of a Hilbert space H.

Definition 3.1. *A bounded operator $B \in \mathcal{L}(H)$ is said to be* coercive *on $\mathcal{U}$, if there exists $\kappa > 0$ such that*

$$\mathrm{Re}\,[Bu, u] \geq \kappa \|u\|^2, \qquad u \in \mathcal{U}. \tag{3.1}$$

An operator which is coercive on H is simply called coercive.

We define the angle of coercivity of B by

$$\omega_c(B) := \sup_{u \in \mathcal{U}} \arg[Bu, u]. \tag{3.2}$$

Note that $\omega_c(B) < \frac{1}{2}\pi$.

Remark 3.2. It follows directly from the definition that an operator B which is coercive on $\mathcal{U}$ has the following properties:

(i) B is coercive on $\overline{\mathcal{U}}$.
(ii) B^* is coercive on $\overline{\mathcal{U}}$.
(iii) $\|Bu\| \eqsim \|u\| \eqsim \|B^*u\|$ for $u \in \overline{\mathcal{U}}$. This follows from the fact that

$$\|u\|^2 \lesssim \mathrm{Re}\,[Bu, u] \leq \|Bu\|\,\|u\|, \qquad u \in \overline{\mathcal{U}},$$

hence $\|u\| \lesssim \|Bu\|$. The estimate $\|u\| \lesssim \|B^*u\|$ is obtained similarly.

Compositions of closed operators

Let X, Y, Z be Banach spaces. Compositions of operators are defined in the following "naive" way. For operators $A : \mathsf{D}(A) \subseteq Y \to Z$ and $B : \mathsf{D}(B) \subseteq X \to Y$ we define

$$\begin{aligned} \mathsf{D}(AB) &:= \{x \in \mathsf{D}(B) : Bx \in \mathsf{D}(A)\}, \\ (AB)x &:= A(Bx), \quad x \in \mathsf{D}(AB). \end{aligned}$$

The following result can be found in many textbooks.

Lemma 3.3. *Let X and Y be reflexive Banach spaces and let $A : \mathsf{D}(A) \subseteq X \to Y$ be closed and densely defined. Then A^* is closed and densely defined, and $A^{**} = A$.*

Proposition 3.4. *Let X and Y be reflexive Banach spaces, let $A : \mathsf{D}(A) \subseteq X \to Y$ be closed and densely defined, and let $S \in \mathcal{L}(X)$ and $T \in \mathcal{L}(Y)$ satisfy*

$$\|S^*x^*\| \eqsim \|x^*\|, \quad x^* \in \mathsf{R}(A^*), \qquad \|Ty\| \eqsim \|y\|, \quad y \in \mathsf{R}(A).$$

Then TAS is closed and densely defined, and $(TAS)^* = S^*A^*T^*$.

Proof. We will first prove the following claim: let $C : \mathsf{D}(C) \subseteq X \to Y$ be closed and densely defined, and let $R \in \mathcal{L}(Y)$ satisfy $\|Ry\| \eqsim \|y\|$ for $y \in \mathsf{R}(C)$. Then

$$RC \text{ is closed and densely defined, and } (RC)^* = C^* R^*. \tag{3.3}$$

First note that $\mathsf{D}(RC) = \mathsf{D}(C)$ is dense in X. To prove that RC is closed, suppose that and $x_n \to x$ and $RCx_n \to y$. Since $\|Cx_n - Cx_m\| \lesssim \|RCx_n - RCx_m\|$, it follows that Cx_n converges to some $\tilde{y} \in Y$. Since C is closed, it follows that $x \in \mathsf{D}(C)$ and $Cx = \tilde{y}$, hence $x \in \mathsf{D}(RC)$ and $RCx = \lim_{n\to\infty} RCx_n = y$, which shows that RC is closed.

It remains to show that $(RC)^* = C^*R^*$. Take $y^* \in \mathsf{D}(RC)^*$ and put $x^* := (RC)^* y^*$. For all $x \in \mathsf{D}(RC) = \mathsf{D}(C)$ we have $\langle Cx, R^*y^*\rangle = \langle RCx, y^*\rangle = \langle x, x^*\rangle$, hence $R^*y^* \in \mathsf{D}(C^*)$ and $C^*R^*y^* = x^*$.

Conversely, take $y^* \in \mathsf{D}(C^*R^*)$ and put $x^* := C^*R^*y^*$. For all $x \in \mathsf{D}(C) = \mathsf{D}(RC)$ we have $\langle RCx, y^*\rangle = \langle Cx, R^*y^*\rangle = \langle x, x^*\rangle$, hence $y^* \in \mathsf{D}((RC)^*)$ and $(RC)^*y^* = x^*$. This completes the proof of the claim.

Applying (3.3) to $R = S^*$ and $C = A^*$, we obtain that S^*A^* is closed and densely defined, and $(S^*A^*)^* = AS$. By Lemma 3.3 it follows that AS is closed and densely defined and $(AS)^* = S^*A^*$.

By another application of (3.3), this time to $C = AS$ and $R = T$, we obtain that TAS is closed and densely defined, and $TAS = (AS)^*T^* = S^*A^*T^*$. $\square$

The general setup

We will now present the setup in which we will work throughout this chapter. Let H and $\underline{H}$ be separable Hilbert spaces. We are given the following operators:

- $V : \mathsf{D}(V) \subseteq H \to \underline{H}$ is closed and densely defined,
- $B_1 \in \mathcal{L}(H)$ is coercive on $\mathsf{R}(V^*)$,
- $B_2 \in \mathcal{L}(\underline{H})$ is coercive on $\mathsf{R}(V)$.

In our application we have $B_1 = I_H$, but the duality argument in the proof of Proposition 3.9 forces us to consider the general case.

These operators can be naturally extended to the direct sum $\mathscr{H} := H \oplus \underline{H}$ by defining

$$\widehat{V} := \begin{bmatrix} 0 & 0 \\ V & 0 \end{bmatrix}, \qquad \widehat{B}_1 := \begin{bmatrix} B_1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad \widehat{B}_2 := \begin{bmatrix} 0 & 0 \\ 0 & B_2 \end{bmatrix}.$$

The operator $\widehat{V}$ defined on its natural domain $\mathsf{D}(V) \oplus \underline{H}$ is a closed operator on $\mathscr{H}$. The operators $\widehat{B}_1$ and $\widehat{B}_2$ are bounded on $\mathscr{H}$, and coercive on $\overline{\mathsf{R}(\widehat{V}^*)}$ and $\overline{\mathsf{R}(\widehat{V})}$ respectively. Moreover, $\omega_c(\widehat{B}_1) = \omega_c(B_1)$ and $\omega_c(\widehat{B}_2) = \omega_c(B_2)$.

We will consider the perturbed operators

$$V_B := B_2^* V B_1^*, \qquad \widehat{V}_B := \widehat{B}_2^* \widehat{V} \widehat{B}_1^* = \begin{bmatrix} 0 & 0 \\ B_2^* V B_1^* & 0 \end{bmatrix}.$$

Proposition 3.4 implies that their adjoints are given by

$$V_B^* := B_1 V^* B_2, \qquad \widehat{V}_B^* := \widehat{B}_1 \widehat{V}^* \widehat{B}_2 = \begin{bmatrix} 0 & B_1 V^* B_2 \\ 0 & 0 \end{bmatrix}.$$

The perturbed *Hodge-Dirac operators* associated with the triple (V, B_1, B_2) are defined by

$$T_B := \widehat{V} + \widehat{V}_B^* = \begin{bmatrix} 0 & B_1 V^* B_2 \\ V & 0 \end{bmatrix}, \qquad T_B^* := \widehat{V}_B + \widehat{V}^* = \begin{bmatrix} 0 & V^* \\ B_2^* V B_1^* & 0 \end{bmatrix}.$$

3.2 The Hodge decomposition

In this section we will be concerned with decompositions of the Hilbert spaces H and $\underline{H}$ induced by perturbed Hodge-Dirac operators.

First we recall some general facts on decompositions. Let $\mathcal{U}$ and $\mathcal{V}$ be closed linear subspaces of a Banach space X. Recall that $X = \mathcal{U} \oplus \mathcal{V}$ means that for every $x \in X$ there exist unique elements $u \in \mathcal{U}$ and $v \in \mathcal{V}$ such that $x = u + v$. An equivalent way to state this is that $\mathcal{U} + \mathcal{V} = X$ and $\mathcal{U} \cap \mathcal{V} = \varnothing$. It is well known that this algebraic property is equivalent to the following topological one: there exists a bounded projection P on X such that $\mathsf{R}(P) = \mathcal{U}$ and $\mathsf{N}(P) = \mathsf{R}(I - P) = \mathcal{V}$.

If X is a Hilbert space and $\mathcal{U}$ and $\mathcal{V}$ are orthogonal subspaces satisfying $X = \mathcal{U} \oplus \mathcal{V}$, we write $X = \mathcal{U} \oplus^\perp \mathcal{V}$.

The following result follows from elementary Hilbert space theory.

Proposition 3.5. *Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be Hilbert spaces, and let $C : \mathscr{H}_1 \to \mathscr{H}_2$ be closed and densely defined. Then*

$$\mathscr{H}_1 = \mathsf{N}(C) \oplus^\perp \overline{\mathsf{R}(C^*)}, \qquad \mathscr{H}_2 = \mathsf{N}(C^*) \oplus^\perp \overline{\mathsf{R}(C)}.$$

Proof. The second decomposition is trivial. Since $C^{**} = C$, the first one follows immediately. □

Now we return to the setting of this chapter and prove a useful decomposition of the Hilbert spaces H and $\underline{H}$. In the unperturbed case where B_1 and B_2 are the identity operators on H and $\underline{H}$, the result follows immediately from Proposition 3.5, and the decompositions are orthogonal. It is remarkable that the result remains valid for general B_1 and B_2. However, in this perturbed case the decompositions are not orthogonal in general.

Proposition 3.6 (Hodge decomposition). *The following decompositions hold:*

$$\begin{cases} H = \mathsf{N}(V) \oplus \overline{\mathsf{R}(V_B^*)}, \\ \underline{H} = \mathsf{N}(V_B^*) \oplus \overline{\mathsf{R}(V)}. \end{cases} \tag{3.4}$$

As a consequence,

$$\mathscr{H} = \mathsf{N}(T_B) \oplus \overline{\mathsf{R}(\widehat{V})} \oplus \overline{\mathsf{R}(\widehat{V}_B^*)}. \tag{3.5}$$

The proof relies on the following lemma. For a subset $\mathcal{U}$ of a Banach space X we let $\mathcal{U}^\perp \subseteq X^*$ denote the annihilator of U, defined by

$$\mathcal{U}^\perp := \{x^* \in X^* : \langle u, x^* \rangle = 0 \ \forall u \in \mathcal{U}\}.$$

Lemma 3.7. *Let $\mathcal{U}$ and $\mathcal{V}$ be closed subspaces of a Banach space X. Then $X = \mathcal{U} \oplus \mathcal{V}$ if the following two estimates hold:*

$$\begin{cases} \|u\| \lesssim \|u+v\|, & u \in \mathcal{U},\ v \in \mathcal{V}, \\ \|u^*\| \lesssim \|u^* + v^*\|, & u^* \in \mathcal{U}^\perp,\ v^* \in \mathcal{V}^\perp. \end{cases} \tag{3.6}$$

Proof. Observe that (3.6) immediately implies that

$$\begin{cases} \|u\| + \|v\| \lesssim \|u+v\|, & u \in \mathcal{U},\ v \in \mathcal{V}, \\ \|u^*\| + \|v^*\| \lesssim \|u^* + v^*\|, & u^* \in \mathcal{U}^\perp,\ v^* \in \mathcal{V}^\perp. \end{cases} \tag{3.7}$$

The first inequality in (3.7) implies that for $x \in \mathcal{U} \cap \mathcal{V}$ we have $\|x\| + \|-x\| \lesssim 0$, hence $\|x\| = 0$. This shows that $\mathcal{U} \cap \mathcal{V} = \{0\}$.

By the same argument, the second inequality in (3.7) implies that $\mathcal{U}^\perp \cap \mathcal{V}^\perp = \{0\}$. Since $(\mathcal{U}+\mathcal{V})^\perp = \mathcal{U}^\perp \cap \mathcal{V}^\perp$, the Hahn-Banach theorem implies that $\mathcal{U}+\mathcal{V}$ is dense in X.

It remains to show $\mathcal{U}+\mathcal{V}$ is closed. Take $u_n \in \mathcal{U}$ and $v_n \in \mathcal{V}$ such that $(u_n + v_n)_n$ is a Cauchy sequence in X. The first inequality in (3.7) implies that $(u_n)_n$ and $(v_n)_n$ are Cauchy sequences as well. Since $\mathcal{U}$ and $\mathcal{V}$ are closed, it follows that $\lim_{n\to\infty} u_n + v_n \in \mathcal{U}+\mathcal{V}$, hence $\mathcal{U}+\mathcal{V}$ is closed. ☐

Proof (of Proposition 3.6). It is clear that (3.4) implies (3.5).

To prove the first decomposition in (3.4), we apply Lemma 3.7. In view of Proposition 3.5 it suffices to show that

$$\begin{cases} \|V_B^* u\| \lesssim \|V_B^* u + v\|, & u \in \mathsf{D}(V_B^*),\ v \in \mathsf{N}(V), \\ \|V^* u\| \lesssim \|V^* u + v\|, & u \in \mathsf{D}(V^*),\ v \in \mathsf{N}(V_B). \end{cases} \tag{3.8}$$

For $u \in \mathsf{D}(V_B^*)$ and $v \in \mathsf{N}(V)$ we obtain, using the coercivity of B_1 on $\mathsf{R}(V^*)$,

$$\begin{aligned} \|V_B^* u\|^2 \lesssim \|V^* B_2 u\|^2 &\lesssim \mathrm{Re}\,[B_1 V^* B_2 u, V^* B_2 u] \\ &= \mathrm{Re}\,[V_B^* u + v, V^* B_2 u] \lesssim \|V_B^* u + v\|\,\|V_B^* u\|. \end{aligned}$$

On the other hand, since $\mathsf{N}(V_B) = \mathsf{N}(V B_1^*)$ by Remark 3.2, for $u \in \mathsf{D}(V^*)$ and $v \in \mathsf{N}(V_B)$ we obtain

$$\|V^* u\|^2 \lesssim \mathrm{Re}\,[B_1 V^* u, V^* u] = \mathrm{Re}\,[B_1 V^* u, V^* u + v] \lesssim \|V^* u\|\,\|V^* u + v\|.$$

This proves the first decomposition in (3.4).

We argue similarly to prove the second decomposition in (3.4). It suffices to show that

$$\begin{cases} \|V u\| \lesssim \|V u + v\|, & u \in \mathsf{D}(V), v \in \mathsf{N}(V_B^*), \\ \|V_B u\| \lesssim \|V_B u + v\|, & u \in \mathsf{D}(V_B), v \in \mathsf{N}(V^*). \end{cases} \tag{3.9}$$

For $u \in \mathsf{D}(V)$ and $v \in \mathsf{N}(V_B^*) = \mathsf{N}(V^* B_2)$ we obtain

$$\|Vu\|^2 \lesssim \mathrm{Re}\,[B_2^* Vu, Vu] = \mathrm{Re}\,[B_2^* Vu, Vu + v] \lesssim \|Vu\|\,\|Vu + v\|,$$

whereas, for $u \in \mathsf{D}(V_B)$ and $v \in \mathsf{N}(V^*)$,

$$\begin{aligned}\|V_B u\|^2 \lesssim \|V B_1^* u\|^2 &\lesssim \mathrm{Re}\,[B_2^* V B_1^* u, V B_1^* u] \\ &= \mathrm{Re}\,[V_B u + v, V B_1^* u] \lesssim \|V_B u + v\|\,\|V_B u\|.\end{aligned}$$

This proves (3.9), hence the proof is complete. □

3.3 Bisectoriality of Hodge-Dirac operators

Our next aim is to show that T_B is bisectorial on $\mathscr{H}$. For this purpose we need the following lemma.

Lemma 3.8. *For every $h \in \overline{\mathsf{R}(\widehat{V}_B^*)}$ there exists a unique $g \in \overline{\mathsf{R}(\widehat{V}^* \widehat{B}_2)}$ satisfying $h = \widehat{B}_1 g$.*

Proof. Take $u_n \in \mathsf{D}(\widehat{V}^* \widehat{B}_2)$ such that $\widehat{B}_1 \widehat{V}^* \widehat{B}_2 u_n$ converges to h. Then

$$\begin{aligned}\|\widehat{V}^* \widehat{B}_2 (u_n - u_m)\|^2 &\lesssim \mathrm{Re}\,[\widehat{B}_1 \widehat{V}^* \widehat{B}_2 (u_n - u_m), \widehat{V}^* \widehat{B}_2 (u_n - u_m)] \\ &\le \|\widehat{B}_1 \widehat{V}^* \widehat{B}_2 (u_n - u_m)\|\,\|\widehat{V}^* \widehat{B}_2 (u_n - u_m)\|,\end{aligned}$$

hence $\|\widehat{V}^* \widehat{B}_2 (u_n - u_m)\| \lesssim \|\widehat{B}_1 \widehat{V}^* \widehat{B}_2 (u_n - u_m)\|$. We infer that $\widehat{V}^* \widehat{B}_2 u_n$ converges to some $g \in \mathscr{H}$, and $h = \widehat{B}_1 g$.

To show uniqueness, suppose that $g, g' \in \overline{\mathsf{R}(\widehat{V}^* \widehat{B}_2)}$ are such that $\widehat{B}_1 g = \widehat{B}_1 g'$. Then

$$\|g - g'\|^2 \lesssim \mathrm{Re}\,[\widehat{B}_1 (g - g'), g - g'] = 0,$$

hence $g = g'$. □

Now we can state the main result of this section.

Proposition 3.9. *The operator T_B is bisectorial on $\mathscr{H}$. Moreover, $\omega(T_B) \le \frac{1}{2}(\omega_c(B_1) + \omega_c(B_2))$.*

Proof. Let $\omega \in (\frac{1}{2}(\omega_c(B_1) + \omega_c(B_2)), \frac{1}{2}\pi)$, let $u \in \mathsf{D}(T_B)$, let $z \notin \overline{\Sigma_\omega}$, and put $f := (I + zT_B)u$. We use Proposition 3.6 to write

$$u = u_0 + u_1 + u_2, \qquad f = f_0 + f_1 + f_2 \qquad \in \mathsf{N}(T_B) \oplus \overline{\mathsf{R}(\widehat{V}_B^*)} \oplus \overline{\mathsf{R}(\widehat{V})}.$$

Since $f = u_0 + u_1 + u_2 + z\widehat{V} u_1 + z\widehat{V}_B^* u_2$, it follows that

$$f_0 = u_0, \qquad f_1 = u_1 + z\widehat{V}_B^* u_2, \qquad f_2 = u_2 + z\widehat{V} u_1.$$

Using Lemma 3.8 we may write $u_1 = \widehat{B}_1 \tilde{u}_1$ and $f_1 = \widehat{B}_1 \tilde{f}_1$ for certain $\tilde{u}_1, \tilde{f}_1 \in \overline{\mathsf{R}(\widehat{V}^* \widehat{B}_2)}$. Consequently, since $\widehat{B}_1$ is coercive on $\overline{\mathsf{R}(\widehat{V}^*)}$,

$$\tilde{f}_1 = \tilde{u}_1 + z\widehat{V}^* \widehat{B}_2 u_2.$$

It follows from these identities that

$$\Lambda := -\bar{z}[\tilde{u}_1, \widehat{B}_1 \tilde{u}_1] + z[\widehat{B}_2 u_2, u_2] = -\bar{z}[\tilde{f}_1, \widehat{B}_1 \tilde{u}_1] + z[\widehat{B}_2 u_2, f_2] \tag{3.10}$$

On the other hand, writing

$$\theta_1 := \arg[\tilde{u}_1, \widehat{B}_1 \tilde{u}_1], \qquad \theta_2 := \arg[\widehat{B}_2 u_2, u_2], \qquad \mu := \arg z,$$

we obtain

$$\begin{aligned}
|\Lambda| &\geq \mathrm{Im}\,(e^{-\frac{i}{2}(\theta_1+\theta_2)} \Lambda) \\
&= \mathrm{Im}\,\Big(e^{-\frac{i}{2}(\theta_1+\theta_2)} |z| \big(- e^{i(\theta_1 - \mu)} |[\tilde{u}_1, \widehat{B}_1 \tilde{u}_1]| + e^{i(\theta_2+\mu)} |[\widehat{B}_2 u_2, u_2]| \big) \Big) \\
&= |z| \mathrm{Im}\, \Big(- e^{i(\frac{1}{2}\theta_1 - \frac{1}{2}\theta_2 - \mu)} |[\tilde{u}_1, \widehat{B}_1 \tilde{u}_1]| + e^{i(-\frac{1}{2}\theta_1 + \frac{1}{2}\theta_2 + \mu)} |[\widehat{B}_2 u_2, u_2]| \Big) \\
&= |z| \sin(-\tfrac{1}{2}\theta_1 + \tfrac{1}{2}\theta_2 + \mu) \Big(|[\tilde{u}_1, \widehat{B}_1 \tilde{u}_1]| + |[\widehat{B}_2 u_2, u_2]| \Big).
\end{aligned} \tag{3.11}$$

Assuming that $\mathrm{Im}\, z > 0$ (the case $\mathrm{Im}\, z < 0$ can be treated in the same way), we use the fact that $|\frac{1}{2}\theta_1 - \frac{1}{2}\theta_2| < \omega$ to obtain

$$|z| \sin(-\tfrac{1}{2}\theta_1 + \tfrac{1}{2}\theta_2 + \mu) \geq |z| \sin(\mu - \omega) = \mathrm{dist}(z, \Sigma_\omega). \tag{3.12}$$

Combining (3.10), (3.11), and (3.12), we arrive at

$$| - \bar{z}[\tilde{f}_1, \widehat{B}_1 \tilde{u}_1] + z[\widehat{B}_2 u_2, f_2]| \geq \mathrm{dist}(z, \Sigma_\omega) \Big(|[\tilde{u}_1, \widehat{B}_1 \tilde{u}_1]| + |[\widehat{B}_2 u_2, u_2]| \Big).$$

Since $\tilde{u}_1 \in \overline{\mathsf{R}(\widehat{V}^*)}$ and $u_2 \in \mathsf{R}(\widehat{V})$, this implies that

$$\begin{aligned}
\|\tilde{u}_1\|^2 + \|u_2\|^2 &\lesssim |[\tilde{u}_1, \widehat{B}_1 \tilde{u}_1]| + |[\widehat{B}_2 u_2, u_2]| \\
&\lesssim \frac{|z|}{\mathrm{dist}(z, \Sigma_\omega)} \Big(\|\tilde{f}_1\| \, \|\tilde{u}_1\| + \|u_2\| \, \|f_2\| \Big).
\end{aligned}$$

Setting $u_2 = 0$ we obtain

$$\|u_1\| \eqsim \|\tilde{u}_1\| \lesssim \frac{|z|}{\mathrm{dist}(z, \Sigma_\omega)} \|\tilde{f}_1\| \eqsim \frac{|z|}{\mathrm{dist}(z, \Sigma_\omega)} \|f_1\|,$$

and similarly, putting $\tilde{u}_1 = 0$,

$$\|u_2\| \lesssim \frac{|z|}{\operatorname{dist}(z,\Sigma_\omega)}\|f_2\|.$$

Since the set

$$\left\{\frac{|z|}{\operatorname{dist}(z,\Sigma_\omega)} : z \notin \Sigma_{\omega'}\right\}$$

is bounded for each $\omega' \in (\omega, \frac{1}{2}\pi)$, the decomposition from Proposition 3.6 allows us to obtain:

$$\|u\| \eqsim \|u_0\| + \|u_1\| + \|u_2\| \lesssim \|f_0\| + \|f_1\| + \|f_2\| \eqsim \|f\|.$$

This estimate shows that $I + zT_B$ is injective and has closed range. Since $(I + zT_B)^* = (I + \bar{z}T_B^*)$ and T_B^* is of the same form as T_B (which can be seen by reversing the roles of H and $\underline{H}$ and replacing V by V^*), we find that $I + \bar{z}T_B^*$ is injective as well, hence $I + zT_B$ has dense range. Combining this with the resolvent estimate $\|u\| \lesssim \|f\|$, we obtain that T_B is bisectorial of angle ω. $\square$

3.4 Second order operators

We continue with the setup of the previous section, but specialise to the case $B_1 := I_H$ and we rename $B_2 =: B$. In the previous section we have shown that Hodge-Dirac operator

$$T_B := \begin{bmatrix} 0 & V^*B \\ V & 0 \end{bmatrix},$$

is bisectorial with $\omega(T_B) \le \frac{1}{2}\omega_c(B)$. Combining this with Proposition 5.29 we obtain that the operator

$$T_B^2 := \begin{bmatrix} V^*BV & 0 \\ 0 & VV^*B \end{bmatrix},$$

is sectorial with $\omega^+(T_B^2) \le \omega_c(B)$. By restriction we obtain that the operators

$$A := V^*BV : \mathsf{D}(A) \subseteq H \to H, \qquad \underline{A} := VV^*B : \mathsf{D}(\underline{A}) \subseteq \underline{H} \to \underline{H},$$

are sectorial on H and $\underline{H}$ respectively of angle $\omega(A) = \omega(\underline{A}) = \omega_c(B) < \frac{1}{2}\pi$. Consequently, the operators $-A$ and $-\underline{A}$ generate bounded analytic semigroups on Σ_θ^+ for all $\theta \in (0, \omega_c(B))$. Concerning A even more can be said:

Proposition 3.10. *The operator $-A$ generates an analytic C_0-semigroup on H which is contractive on $\Sigma^+_{\frac{1}{2}\pi-\omega_c(B)}$.*

Proof. See [141, Theorem 1.53]. $\square$

Proposition 3.11. *The operator A has a bounded H^∞-calculus and*

$$\omega^+_{H^\infty(A)} = \omega^+(A) < \frac{1}{2}\pi.$$

Proof. This follows from Theorem 5.50. $\square$

3.5 Kato's square root problem and functional calculus

Associated with the perturbed Hodge-Dirac operators considered in this chapter is the following square root problem: do we have equality of domains $\mathsf{D}(\sqrt{A}) = \mathsf{D}(V)$ and equivalence of norms $\|\sqrt{A}h\| \eqsim \|Vh\|$? In this section we will show that this property can be characterised by means of the H^∞-calculus for the operator $\underline{A}$.

Although the operator $\underline{A}$ does not necessarily have a bounded functional calculus, we always have the following square function estimate.

Proposition 3.12. *For $\psi \in H_0^\infty(\Sigma_\theta^+)$ with $\theta \in (\omega_c(B), \pi)$ we have*

$$\|\sqrt{A}h\| \eqsim \Big(\int_0^\infty \|\psi(t\underline{A})Vh\|^2 \frac{dt}{t}\Big)^{1/2}, \qquad h \in \mathsf{D}(A).$$

Proof. Take $\widetilde{\varphi} \in H_0^\infty(\Sigma_{2\theta}^+)$ and define $\varphi \in H_0^\infty(\Sigma_\theta)$ by $\varphi(z) := \widetilde{\varphi}(z^2)$. We obtain

$$\begin{aligned}
\|\sqrt{A}h\|^2 &\eqsim \int_0^\infty \|\widetilde{\varphi}(tA)\sqrt{A}h\|^2 \frac{dt}{t} && \text{(Proposition 3.11)}\\
&= \int_0^\infty \Big\|\widetilde{\varphi}(tT_B^2)\sqrt{T_B^2}\begin{bmatrix} h \\ 0 \end{bmatrix}\Big\|^2 \frac{dt}{t} \\
&\eqsim \int_0^\infty \Big\|\varphi(tT_B)\sqrt{T_B^2}\begin{bmatrix} h \\ 0 \end{bmatrix}\Big\|^2 \frac{dt}{t} && \text{(Proposition 5.32)}\\
&= \int_0^\infty \Big\|\operatorname{sgn}(tT_B)\varphi(tT_B)T_B\begin{bmatrix} h \\ 0 \end{bmatrix}\Big\|^2 \frac{dt}{t} && \text{(Proposition 5.30)}\\
&\eqsim \int_0^\infty \Big\|\varphi(tT_B)T_B\begin{bmatrix} h \\ 0 \end{bmatrix}\Big\|^2 \frac{dt}{t} && \text{(Corollary 5.39)}\\
&= \int_0^\infty \Big\|\varphi(tT_B)\begin{bmatrix} 0 \\ Vh \end{bmatrix}\Big\|^2 \frac{dt}{t} \\
&\eqsim \int_0^\infty \Big\|\widetilde{\varphi}(tT_B^2)\begin{bmatrix} 0 \\ Vh \end{bmatrix}\Big\|^2 \frac{dt}{t} && \text{(Proposition 5.32)}\\
&\eqsim \int_0^\infty \|\widetilde{\varphi}(t\underline{A})Vh\|^2 \frac{dt}{t}.
\end{aligned}$$

The extension to arbitrary $\psi \in H_0^\infty(\Sigma_\theta^+)$ follows from Corollary 5.39. $\square$

The next result is a variation of [9, Theorem 10.1]. We write $\underline{A}_* := VV^*B^*$ and we let $\underline{S}_*$ denote the semigroup generated by $-\underline{A}_*$.

Theorem 3.13. *The following assertions are equivalent:*

(1) $\mathsf{D}(\sqrt{A}) \subseteq \mathsf{D}(V)$ *with* $\|\sqrt{A}h\| \gtrsim \|Vh\|$, $h \in \mathsf{D}(\sqrt{A})$;
(2) $\underline{A}$ *satisfies a square function estimate on* $\overline{\mathsf{R}(V)}$:

$$\Big(\int_0^\infty \|t\underline{A}\,\underline{S}(t)u\|^2 \frac{dt}{t}\Big)^{1/2} \gtrsim \|u\|;$$

(3) $\mathsf{D}(\sqrt{A^*}) \supseteq \mathsf{D}(V)$ *with* $\|\sqrt{A^*}h\| \lesssim \|Vh\|$, $h \in \overline{\mathsf{D}(V)}$*;*
(4) $\underline{A}_*$ *satisfies a square function estimate on* $\overline{\mathsf{R}(V)}$*:*

$$\Big(\int_0^\infty \|t\underline{A}_*\underline{S}_*(t)u\|^2\,\frac{dt}{t}\Big)^{1/2} \lesssim \|u\|.$$

As a consequence, the following assertions are equivalent:

(1′) $\mathsf{D}(\sqrt{A}) = \mathsf{D}(V)$ *with equivalence of norms* $\|\sqrt{A}h\| \eqsim \|Vh\|$*;*
(2′) $\underline{A}$ *admits a bounded* H^∞*-functional calculus on* $\overline{\mathsf{R}(V)}$*;*
(3′) $\mathsf{D}(\sqrt{A^*}) = \mathsf{D}(V)$ *with equivalence of norms* $\|\sqrt{A^*}h\| \eqsim \|Vh\|$*;*
(4′) $\underline{A}_*$ *admits a bounded* H^∞*-functional calculus on* $\overline{\mathsf{R}(V)}$*.*

Proof. The equivalences (1) ⇔ (2) and (3) ⇔ (4) follow from Proposition 3.12 with $\psi(z) = ze^{-z}$, taking into account that $\mathsf{D}(A)$ is a core for both $\mathsf{D}(\sqrt{A})$ and $\mathsf{D}(V)$.

We will now show that (1) implies (3). Taking Proposition 5.20 into account, and using and the fact that $\mathsf{D}(A)$ is a core for $\mathsf{D}(\sqrt{A})$, we infer that the collection of all $\widetilde{f} \in H$ of the form $\widetilde{f} = \widetilde{f}_0 + \sqrt{A}f$ with $\widetilde{f}_0 \in \mathsf{N}(\sqrt{A})$ and $f_1 \in \mathsf{D}(A)$ is dense in H. Using that $\widetilde{f}_0 \in \mathsf{N}(\sqrt{A})$ and $\|\sqrt{A}f\| \lesssim \|\widetilde{f}\|$, we obtain for any $g \in \mathsf{D}(A^*)$,

$$\begin{aligned}
\|\sqrt{A^*}g\| &= \sup_{\|\widetilde{f}_0+\sqrt{A}f\|\le 1} |\langle \sqrt{A^*}g, \widetilde{f}_0 + \sqrt{A}f\rangle| \\
&\lesssim \sup_{\|\sqrt{A}f\|\le 1} |\langle \sqrt{A^*}g, \sqrt{A}f\rangle| \lesssim \sup_{\|Vf\|\le 1} |\langle A^*g, f\rangle| \\
&= \sup_{\|Vf\|\le 1} |\langle B^*Vg, Vf\rangle| \le \sup_{\|Vf\|\le 1} \|B\|\,\|Vg\|\,\|Vf\| \\
&= \|B\|\,\|Vg\|.
\end{aligned}$$

Since $\mathsf{D}(A^*)$ is a core for $\mathsf{D}(\sqrt{A^*})$ and $\mathsf{D}(V)$, we obtain (3).

The reverse implication (3) ⇒ (1) is obtained by reversing the roles of A and A^*.

The equivalences (1′)⇔ (2′) and (3′) ⇔ (4′) follow from Proposition 3.12 and Theorem 5.40, and the equivalence of (1′) and (3′) is obtained by applying (1) ⇔ (3) to both A and A^*. □

It is possible to give additional equivalent conditions in terms of the operator $\underline{A}$.

Proposition 3.14. *The assertions* (1)–(4) *of Theorem 3.13 are equivalent to*

(5) $\mathsf{D}(\sqrt{\underline{A}}) \supseteq \mathsf{D}(V^*B)$ *with* $\|\sqrt{\underline{A}}u\| \lesssim \|V^*Bu\|$, $u \in \mathsf{D}(V^*B)$*;*
(6) $\mathsf{D}(\sqrt{\underline{A}_*}) \supseteq \mathsf{D}(V^*B^*)$ *with* $\|\sqrt{\underline{A}_*}u\| \lesssim \|V^*B^*u\|$, $u \in \mathsf{D}(V^*B^*)$.

Similarly, the conditions (1′)–(4′) *of Theorem 3.13 are equivalent to*

(5′) $\mathsf{D}(\sqrt{\underline{A}}) = \mathsf{D}(V^*B)$ *with* $\|\sqrt{\underline{A}}u\| \eqsim \|V^*Bu\|$, $u \in \mathsf{D}(V^*B)$.

(6′) $\mathsf{D}(\sqrt{\underline{A}_*}) = \mathsf{D}(V^*B)$ *with* $\|\sqrt{\underline{A}_*}u\| \eqsim \|V^*B^*u\|$, $u \in \mathsf{D}(V^*B^*)$.

Proof. To see that (1) implies (5), note that for $h \in \mathsf{D}(A)$ we have

$$\|(V^*B)Vh\| = \|Ah\| \gtrsim \|V\sqrt{A}h\| = \|\sqrt{\underline{A}}Vh\|.$$

Since $V(\mathsf{D}(A))$ is a core for both $\mathsf{D}(V^*B)$ and $\mathsf{D}(\sqrt{\underline{A}})$, (5) follows. The converse implication that (5) implies (1) is proved similarly.

Replacing B by B^*, we obtain the equivalence of (3) and (6). The equivalence of the primed statements follows from the same argument. □

Remark 3.15. One might ask whether the equivalent conditions in the theorem above are always fulfilled. This is not the case. A counterexample has been constructed by McIntosh [121] by gluing together finite dimensional examples in a clever way.

One might wonder if the assertions are always fulfilled when we rule out such "artificial" counterexamples and restrict ourselves to a class of more natural and physically meaningful operators such as in the following example.

Example 3.16 (Kato's square root problem).

Consider the following situation: let $H = L^2(\mathbb{R}^n)$, let $\underline{H} = L^2(\mathbb{R}^n; \mathbb{R}^n)$, let $V := \nabla : W^{1,2}(\mathbb{R}^n) \subseteq H \to \underline{H}$ be the gradient, and let $b \in L^\infty(\mathbb{R}^n; M_n(\mathbb{C}))$ be a matrix-valued function satisfying $[b(x)\xi, \xi] \geq \kappa|\xi|^2$ for some $\kappa > 0$ and all $x \in \mathbb{R}^n$ and $\xi \in \mathbb{R}^n$. Consider the multiplication operator $B \in \mathcal{L}(L^2(\mathbb{R}^n; \mathbb{R}^n))$ defined by $(Bu)(x) := b(x)u(x)$ for $x \in \mathbb{R}^n$.

In this case $A := V^*BV = -\nabla \cdot B\nabla$ is a second order elliptic differential operator with L^∞-coefficients. The question whether

$$\mathsf{D}(\sqrt{A}) = \mathsf{D}(V)$$

is the famous Kato square root problem, which remained unsolved for several decades. In [8] it has been solved positively by Auscher, Hofmann, Lacey, McIntosh, and Tchamitchian.

3.6 Notes

As mentioned before, the operator theoretic framework described in this chapter has been developed by Axelsson, Keith, and McIntosh [12]. Some of the ideas can be traced back to earlier papers such as [9, 10]. It is shown in [12] that this framework provides a unified view on many results in harmonic analysis, including the Cauchy integral on Lipschitz curves and surfaces and the Kato square root problem. Of course, the actual proofs of these results involve deep harmonic analysis.

We have chosen to present a slightly less general framework compared to [12], in order to make the application to elliptic operators in Wiener spaces more streamlined.

The proof of Proposition 3.9 is taken from [12]. Theorem 3.13 is a variation of a result in [9]. The proof that we present here, based on Proposition 3.12, has been demonstrated to us by M^{c}Intosh.

4

L^p-Theory for Elliptic Operators on Wiener Spaces

In this chapter we consider a class of elliptic operators on Wiener spaces, which contains the Ornstein-Uhlenbeck operators considered in Chapter 2. These operators will be studied in an L^p-setting. Our first main result (Theorem 4.37) provides necessary and sufficient conditions for the boundedness of the Riesz transforms associated with these operators. Our second main result (Theorem 4.42) gives a characterisation of their L^p-domains.

4.1 Elliptic operators on Wiener spaces

We begin by introducing a class of elliptic operators on Wiener spaces.

The setup

Let us present the setup in which we will work throughout this chapter. This setup is an extension of the framework considered in Chapter 3. Our data are the following:

- (E, H, μ) is an abstract Wiener space.
- $\underline{H}$ is a real separable Hilbert space.
- V is a closed and densely defined linear operator from H into $\underline{H}$.
- B is a bounded operator on $\underline{H}$ which is coercive on $\overline{\mathsf{R}(V)}$, i.e., there exists $\kappa > 0$ such that

$$[Bu, u] \geq \kappa \|u\|^2, \qquad u \in \overline{\mathsf{R}(V)}. \tag{4.1}$$

We have seen in Chapter 3 that it is natural to introduce the first order operator

$$T_B := \begin{bmatrix} 0 & V^*B \\ V & 0 \end{bmatrix} \text{ on } \mathscr{H} := H \oplus \underline{H},$$

and the second order operators

$$A := V^*BV \text{ on } H, \qquad \underline{A} := VV^*B \text{ on } \underline{H}.$$

The operators $-A$ and $-\underline{A}$ are the generators of bounded analytic C_0-semigroups

$$(S(t))_{t\geq 0} \subseteq \mathcal{L}(H), \qquad (\underline{S}(t))_{t\geq 0} \subseteq \mathcal{L}(\underline{H}).$$

Having the extra structure of an abstract Wiener space and a gradient D_V, it is natural to introduce additional operators. Fix $1 < p < \infty$. We have seen in Theorem 1.33 that the gradient

$$D_V : \mathsf{D}_p(D_V) \subseteq L^p(\mu) \to L^p(\mu; \underline{H})$$

is a closed and densely defined operator. We let D_V^* denote the adjoint of the operator $D_V : \mathsf{D}_{p'}(D_V) \subseteq L^{p'}(\mu) \to L^{p'}(\mu; \underline{H})$, where $\frac{1}{p} + \frac{1}{p'} = 1$, thus

$$D_V^* : \mathsf{D}_p(D_V^*) \subseteq L^p(\mu; \underline{H}) \to L^p(\mu).$$

We consider the first order operator

$$\Pi_B := \begin{bmatrix} 0 & D_V^*B \\ D_V & 0 \end{bmatrix}$$

on the space

$$\mathcal{L}^p(\mu) := L^p(\mu) \oplus L^p(\mu; \underline{H}),$$

and the second order operators

$$L := D_V^*BD_V \text{ on } L^p(\mu), \qquad \underline{L} := D_V D_V^*B \text{ on } L^p(\mu; \underline{H}).$$

For $p = 2$, this construction is a special case of the construction in Chapter 3. The results presented there imply that the operator Π_B is bisectorial and the operators L and $\underline{L}$ are sectorial. For $1 < p < \infty$, $p \neq 2$, the situation is more delicate. A detailed analysis of this situation is the topic of the next section, where L^p-(bi)sectoriality results will be proved. For the moment we have the following result for the first order operator Π_B.

Lemma 4.1. *Let $1 < p < \infty$. The operators $D_V^*B : \mathsf{D}(D_V^*B) \subseteq L^p(\mu; \underline{H}) \to L^p(\mu)$ and $B^*D_V : \mathsf{D}(B^*D_V) \subseteq L^{p'}(\mu) \to L^{p'}(\mu; \underline{H})$ are closed and densely defined operators satisfying*

$$(D_V^*B)^* = B^*D_V, \qquad D_V^*B = (B^*D_V)^*.$$

As a consequence, the operator $\Pi_B : \mathsf{D}(\Pi_B) \subseteq \mathcal{L}^p(\mu) \to \mathcal{L}^p(\mu)$ is closed and densely defined. Its adjoint is the closed and densely defined operator on $\mathcal{L}^{p'}(\mu)$ given by

$$\Pi_B^* = \begin{bmatrix} 0 & D_V^* \\ B^*D_V & 0 \end{bmatrix}.$$

Proof. Since $(D_V^*)^* = D_V$ and $\|B^* D_V f\|_p \eqsim \|D_V f\|_p$ for $f \in \mathsf{D}_p(D_V)$, the result is a consequence of Proposition 3.4. $\square$

In the remainder of this section we give a rigourous L^p-definition of the second order operators L and $\underline{L}$.

The operator L in $L^p(\mu)$

In $L^2(\mu)$, it follows from the theory presented in Section 3.4 that the operator $L := D_V^* B D_V$ is sectorial of angle $\omega^+(L) = \omega_c(B)$. In the present section we will rigorously define L as a closed and densely defined operator acting in $L^p(\mu)$, for $1 < p < \infty$.

Lemma 4.2. *Identifying H with its image $\phi(H)$ in $L^2(\mu)$, A is the part of L in H.*

Proof. Let $h \in \mathsf{D}(A)$. Then $h \in \mathsf{D}(V)$ and $BVh \in \mathsf{D}(V^*)$. It follows that $\phi_h \in \mathsf{D}(D_V)$ and $BD_V\phi_h = 1 \otimes BVh$. Lemma 1.32 implies that $BD_V\phi_h \in \mathsf{D}(D_V^*)$ and $D_V^* BD_V \phi_h = 1 \otimes Ah$. Denoting the part of L in H by L^H for the moment, this argument shows that $A \subseteq L^H$.

On the other hand, if $\phi_h \in \mathsf{D}(L^H)$, then $\phi_h \in \mathsf{D}(L)$ and $L\phi_h = \phi_{h'}$ for some $h' \in H$. Hence for all $g \in \mathsf{D}(V)$ we obtain

$$[BVh, Vg] = [BD_V\phi_h, D_V\phi_g] = [L\phi_h, \phi_g] = [\phi_{h'}, \phi_g] = [h', g].$$

It follows that $h \in \mathsf{D}(A)$ and $[Ah, g] = [h', g]$. This shows that $Ah = h'$, and we have proved the opposite inclusion $A \supseteq L^H$. $\square$

The next result identifies P as the second quantisation of S.

Theorem 4.3. *For all $t \geq 0$ we have $P(t) = \Gamma(S(t))$.*

Proof. We recall from Lemma 4.2 that $P(t)\phi_h = S(t)h$ for all $h \in H$.

First we check that for all $h \in \mathsf{D}(A)$, the functions $E_h \in L^2(\mu)$ are in the domains of L and $\widetilde{L}$, where $-\widetilde{L}$ is the generator of $\Gamma(S)$, and that both generators agree on those functions. Using (1.16) and Lemma 1.32 we obtain

$$\begin{aligned} LE_h &= D_V^* BD_V E_h \\ &= D_V^*(E_h \otimes BVh) \\ &= E_h \phi_{V^* BVh} - [BVh, Vh]E_h \\ &= (\phi_{Ah} - [Ah, h])E_h, \end{aligned}$$

while on the other hand, using (1.6) and (1.12) combined with a simple approximation argument, we have

$$
\begin{aligned}
\widetilde{L}E_h &= \lim_{t\downarrow 0} \tfrac{1}{t}(E_{S(t)h} - E_h)\\
&= E_h \frac{d}{dt}\Big|_{t=0}\Big(\phi_{S(t)h} - \frac{1}{2}\|S(t)h\|^2\Big)\\
&= (\phi_{Ah} - [Ah,h])E_h.
\end{aligned}
$$

The set $\mathrm{lin}\{E_h : h \in \mathsf{D}(A)\}$ is dense in $L^2(\mu)$ and invariant under the semigroup $\Gamma(S)$. As a consequence, this set is a core for $\mathsf{D}(\widetilde{L})$. It follows that $\mathsf{D}(\widetilde{L}) \subseteq \mathsf{D}(L)$. Since both $-\widetilde{L}$ and $-L$ are generators this implies $\mathsf{D}(\widetilde{L}) = \mathsf{D}(L)$ and therefore $\widetilde{L} = L$. □

So far we have considered P as a C_0-semigroup in $L^2(\mu)$. Having identified P as a second quantised semigroup on $L^2(\mu)$, we are in a position to prove that P extends to the spaces $L^p(\mu)$.

Theorem 4.4. *For $1 \le p < \infty$, the semigroup P extends to a C_0-semigroup of positive contractions on $L^p(\mu)$ satisfying $\|P(t)f\|_\infty \le \|f\|_\infty$ for $f \in L^\infty(\mu)$. The measure μ is an invariant measure for P, i.e.,*

$$
\int_E P(t)f\,d\mu = \int_E f\,d\mu, \qquad f \in L^p(\mu),\ t \ge 0.
$$

For $1 < p < \infty$, P is an analytic C_0-contraction semigroup on $L^p(\mu)$.

Proof. This follows immediately from Proposition 1.25 and Theorem 1.26. □

Remark 4.5. The precise angle of sectoriality $\omega^+(L)$ in $L^p(\mu)$ has been obtained in the case of Ornstein-Uhlenbeck semigroups in [28, 108]. The argument also works in the more general setting considered here.

Definition 4.6. *On $L^p(\mu)$ we define the operator L as the negative generator of the semigroup P.*

We finish this section with some algebraic properties of the operator L.

Lemma 4.7. *For all $1 < p < \infty$, $\mathcal{F}C_{\mathrm{b}}^\infty(E;\mathsf{D}(A))$ is a P-invariant core for $\mathsf{D}_p(L)$. Moreover, for $f \in \mathcal{F}C_{\mathrm{b}}^\infty(E;\mathsf{D}(A))$ of the form $f = \varphi(\phi_{h_1},\dots,\phi_{h_n})$, we have the explicit expression*

$$
\begin{aligned}
Lf(x) = -\sum_{j,k=1}^n [BVh_j, Vh_k]\partial_j\partial_k\varphi(\phi_{h_1},\dots,\phi_{h_n})&\\
+\sum_{j=1}^n \partial_j\varphi(\phi_{h_1},\dots,\phi_{h_n})\cdot\phi_{Ah_j}&.
\end{aligned}
\tag{4.2}
$$

Furthermore, for $f,g \in \mathcal{F}C_{\mathrm{b}}^\infty(E;\mathsf{D}(A))$ and $\psi \in C_{\mathrm{b}}^\infty(\mathbb{R})$ we have

(1) (Product rule) *$fg \in \mathcal{F}C_{\mathrm{b}}^\infty(E;\mathsf{D}(A))$ and*

$$
L(fg) = fLg + gLf - [(B+B^*)D_V f, D_V g];
$$

(2) (Chain rule) $\psi \circ f \in \mathcal{F}C_{\mathrm{b}}^{\infty}(E;\mathsf{D}(A))$ *and*

$$L(\psi \circ f) = (\psi' \circ f)Lf - (\psi'' \circ f)[BD_V f, D_V f].$$

Proof. First we show that $\mathcal{F}C_{\mathrm{b}}^{\infty}(E;\mathsf{D}(A))$ is contained in $\mathsf{D}_p(L)$; we thank Vladimir Bogachev for pointing out an argument which simplifies our orinig-inal proof. Pick a function $f \in \mathcal{F}C_{\mathrm{b}}^{\infty}(E;\mathsf{D}(A))$ and notice that $f \in \mathsf{D}(L) \cap L^p(\mu)$. The space $L^p(\mu)$ being reflexive, by a standard result from semigroup theory (cf. [22]) it suffices to show that

$$\overline{\lim_{t\downarrow 0}} \frac{1}{t}\|P(t)f - f\| < \infty.$$

Using that $L = D_V^* B D_V$ in $L^2(\mu)$, an explicit calculation using Lemma 1.32 shows that $Lf \in L^2(\mu) \cap L^p(\mu)$. Moreover, in $L^2(\mu)$ we have the identity

$$\frac{1}{t}(P(t)f - f) = \frac{1}{t}\int_0^t P(s)Lf\,ds.$$

Since $Lf \in L^p(\mu)$, the right-hand side can be interpreted as a Bochner integral in $L^p(\mu)$, which for $0 < t \le 1$ can be estimated in $L^p(\mu)$ by

$$\Big\|\frac{1}{t}\int_0^t P(s)Lf\,ds\Big\| \le \|Lf\|.$$

This gives the desired bound for the limes superior.

To show that $\mathcal{F}C_{\mathrm{b}}^{\infty}(E;\mathsf{D}(A))$ is invariant under P, we take f of the form

$$f = \varphi(\phi_{h_1}, \dots, \phi_{h_n}),$$

with $\varphi \in C_{\mathrm{b}}^{\infty}(\mathbb{R}^n)$ and $h_1, \dots, h_n \in \mathsf{D}(A)$. Let $R(t) := \sqrt{I - S^*(t)S(t)}$. By Mehler's formula, for μ-almost all $x \in E$ we have

$$\begin{aligned} P(t)f(x) &= \int_E \varphi(\phi_{S(t)h_1}(x) + \phi_{R(t)h_1}(y), \dots \\ &\qquad\qquad \dots, \phi_{S(t)h_n}(x) + \phi_{R(t)h_n}(y))\,d\mu(y) \\ &= \psi_t(\phi_{S(t)h_1}(x), \dots, \phi_{S(t)h_n}(x)), \end{aligned} \tag{4.3}$$

where

$$\psi_t(\xi_1, \dots, \xi_n) = \int_E \varphi(\xi_1 + \phi_{R(t)h_1}(y), \dots, \xi_n + \phi_{R(t)h_n}(y))\,d\mu(y).$$

Since $\psi_t \in C_{\mathrm{b}}^{\infty}(\mathbb{R}^n)$ and $S(t)h_j \in \mathsf{D}(A)$ for $j = 1, \dots, n$, it follows that the subspace $\mathcal{F}C_{\mathrm{b}}^{\infty}(E;\mathsf{D}(A))$ is invariant under P. Since it is dense in $L^p(\mu)$ and contained in $\mathsf{D}_p(L)$, it is a core for $\mathsf{D}_p(L)$.

The expression (4.2) and the identities (1) and (2) follow by direct computation, using the identity $L = D_V^* B D_V$ and Lemma 1.32. $\square$

Remark 4.8. The same proof shows that $\mathcal{F}C_{\mathrm{b}}^{\infty}(E;\mathsf{D}(A^k))$ is a P-invariant core for $\mathsf{D}_p(L)$ for every $k \ge 1$.

The operator $\underline{L}$ in $L^p(\mu; \underline{H})$

Having defined L as an operator acting on $L^p(\mu)$, we will now turn to the operator $\underline{L}$. The following result follows from the theory presented in Section 3.4.

Proposition 4.9. *The operator $\underline{L}$ is sectorial on $L^2(\mu; \underline{H})$ of angle $\omega^+(\underline{L}) \le \omega_c(B)$.*

As a consequence, $-\underline{L}$ generates a bounded analytic C_0-semigroup on $L^2(\mu; \underline{H})$. In what follows we denote this semigroup by $\underline{P}$. Our next aim is to give a meaning to the operator $\underline{L}$ on the spaces $\overline{\mathsf{R}_p(D_V)}$, $1 < p < \infty$, where the closure is taken in $L^p(\mu; \underline{H})$. For this purpose, we need some a couple of lemmas.

Lemma 4.10. *For all $u \in \mathsf{D}(\underline{A})$ we have $\mathbf{1} \otimes u \in \mathsf{D}(\underline{L})$ and*

$$\underline{L}(\mathbf{1} \otimes u) = \mathbf{1} \otimes \underline{A}u.$$

Proof. We have

$$\underline{L}(\mathbf{1} \otimes u) = D_V D_V^*(\mathbf{1} \otimes Bu) = D_V(\phi_{V^*Bu}) = \mathbf{1} \otimes VV^*Bu = \mathbf{1} \otimes \underline{A}u.$$

□

Lemma 4.11. *For all $h \in \mathsf{D}(V)$ and $t \ge 0$ we have $S(t)h \in \mathsf{D}(V)$ and*

$$VS(t)h = \underline{S}(t)Vh.$$

Proof. We may assume that $t > 0$.

First let $g \in \mathsf{D}(A^2)$. Since $Ag \in \mathsf{D}(A) \subseteq \mathsf{D}(V)$ we find that $Vg \in \mathsf{D}(\underline{A})$ and $\underline{A}Vg = VAg$. For $\lambda > 0$ it follows that $(I + \lambda\underline{A})Vg = V(I + \lambda A)g$. Applying this to $g = (I + \lambda A)^{-1}h$ with $h \in \mathsf{D}(A)$ we obtain

$$V(I + \lambda A)^{-1}h = (I + \lambda\underline{A})^{-1}Vh.$$

Taking $\lambda = \frac{t}{n}$ and repeating this argument n times we obtain, for all $h \in \mathsf{D}(A)$,

$$V(I + \tfrac{t}{n}A)^{-n}h = (I + \tfrac{t}{n}\underline{A})^{-n}Vh.$$

Taking limits $n \to \infty$ and using the closedness of V, we obtain $S(t)h \in D(V)$ and

$$VS(t)h = \underline{S}(t)Vh.$$

We are still assuming that $h \in \mathsf{D}(A)$. However, this assumption may now be removed by recalling the fact that $\mathsf{D}(A)$ is a core for $\mathsf{D}(V)$. □

Lemma 4.12. *For all $t \ge 0$ we have $\underline{S}(t)\overline{\mathsf{R}(V)} \subseteq \overline{\mathsf{R}(V)}$. Moreover, the part of $\underline{A}$ in $\overline{\mathsf{R}(V)}$ is injective.*

Proof. The first assertion follows from Lemma 4.11. Suppose that $\underline{A}u = VV^*Bu = 0$ for some u belonging to the domain of the part of $\underline{A}$ in $\overline{\mathsf{R}(V)}$. Then $\|V^*Bu\|^2 = 0$, so $Bu \in \mathsf{N}(V^*)$. Thus $[Bu, Vh] = 0$ for all $h \in \mathsf{D}(V)$. Since $u \in \overline{\mathsf{R}(V)}$ it follows that $[Bu, u] = 0$, and therefore $u = 0$ by the coercivity of B on $\overline{\mathsf{R}(V)}$. □

Next we show that the semigroups $\underline{P}$ and $P \otimes \underline{S}$ agree on $\overline{\mathsf{R}(D_V)}$. We will use the following lemma.

Lemma 4.13. *For $1 < p < \infty$, $\mathcal{F}C_{\mathrm{b}}^{\infty}(E; \mathsf{D}(A))$ is a core for $\mathsf{D}_p(D_V)$.*

Proof. First let $f = \varphi(\phi_{h_1}, \ldots, \phi_{h_n})$ with $\varphi \in C_{\mathrm{b}}^1(\mathbb{R}^n)$ and $h_1, \ldots, h_n \in \mathsf{D}(V)$. Choose sequences $(h_{jk})_{k\geq 1}$ in $\mathsf{D}(A)$ with $h_{jk} \to h_j$ in $\mathsf{D}(V)$ as $k \to \infty$. Then $f_k \to f$ in $L^p(\mu)$ and $D_V f_k \to D_V f$ in $L^p(\mu; \underline{H})$, where $f_k = \varphi(\phi_{h_{1k}}, \ldots . \phi_{h_{nk}})$. Since $\mathcal{F}C_{\mathrm{b}}^1(E; \mathsf{D}(V))$ is a core for $\mathsf{D}_p(D_V)$, this proves that $\mathcal{F}C_{\mathrm{b}}^1(E; \mathsf{D}(A))$ is a core for $\mathsf{D}_p(D_V)$. Now a standard mollifier argument, convolving φ with a smooth function of compact support, shows that $\mathcal{F}C_{\mathrm{b}}^{\infty}(E; \mathsf{D}(A))$ is a core for $\mathsf{D}_p(D_V)$. □

The next result is well known in the context of Ornstein-Uhlenbeck semigroups; see, e.g., [32, Lemma 2.7], [110, Proposition 3.5].

Theorem 4.14. *For all $1 < p < \infty$, the semigroup $P \otimes \underline{S}$ restricts to a bounded analytic C_0-semigroup on $\overline{\mathsf{R}_p(D_V)}$. For $f \in \mathsf{D}_p(D_V)$ and $t \geq 0$ we have $P(t)f \in \mathsf{D}_p(D_V)$ and*

$$D_V P(t) f = (P(t) \otimes \underline{S}(t)) D_V f.$$

Proof. First we show that for all $f \in \mathsf{D}_p(D_V)$ we have $P(t)f \in \mathsf{D}_p(D_V)$ and $D_V P(t) f = (P(t) \otimes \underline{S}(t)) D_V f$. Since D_V is closed and $\mathcal{F}C_{\mathrm{b}}^{\infty}(E; \mathsf{D}(A))$ is a core for $\mathsf{D}_p(D_V)$ by Lemma 4.13, it suffices to check this for functions $f \in \mathcal{F}C_{\mathrm{b}}^{\infty}(E; \mathsf{D}(A))$.

We use the notations of Lemma 4.7. By (4.3) and Lemma 4.11, for functions $f = \varphi(\phi_{h_1}, \ldots, \phi_{h_n})$ we have, for μ-almost all $x \in E$,

$$\begin{aligned} D_V P(t) f(x) &= \sum_{j=1}^{n} \partial_j \psi_t(\phi_{S(t)h_1}(x), \ldots, \phi_{S(t)h_n}(x)) \otimes V S(t) h_j \\ &= \sum_{j=1}^{n} \int_E \partial_j \varphi(\phi_{S(t)h_1}(x) + \phi_{R(t)h_1}(y), \ \ldots \\ &\qquad\qquad \ldots \ , \phi_{S(t)h_n}(x) + \phi_{R(t)h_n}(y)) \, d\mu(y) \otimes \underline{S}(t) V h_j \\ &= (P(t) \otimes \underline{S}(t)) D_V f(x). \end{aligned}$$

This identity shows that $P(t) \otimes \underline{S}$ maps $\mathsf{R}_p(D_V)$ into itself, and therefore $P \otimes \underline{S}$ restricts to a bounded C_0-semigroup on $\overline{\mathsf{R}_p(D_V)}$. The invariance of $\overline{\mathsf{R}_p(D_V)}$ under the operators $P(z) \otimes \underline{S}(z)$, where $z \in \mathbb{C}$ is in the sector of bounded analyticity of P, follows by uniqueness of analytic continuation (consider the quotient mapping from $L^p(\mu; \underline{H})$ to $L^p(\mu; \underline{H})/\overline{\mathsf{R}_p(D_V)}$). □

In the next result we return to the L^2-setting and show that the semigroups $P \otimes \underline{S}$ and $\underline{P}$ on $L^2(\mu; \underline{H})$ agree on $\overline{\mathsf{R}(D_V)}$.

Theorem 4.15. *Both $\underline{P}$ and $P \otimes \underline{S}$ restrict to bounded analytic C_0-semigroups on $\overline{\mathsf{R}(D_V)}$, and their restrictions coincide:*

$$\underline{P}(t)F = P(t) \otimes \underline{S}(t)F, \qquad F \in \overline{\mathsf{R}(D_V)}.$$

Proof. The invariance of $\overline{\mathsf{R}(D_V)}$ under $P \otimes \underline{S}$ follows from the previous theorem. Let us write $-N$ for the generator of $P \otimes \underline{S}$ on $\overline{\mathsf{R}(D_V)}$. From $V(\mathsf{D}(A^2)) \subseteq \mathsf{D}(\underline{A})$ (cf. the proof of Lemma 4.11) and $\mathcal{F}C_{\mathrm{b}}^{\infty}(E; \mathsf{D}(A^2)) \otimes \mathsf{D}(\underline{A}) \subseteq \mathsf{D}(L) \otimes \mathsf{D}(\underline{A})$ we see that the subspace $U := \{D_V f : f \in \mathcal{F}C_{\mathrm{b}}^{\infty}(E; \mathsf{D}(A^2))\}$ is contained in $\mathsf{D}(N)$. This subspace is dense in $\overline{\mathsf{R}(D_V)}$ since $\mathcal{F}C_{\mathrm{b}}^{\infty}(E; \mathsf{D}(A^2))$ is a core for $\mathsf{D}(L)$ (by Lemma 4.7 and the remark following it) and $\mathsf{D}(L)$ is a core for $\mathsf{D}(D_V)$. Since $(P \otimes \underline{S})U \subseteq U$ by Theorem 4.14, it follows that U is a core for $\mathsf{D}(N)$.

For functions $f \in \mathcal{F}C_{\mathrm{b}}^{\infty}(E; \mathsf{D}(A^2))$ we obtain

$$N D_V f = D_V L f = \underline{L} D_V f.$$

The first identity follows from Theorem 4.14 and the second from a direct computation. Thus $N = \underline{L}$ on the core U of $\mathsf{D}(N)$. It follows that $\mathsf{D}(N) \subseteq \mathsf{D}(\underline{L})$ and $N = \underline{L}$ on $\mathsf{D}(N)$. Let $\lambda > 0$. Multiplying the identity $\lambda + N = \lambda + \underline{L}$ from the right with $(\lambda + N)^{-1}$ and from the left with $(\lambda + \underline{L})^{-1}$, we obtain $(\lambda + N)^{-1} = (\lambda + \underline{L})^{-1}$ on $\overline{\mathsf{R}(D_V)}$. In particular, $(\lambda + \underline{L})^{-1}$ maps $\overline{\mathsf{R}(D_V)}$ into itself. As in Lemma 4.11 it follows that $\underline{P}$ leaves $\overline{\mathsf{R}(D_V)}$ invariant and that the restriction of $\underline{P}$ to $\overline{\mathsf{R}(D_V)}$ equals the semigroup generated by $-N$, which is $P \otimes \underline{S}|_{\overline{\mathsf{R}(D_V)}}$. □

Definition 4.16. *Let $1 < p < \infty$. On $\overline{\mathsf{R}_p(D_V)}$ we define $\underline{P} := P \otimes \underline{S}|_{\overline{\mathsf{R}_p(D_V)}}$. The negative generator of $\underline{P}$ is denoted by $\underline{L}$.*

By Theorem 4.15, for $p = 2$ this definition is consistent with the one given after Proposition 4.9.

We close this section with some consequences of the theory presented in Chapter 5.

Proposition 4.17. *Let $1 < p < \infty$. The operator L is γ-sectorial and admits a bounded H^∞-calculus on $L^p(\mu)$ of angle $\omega_{H^\infty}^+(L) = \omega_\gamma^+(L) < \frac{1}{2}\pi$. Moreover,*

(1) *The family $\{P(t) : t \geq 0\}$ is γ-bounded in $\mathcal{L}(L^p(\mu))$;*
(2) *The family $\{\underline{P}(t) : t \geq 0\}$ is γ-bounded in $\mathcal{L}(\overline{\mathsf{R}_p(D_V)})$.*

Proof. Since $-L$ generates an analytic C_0-semigroup of positive contractions on $L^p(\mu)$, the first part follows from Theorem 5.53. Assertion (1) follows from Lemma 5.46, and assertion (2) follows by combining (1) with the identity $\underline{P} = P \otimes \underline{S}$ and Proposition 5.6. □

4.2 Randomised gradient bounds and LPS inequalities

In this section we will prove randomised gradient bounds and Littlewood-Paley-Stein inequalities for the semigroup P. These results form the core of the proof of Theorem 4.37, which is an extended version of Theorem 0.1 presented in the introduction.

For functions $f \in \mathcal{F}C_{\mathrm{b}}^{\infty}(E;\mathsf{D}(A))$ we consider the Littlewood-Paley-Stein square functions

$$\mathscr{H} f(x) := \Big(\int_0^\infty \|\sqrt{t} D_V P(t) f(x)\|^2 \,\frac{dt}{t}\Big)^{1/2}, \qquad x \in E,$$
$$\mathscr{G} f(x) := \Big(\int_0^\infty \|t D_V Q(t) f(x)\|^2 \,\frac{dt}{t}\Big)^{1/2}, \qquad x \in E.$$

where Q denotes the analytic C_0-semigroup generated by $-\sqrt{L}$.

The functions $t \mapsto D_V P(t) f$ are analytic in a sector containing $\mathbb{R}_+$, and therefore a well-known result of Stein [156] allows us to select a pointwise version $(t,x) \mapsto D_V P(t) f(x)$ which is analytic in t for every fixed x. Using such a version, we see that $\mathscr{H} f$ is well defined almost everywhere (but possibly infinite). The square function $\mathcal{G} f$ is well defined by similar reasoning.

The main results of this section are the following two theorems, which together imply parts (2) and (3) of Theorem 0.3 announced in the introduction. Part (1) of Theorem 0.3 is contained in Theorem 4.25.

Theorem 4.18 (Randomised gradient bounds). *Let $1 < p < \infty$. Then $\mathsf{D}_p(L)$ is a core for $\mathsf{D}_p(D_V)$ and the families*

$$\big\{\sqrt{t} D_V P(t) :\ t > 0\big\} \quad \textit{and} \quad \big\{t D_V (I + t^2 L)^{-1} :\ t > 0\big\}$$

are γ-bounded in $\mathcal{L}(L^p(\mu), L^p(\mu; \underline{H}))$.

Theorem 4.19 (Littlewood-Paley-Stein inequality). *Let $1 < p < \infty$. The following estimate holds for all $f \in \mathcal{F}C_{\mathrm{b}}^{\infty}(E;\mathsf{D}(A))$:*

$$\|\mathscr{H} f\|_p \lesssim \|f\|_p.$$

By Theorem 4.18 the square functions $\mathscr{H} f$ and $\mathscr{G} f$ are actually well-defined for arbitrary $f \in L^p(\mu)$, and by approximation the estimate of Theorem 4.19 extends to all of $L^p(\mu)$. Since we do not need these observations we leave the details to the reader.

For the proofs of both theorems we distinguish between the cases $1 < p \le 2$ and $2 < p < \infty$. For $1 < p \le 2$ we show by a direct argument that $\mathscr{H}$ is L^p-bounded and deduce from this that $\mathsf{D}_p(L)$ is a core for $\mathsf{D}_p(D_V)$. Theorem 4.18 is then a consequence of Theorem 5.47. For $2 < p < \infty$ we first derive Theorem 4.18 from a pointwise gradient bound and a duality argument involving maximal functions. Since L has a bounded H^∞-calculus of angle

$< \frac{1}{2}\pi$ by Proposition 4.17, Theorem 4.19 then follows by an application of Theorem 5.47.

Both square functions are related by the following inequality. The argument is taken from [39].

Lemma 4.20. *For $f \in \mathscr{F}C_{\mathrm{b}}^{\infty}(E;\mathsf{D}(A))$ we have $\mathscr{G}f \le \mathscr{H}f$ μ-a.e.*

Proof. Using the representation

$$Q(t)f = \frac{1}{\sqrt{\pi}}\int_0^\infty \frac{e^{-u}}{\sqrt{u}} P\big(\frac{t^2}{4u}\big) f\,du$$

and the closedness of D_V,

$$\begin{aligned}\mathscr{G}^2 f(x) &= \int_0^\infty \|tD_V Q(t)f(x)\|^2 \,\frac{dt}{t}\\ &\le \frac{1}{\pi}\int_0^\infty \Big(\int_0^\infty \Big\|tD_V P\big(\frac{t^2}{4u}\big)f(x)\Big\| \frac{e^{-u}}{\sqrt{u}}\,du\Big)^2 \frac{dt}{t}.\end{aligned}$$

Since $\int_0^\infty \frac{e^{-u}}{\sqrt{u}}\,du = \sqrt{\pi}$ we may apply Jensen's inequality to obtain

$$\begin{aligned}\mathscr{G}^2 f(x) &\le \frac{1}{\sqrt{\pi}}\int_0^\infty \Big(\int_0^\infty \Big\|tD_V P\big(\frac{t^2}{4u}\big)f(x)\Big\|^2 \frac{e^{-u}}{\sqrt{u}}\,du\Big)\,\frac{dt}{t}\\ &= \frac{1}{\sqrt{\pi}}\int_0^\infty \Big(\int_0^\infty \Big\|tD_V P\big(\frac{t^2}{4u}\big)f(x)\Big\|^2 \frac{dt}{t}\Big)\frac{e^{-u}}{\sqrt{u}}\,du\\ &= \frac{2}{\sqrt{\pi}}\int_0^\infty \Big(\int_0^\infty \big\|\sqrt{s}D_V P(s)f(x)\big\|^2 \frac{ds}{s}\Big)\sqrt{u}e^{-u}\,du\\ &= \mathscr{H}^2 f(x).\end{aligned}$$

□

The case $1 < p \le 2$

We begin with some preliminary observations.

Lemma 4.21. *For $h \in \mathsf{D}(V)$ we have*

$$\int_0^\infty \|\underline{S}(t)Vh\|^2\,dt \le (2\kappa)^{-1}\|h\|^2.$$

Proof. Let $t > 0$. Using Lemma 4.11 and the fact that $S(t)h \in \mathsf{D}(A)$ by analyticity, we obtain

$$\begin{aligned}\|\underline{S}(t)Vh\|^2 = \|VS(t)h\|^2 &\le \kappa^{-1}[BVS(t)h, VS(t)h]\\ &= \kappa^{-1}[AS(t)h, S(t)h]\\ &= -(2\kappa)^{-1}\frac{d}{dt}\|S(t)h\|^2.\end{aligned}$$

Hence

$$\begin{aligned}\int_0^\infty \|\underline{S}(t)Vh\|^2\,dt &\le (2\kappa)^{-1}\varlimsup_{T\to\infty}\int_0^T -\frac{d}{dt}\|S(t)h\|^2\,dt\\ &= (2\kappa)^{-1}\Big(\|h\|^2 - \varliminf_{T\to\infty}\|S(T)h\|^2\Big)\\ &\le (2\kappa)^{-1}\|h\|^2.\end{aligned}$$

□

Lemma 4.22. *Let $f \in \mathcal{F}C_{\mathrm{b}}^\infty(E;\mathsf{D}(A))$ and $F \in \mathcal{F}C_{\mathrm{b}}^\infty(E;\mathsf{D}(A)) \otimes \mathsf{D}(A)$ be such that $D_V f = (I\otimes V)F$. Then for all $1<p<\infty$ we have $\mathscr{H}f \in L^p(\mu)$ and $\|\mathscr{H}f\|_p \lesssim \|F\|_p$.*

Proof. By Proposition 5.6 and Lemma 5.46, the set $\{P(t)\otimes I :\ t\ge 0\}$ is γ-bounded in $\mathcal{L}(L^p(\mu;\underline{H}))$. Hence, by Propositions 5.16, 5.15, and Lemma 4.21,

$$\begin{aligned}\|\mathscr{H}f\|_p &= \Big\|\Big(\int_0^\infty \|\underline{P}(t)D_V f\|^2\,dt\Big)^{1/2}\Big\|_p\\ &= \Big\|\Big(\int_0^\infty \|(P(t)\otimes I)(I\otimes \underline{S}(t))(I\otimes V)F\|^2\,dt\Big)^{1/2}\Big\|_p\\ &\lesssim \Big\|\Big(\int_0^\infty \|(I\otimes \underline{S}(t))(I\otimes V)F\|^2\,dt\Big)^{1/2}\Big\|_p\\ &\le (2k)^{-1/2}\|F\|_p.\end{aligned}$$

□

The following proof is based on a classical argument which goes back to Stein [156]. The same idea has been applied in the related works [32, 39, 110, 152].

Proof (of Theorem 4.19, $1<p\le 2$). First we show that it suffices to prove the estimate for functions $f \in \mathcal{F}C_{\mathrm{b}}^\infty(E;\mathsf{D}(A))$ satisfying $f\ge\varepsilon$ for some $\varepsilon>0$.

Fix $f = \varphi(\phi_{h_1},\dots,\phi_{h_k}) \in \mathcal{F}C_{\mathrm{b}}^\infty(E;\mathsf{D}(A))$ of the usual form. Pick functions $m_n \in C_{\mathrm{b}}^\infty(\mathbb{R}^k)$ satisying $m_n\ge 0$, $\mathrm{supp}(m_n)\subseteq[-\frac1n,\frac1n]^k$, and $\|m_n\|_1=1$, and put

$$\begin{aligned}\psi_{n,\pm} &:= \big(\varphi^\pm + \frac1n\big) * m_n,\\ g_{n,\pm} &:= \psi_{n,\pm}(\phi_{h_1},\dots,\phi_{h_k}),\\ g_{n,\pm,j} &:= \partial_j\psi_{n,\pm}(\phi_{h_1},\dots,\phi_{h_k}).\end{aligned}$$

Clearly $g_{n,\pm} \in \mathcal{F}C_{\mathrm{b}}^\infty(E;\mathsf{D}(A))$ satisfy $\frac1n \le g_{n,\pm} \le \|\varphi\|_\infty + 1$, and

$$\big\|\big(f^\pm + \frac1n\big) - g_{n,\pm}\big\|_p \to 0$$

by dominated convergence. From Lemma 4.22 it follows that

$$\begin{aligned}\|\mathscr{H}f - \mathscr{H}(g_{n,+} - g_{n,-})\|_p &\le \|\mathscr{H}(f - (g_{n,+} - g_{n,-}))\|_p \\ &\lesssim \Big\| \sum_{j=1}^{k} (f_j - (g_{n,+,j} - g_{j,n,-,j})) \otimes h_j \Big\|_p,\end{aligned}$$

where $f_j = \partial_j \varphi(\phi_{h_1}, \ldots, \phi_{h_k})$. Since the functions

$$g_{n,\pm,j} = \big(\partial_j \varphi(\phi_{h_1}, \ldots, \phi_{h_k}) \mathbf{1}_{\{\pm\varphi(\phi_{h_1},\ldots,\phi_{h_k})>0\}}\big) * m_n,$$

belong to L^∞ uniformly in n, we conclude by dominated convergence that $\|f_j - (g_{n,+,j} - g_{n,-,j})\|_p \to 0$. Therefore $\mathscr{H}(g_{n,+} - g_{n,-}) \to \mathscr{H}f$ in $L^p(\mu)$ as $n \to \infty$. Hence if $\|\mathscr{H}g_{n,\pm}\|_p \lesssim \|g_{n,\pm}\|_p$ with constants not depending on n, then

$$\begin{aligned}\|\mathscr{H}f\|_p &= \lim_{n\to\infty} \|\mathscr{H}(g_{n,+} - g_{n,-})\|_p \\ &\le \overline{\lim_{n\to\infty}} (\|\mathscr{H}g_{n,+}\|_p + \|\mathscr{H}g_{n,-}\|_p) \\ &\lesssim \overline{\lim_{n\to\infty}} (\|g_{n,+}\|_p + \|g_{n,-}\|_p) \\ &= \|f^+\|_p + \|f^-\|_p \\ &\le 2\|f\|_p.\end{aligned}$$

Thus it suffices to prove the result for $f \in \mathcal{F}C_{\mathrm{b}}^\infty(E; \mathsf{D}(A))$ satisfying $f \ge \varepsilon$ for some $\varepsilon > 0$. Set

$$u(t,x) := P(t)f(x), \qquad x \in E,\ t > 0,$$

and notice that by Mehler's formula (1.13) we have $u(t,x) \ge \varepsilon$ for all $x \in E$ and $t \ge 0$. By Lemma 4.7 we have $u(t,\cdot) \in \mathcal{F}C_{\mathrm{b}}^\infty(E; \mathsf{D}(A)) \subseteq \mathsf{D}_p(L)$ for all $t \ge 0$. Arguing as in [32, 39, 152], for $1 < p \le 2$ we use Lemma 4.7 and a truncation argument to obtain that $u(t,\cdot)^p \in \mathsf{D}_p(L)$ and

$$\begin{aligned}(\partial_t + L)u(t,x)^p &= pu(t,x)^{p-1}(\partial_t + L)u(t,x) \\ &\qquad - p(p-1)u(t,x)^{p-2}[BD_V u(t,x), D_V u(t,x)] \\ &= -p(p-1)u(t,x)^{p-2}[BD_V u(t,x), D_V u(t,x)].\end{aligned}$$

Hence, using the coercivity Assumption (A3),

$$\begin{aligned}\|D_V u(t,x)\|^2 &\le k^{-1}[BD_V u(t,x), D_V u(t,x)] \\ &= -\frac{1}{kp(p-1)} u(t,x)^{2-p}(\partial_t + L)u(t,x)^p.\end{aligned}$$

Now we set

$$K(x) := -\int_0^\infty (\partial_t + L)u(t,x)^p \, dt$$

and

$$u_\star(x) := \sup_{t>0} u(t,x)$$

to obtain

$$\begin{aligned} \mathscr{H} f(x)^2 &= \int_0^\infty \|D_V u(t,x)\|^2 \, dt \\ &\le -C_{p,k} \int_0^\infty u(t,x)^{2-p} (\partial_t + L) u(t,x)^p \, dt \\ &\le C_{p,k} u_\star(x)^{2-p} K(x). \end{aligned}$$

Hölder's inequality with exponents $\frac{2}{2-p}$ and $\frac{2}{p}$ implies

$$\begin{aligned} \int_E \mathscr{H} f(x)^p \, d\mu(x) &\le C_{p,k}^{\frac{p}{2}} \int_E u_\star(x)^{\frac{(2-p)p}{2}} K(x)^{\frac{p}{2}} \, d\mu(x) \\ &\le C_{p,k}^{\frac{p}{2}} \Big(\int_E u_\star(x)^p \, d\mu(x) \Big)^{\frac{2-p}{2}} \Big(\int_E K(x) \, d\mu(x) \Big)^{\frac{p}{2}}. \end{aligned} \tag{4.4}$$

Using the invariance of μ and the L^p-contractivity of P we obtain

$$\begin{aligned} \int_E K(x) \, d\mu(x) &= -\int_0^\infty \int_E (\partial_t + L) u(t,x)^p \, d\mu(x) \, dt \\ &= -\int_0^\infty \int_E \partial_t u(t,x)^p \, d\mu(x) \, dt \\ &= -\int_0^\infty \partial_t \int_E u(t,x)^p \, d\mu(x) \, dt \\ &\le \varlimsup_{t\to\infty} \big(\|f\|_p^p - \|u(t,\cdot)\|_p^p \big) \\ &\le \|f\|_p^p, \end{aligned} \tag{4.5}$$

where the use of Fubini's theorem is justified by the non-negativity of the integrand K, and the interchange of differentiation and integration by the fact that $f \in \mathcal{F}C_{\mathrm{b}}^\infty(E; \mathsf{D}(A))$.

Combining (4.4), (4.5) and Proposition 5.54 we conclude that

$$\|\mathscr{H} f\|_p^p \lesssim \|u_\star\|_p^{\frac{(2-p)p}{2}} \|f\|_p^{\frac{p^2}{2}} \lesssim \|f\|_p^p.$$

$\square$

Proof (Proof of Theorem 4.18, $1 < p \le 2$). First we show that $\mathsf{D}_p(L)$ is contained in $\mathsf{D}_p(D_V)$. Once we know this, Lemmas 4.7 and 4.13 imply that $\mathsf{D}_p(L)$ is a even core for $\mathsf{D}_p(D_V)$.

Fix a function $f \in \mathcal{F}C_{\mathrm{b}}^\infty(E; \mathsf{D}(A))$. From Theorem 4.14 it follows that $s \mapsto e^{-s} D_V P(s) f = e^{-s} \underline{P}(s) D_V f$ is Bochner integrable in $L^p(\mu; \underline{H})$ and

$$\int_0^\infty e^{-s} D_V P(s) f \, ds = (I + \underline{L})^{-1} D_V f.$$

Since $s \mapsto e^{-s}P(s)f$ is Bochner integrable in $L^p(\mu)$, the closedness of D_V implies that $(I+L)^{-1}f = \int_0^\infty e^{-s}P(s)f\,ds \in \mathsf{D}_p(D_V)$ and

$$D_V(I+L)^{-1}f = D_V \int_0^\infty e^{-s}P(s)f\,ds = \int_0^\infty e^{-s}D_V P(s)f\,ds.$$

Moreover, by the Cauchy-Schwarz inequality,

$$\begin{aligned}
\|D_V(I+L)^{-1}f\|_p &\le \Big\| \int_0^\infty e^{-s}\|D_V P(s)f\|\,ds\Big\|_p \\
&\le \frac{1}{\sqrt{2}}\Big\|\Big(\int_0^\infty \|D_V P(s)f\|^2\,ds\Big)^{1/2}\Big\|_p \\
&= \frac{1}{\sqrt{2}}\|\mathscr{H}f\|_p \lesssim \|f\|_p.
\end{aligned}$$

It follows that $D_V(I+L)^{-1}$ extends to a bounded operator from $L^p(\mu)$ to $L^p(\mu;\underline{H})$. In view of the closedness of D_V and Lemma 4.7, the desired inclusion follows from this. This concludes the proof that $\mathsf{D}_p(L)$ is a core for $\mathsf{D}_p(D_V)$.

The γ-boundedness assertions follow from Theorem 5.47 and Remark 5.48. $\square$

The case $2 < p < \infty$

In case that P is symmetric it is possible to use a variant of a duality argument of Stein [156] to prove the boundedness of $\mathscr{H}$. This approach has been taken in [32], but the proof breaks down if L is non-symmetric and we have to proceed in a different way.

First we derive an explicit formula for the semigroup P which allows us to prove suitable gradient bounds. Having obtained those gradient bounds we give a general argument involving a maximal inequality for P^* to prove the γ-boundedness of the collection $\{\sqrt{t}D_V P(t) : \ t > 0\}$. Since L has a bounded H^∞-calculus, we obtain the boundedness of $\mathscr{H}$ by an appeal to Theorem 5.47.

We begin with some preliminary observations. For $0 < t < \infty$ we define the operators $Q_t \in \mathcal{L}(E^*, E)$ by

$$Q_t x^* := ii^*x^* - iS^*(t)S(t)i^*x^*,$$

where $i : H \hookrightarrow E$ is the inclusion operator. The operators Q_t are positive and symmetric, i.e., for all $x^*, y^* \in E^*$ we have $\langle Q_t x^*, x^*\rangle \ge 0$ and $\langle Q_t x^*, y^*\rangle = \langle Q_t y^*, x^*\rangle$. Let H_t be the reproducing kernel Hilbert space associated with Q_t and let $i_t : H_t \hookrightarrow E$ be the inclusion mapping. Then,

$$i_t i_t^* = Q_t.$$

Since $\langle Q_t x^*, x^*\rangle \le \langle Qx^*, x^*\rangle$ for all $x^* \in E^*$, the operators Q_t are covariances of Gaussian measures μ_t on E by Proposition 1.5. This estimate also implies that we have a continuous inclusion $H_t \hookrightarrow H$ and that the mapping

$$V_t : i^*x^* \mapsto i_t^* x^*, \qquad x^* \in E^*,$$

is well defined and extends to a contraction from H into H_t. It is easy to check that the adjoint operator V_t^* is the inclusion from H_t into H.

Let us also note that for $s \le t$ and $x^* \in E^*$ we have

$$\langle Q_s x^*, x^*\rangle = \|i^*x\|^2 - \|S(s)i^*x^*\|^2 \le \|i^*x\|^2 - \|S(t)i^*x^*\|^2 = \langle Q_t x^*, x^*\rangle$$

by the contractivity of S.

In the next proposition we fix $t > 0$ and $h \in H_t$ and denote by $\phi_h^{\mu_t} : E \to \mathbb{R}$ the (μ_t-essentially unique; see Remark 1.13) μ_t-measurable linear extension of the function $\phi_h^{\mu_t}(i_t g) := [g, h]_{H_t}$.

Proposition 4.23. *For all $f = \varphi(\phi_{h_1}, \dots, \phi_{h_n}) \in \mathcal{F}C_{\mathrm{b}}(E)$ the following identity holds for μ-almost all $x \in E$:*

$$P(t)f(x) = \int_E \varphi(\phi_{S(t)h_1}(x) + \phi^{\mu_t}_{V_t h_1}(y), \dots, \phi_{S(t)h_n}(x) + \phi^{\mu_t}_{V_t h_n}(y))\, d\mu_t(y).$$

Proof. Defining $\psi : E \times \mathbb{R}^n \to \mathbb{R}$ by

$$\psi(x,\xi) := \varphi(\phi_{S(t)h_1}(x) + \xi_1, \dots, \phi_{S(t)h_n}(x) + \xi_n),$$

we have

$$\begin{aligned}
&\int_E \varphi(\phi_{S(t)h_1}(x) + \phi^{\mu_t}_{V_t h_1}(y), \dots, \phi_{S(t)h_n}(x) + \phi^{\mu_t}_{V_t h_n}(y))\, d\mu_t(y) \\
&\quad = \int_E \psi\big(x, (\phi^{\mu_t}_{V_t h_1}(y), \dots, \phi^{\mu_t}_{V_t h_n}(y))\big)\, d\mu_t(y) \\
&\quad = \int_{\mathbb{R}^n} \psi(x,\xi)\, d\gamma_t(\xi),
\end{aligned}$$

where γ_t is the centred Gaussian measure on $\mathbb{R}^n$ whose covariance matrix equals $\big([V_t h_i, V_t h_j]\big)_{i,j=1}^n$.

On the other hand, writing $R(t) = \sqrt{I - S^*(t)S(t)}$, by Mehler's formula (1.13) we have

$$\begin{aligned}
P(t)f(x) &= \int_E \varphi(\phi_{S(t)h_1}(x) + \phi_{R(t)h_1}(y), \dots, \phi_{S(t)h_n}(x) + \phi_{R(t)h_n}(y))\, d\mu(y) \\
&= \int_E \psi\big(x, (\phi_{R(t)h_1}(y), \dots, \phi_{R(t)h_n}(y))\big)\, d\mu(y) \\
&= \int_{\mathbb{R}^n} \psi(\xi)\, d\tilde{\gamma}_t(\xi),
\end{aligned}$$

where $\tilde{\gamma}_t$ is the centred Gaussian measure on $\mathbb{R}^n$ whose covariance matrix equals $\big([R(t)h_i, R(t)h_j]\big)_{i,j=1}^n$.

The result follows from the observation that

$$[V_t h_i, V_t h_j] = [h_i, h_j] - [S(t)h_i, S(t)h_j] = [R(t)h_i, R(t)h_j].$$

$\square$

Lemma 4.24. *For all $u \in \underline{H}$ and $t > 0$ we have $\underline{S}^*(t)u \in \mathsf{D}(V^*)$, $V^*\underline{S}^*(t)u \in H_t$, and*

$$\|V^*\underline{S}^*(t)u\|_{H_t} \lesssim \frac{1}{\sqrt{t}}\|u\|.$$

Proof. First we observe that $S(s)$ maps H into $\mathsf{D}(A) \subseteq \mathsf{D}(V)$ for $s > 0$. For $t > 0$ we claim that

$$J_t : V_t h \mapsto VS(\cdot)h$$

extends to a bounded operator from H_t into $L^2(0,t;\underline{H})$ of norm $\le \frac{1}{\sqrt{2k}}$.

Indeed, by the coercivity of B and the definition of H_t, we obtain for $h \in H$,

$$\begin{aligned}\int_0^t \|VS(s)h\|^2\,ds &\le \frac{1}{k}\int_0^t [BVS(s)h, VS(s)h]\,ds\\ &= -\frac{1}{2k}\int_0^t \frac{d}{ds}\|S(s)h\|^2\,ds\\ &= \tfrac{1}{2k}\big(\|h\|^2 - \|S(t)h\|^2\big)\\ &= \tfrac{1}{2k}\|V_t h\|_{H_t}^2.\end{aligned}$$

Recall that V_t^* is the inclusion mapping $H_t \hookrightarrow H$. Noting that $\underline{S}^*(t)$ maps $\underline{H}$ into $\mathsf{D}(\underline{A}^*) \subseteq \mathsf{D}(V^*)$ and using Lemma 4.11, the adjoint mapping $J_t^* : L^2(0,t;\underline{H}) \to H_t$ is given by

$$V_t^* J_t^* f = \int_0^t V^*\underline{S}^*(s)f(s)\,ds, \qquad f \in L^2(0,t;\underline{H}).$$

The resulting identity $V^*\underline{S}^*(t)u = \frac{1}{t}V_t^*J_t^*\big(\underline{S}^*(t-\cdot)u\big)$ shows that $V^*\underline{S}^*(t)u$ can be identified with the element $\frac{1}{t}J_t^*\big(\underline{S}^*(t-\cdot)u\big)$ of H_t and we obtain

$$\begin{aligned}\|V^*\underline{S}^*(t)u\|_{H_t} &= \frac{1}{t}\|J_t^*\big(\underline{S}^*(t-\cdot)u\big)\|_{H_t}\\ &\le \frac{1}{t\sqrt{2k}}\|\underline{S}^*(t-\cdot)u\|_{L^2(0,t;\underline{H})}\\ &\le \frac{1}{\sqrt{2kt}}\sup_{s\ge 0}\|\underline{S}^*(s)\|_{\mathcal{L}(\underline{H})}\|u\|.\end{aligned}$$

$\square$

The following pointwise gradient bound is included for reasons of completeness. We shall only need the special case corresponding to $r = 2$, for which a simpler proof can be given; see Remark 4.26.

Theorem 4.25 (Pointwise gradient bounds). *Let $1 < r < \infty$. For $f \in \mathcal{F}C_{\rm b}(E)$ and $t > 0$ we have, for μ-almost all $x \in E$,*

$$\sqrt{t}\|D_V P(t)f(x)\| \lesssim (P(t)|f|^r(x))^{1/r}.$$

Proof. For notational simplicity we take f of the form $f = \varphi(\phi_h)$ with $\varphi \in C_{\rm b}(\mathbb{R})$ and $h \in H$. It is immediate to check that the argument carries over to general cylindrical functions in $\mathcal{F}C_{\rm b}(E)$.

By Lemma 4.24 we have $S^*(t)V^*u \in H_t$ for $u \in \mathsf{D}(V^*)$ and therefore, for all $h \in H$,

$$\phi_{S(t)h}(iV^*u) = [S(t)h, V^*u] = [h, S^*(t)V^*u] = \phi^{\mu_t}_{V_t h}(iS^*(t)V^*u).$$

By Proposition 4.23 (with $\mathscr{H} = \mathbb{R}$) we find that for all $g \in H$,

$$\begin{aligned} P(t)f(x + iV^*u) &= \int_E \varphi(\phi_{S(t)h}(x + iV^*u) + \phi^{\mu_t}_{V_t h}(y))\, d\mu_t(y) \\ &= \int_E \varphi(\phi_{S(t)h}(x) + \phi^{\mu_t}_{V_t h}(y + iS^*(t)V^*u))\, d\mu_t(y). \end{aligned}$$

Recalling that D denotes the Malliavin derivative we have, for all $u \in \mathsf{D}(V^*)$,

$$\begin{aligned} [D_V P(t)f(x), u] &= [DP(t)f(x), V^*u] \\ &= \lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon}\Big(P(t)f(x + \varepsilon iV^*u) - P(t)f(x)\Big) \\ &= \lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon}\int_E \varphi(\phi_{S(t)h}(x) + \phi^{\mu_t}_{V_t h}(y + \varepsilon iS^*(t)V^*u)) \\ &\qquad\qquad - \varphi(\phi_{S(t)h}(x) + \phi^{\mu_t}_{V_t h}(y))\, d\mu_t(y). \end{aligned}$$

Using Lemma 4.24 and the Cameron-Martin Theorem 1.17 we obtain

$$[D_V P(t)f(x), u] = \lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon}\int_E \big(E^{\mu_t}_{\varepsilon S^*(t)V^*u}(y) - 1\big)\varphi(\phi_{S(t)h}(x) + \phi^{\mu_t}_{V_t h}(y))\, d\mu_t(y),$$

where $E^{\mu_t}_h(y) = \exp\big(\phi^{\mu_t}_h(y) - \frac{1}{2}\|h\|^2_{H_t}\big)$. It is easy to see that for each $h \in H_t$ the family $\big(\frac{1}{\varepsilon}(E^{\mu_t}_{\varepsilon h} - 1)\big)_{0<\varepsilon<1}$ is uniformly bounded in $L^2(\mu_t)$, and therefore uniformly integrable in $L^1(\mu_t)$. Passage to the limit $\varepsilon \downarrow 0$ now gives

$$[D_V P(t)f(x), u] = \int_E \phi^{\mu_t}_{S^*(t)V^*u}(y)\varphi(\phi_{S(t)h}(x) + \phi^{\mu_t}_{V_t h}(y))\, d\mu_t(y).$$

By Hölder's inequality with $\frac{1}{q} + \frac{1}{r} = 1$, using the Gaussianity of $\phi^{\mu_t}_{S^*(t)V^*u}$ on (μ_t) and the Kahane-Khintchine inequality, Proposition 4.23, and Lemma 4.24 we find that

$$
\begin{aligned}
&|[D_V P(t)f(x), u]| \\
&\quad \le \Big(\int_E |\phi^{\mu_t}_{S^*(t)V^*u}(y)|^q \, d\mu_t(y) \Big)^{1/q} \Big(\int_E |\varphi(\phi_{S(t)h}(x) + \phi^{\mu_t}_{V_t h}(y))|^r \, d\mu_t(y) \Big)^{1/r} \\
&\quad \lesssim \Big(\int_E |\phi^{\mu_t}_{S^*(t)V^*u}(y)|^2 \, d\mu_t(y) \Big)^{1/2} (P(t)|f|^r(x))^{\frac{1}{r}} \\
&\quad = \|S^*(t)V^*u\|_{H_t} (P(t)|f|^r(x))^{\frac{1}{r}} \\
&\quad \lesssim \frac{1}{\sqrt{t}} \|u\| (P(t)|f|^r(x))^{\frac{1}{r}}.
\end{aligned}
$$

The desired estimate is obtained by taking the supremum over all $u \in \mathsf{D}(V^*)$ with $\|u\| \le 1$. $\square$

Remark 4.26. There is a different argument which we learned from [101, p.328]) which can be used to prove Theorem 4.25 for $r = 2$. Using the product rule from Lemma 4.7, the fact that $\|Bu\| \ge k\|u\|$ for $u \in \overline{\mathsf{R}(V)}$, and the positivity of $P(s)$, we obtain

$$
\begin{aligned}
P(t)f^2 - (P(t)f)^2 &= \int_0^t \partial_s \big(P(s) \big(|P(t-s)f|^2 \big) \big) \, ds \\
&= -\int_0^t P(s) \big(L(P(t-s)f)^2 - 2P(t-s)f \cdot LP(t-s)f \big) \, ds \\
&= 2 \int_0^t P(s) \big(\|BD_V P(t-s)f\|^2 \big) \, ds \\
&\ge 2k \int_0^t P(s) \big(\|D_V P(t-s)f\|^2 \big) \, ds.
\end{aligned}
$$

Next we estimate, for μ-almost all $x \in E$,

$$
\begin{aligned}
M^2 P(r)(\|D_V f\|^2)(x) &\ge P(r)(\|\underline{S}(r) D_V f\|^2)(x) \\
&\overset{(*)}{\ge} \|(P(r) \otimes I)(\underline{S}(r) D_V f)\|^2(x) \\
&= \|\underline{P}(r) D_V f(x)\|^2 \\
&= \|D_V P(r) f(x)\|^2,
\end{aligned}
$$

where $M := \sup_{t \ge 0} \|\underline{S}(t)\|$ and $(*)$ follows from Proposition 4.23 (with $\mathscr{H} = \underline{H}$) and Jensen's inequality. The case $r = 2$ of Theorem 4.25 follows from these two estimates.

The next result is in some sense the dual version of a maximal inequality. It could be compared with the dual version of the non-commutative Doob inequality of [86].

Proposition 4.27. *Let $(T(t))_{t>0}$ be a family of positive operators operators on $L^p := L^p(M, \mu)$, where (M, μ) is a σ-finite measure space and $1 \le p < \infty$.*

Suppose that the maximal function $T^*_\star f := \sup_{t>0} |T^*(t)f|$ *is measurable and* L^q*-bounded, where* $\frac{1}{p}+\frac{1}{q}=1$. *Then, for all* $f_1,\dots,f_n \in L^p$ *and all* $t_1,\dots,t_n > 0$,

$$\Big\| \sum_{k=1}^n T(t_k)|f_k| \Big\|_p \lesssim \Big\| \sum_{k=1}^n |f_k| \Big\|_p.$$

Proof. Taking the supremum over all $g = (g_k)_{k=1}^n \in L^q(\ell^\infty_n)$ of norm one we obtain

$$\begin{aligned}
\Big\| \sum_{k=1}^n T(t_k)|f_k| \Big\|_p &= \|\big(T(t_{(\cdot)})|f_{(\cdot)}|\big)\|_{L^p(\ell^1_n)} \\
&= \sup_g \int_E \sum_{k=1}^n T(t_k)|f_k| \cdot g_k \, d\mu \\
&= \sup_g \int_E \sum_{k=1}^n |f_k| \cdot T^*(t_k) g_k \, d\mu \\
&\le \|\big(|f_{(\cdot)}|\big)\|_{L^p(\ell^1_n)} \sup_g \|\big(T^*(t_{(\cdot)})g_{(\cdot)}\big)\|_{L^q(\ell^\infty_n)}.
\end{aligned}$$

Using the positivity of T^* on L^q to obtain

$$\sup_{1\le k\le n} T^*_\star |g_k| \le T^*_\star\big(\sup_{1\le k\le n} |g_k|\big),$$

we estimate

$$\begin{aligned}
\big\|\big(T^*(t_{(\cdot)})g_{(\cdot)}\big)\big\|_{L^q(\ell^\infty_n)} &= \big\| \sup_{1\le k\le n} |T^*(t_k)g_k| \,\big\|_{L^q} \\
&\le \big\| \sup_{1\le k\le n} T^*_\star |g_k| \,\big\|_{L^q} \\
&\le \big\| T^*_\star(\sup_{1\le k\le n} |g_k|) \,\big\|_{L^q} \\
&\lesssim \big\| \sup_{1\le k\le n} |g_k| \,\big\|_{L^q} \\
&= \|(g_k)\|_{L^q(\ell^\infty_n)}.
\end{aligned}$$

This completes the proof. $\square$

The previous two results are now combined to prove:

Proof (of Theorem 4.18, $2 < p < \infty$*).* Let $\frac{2}{p}+\frac{1}{q}=1$. Proposition 5.54 implies that the maximal function

$$P^*_\star f := \sup_{t>0} |P^*(t)f|$$

is bounded on L^q. Using Theorem 4.25 (for $r=2$) and Proposition 4.27 we obtain, for all $f_1,\dots,f_n \in \mathcal{F}C_{\mathrm{b}}(E)$,

$$\begin{aligned}\Big\|\Big(\sum_{k=1}^n \|\sqrt{t_k}D_V P(t_k)f_k\|^2\Big)^{1/2}\Big\|_p &\lesssim \Big\|\Big(\sum_{k=1}^n P(t_k)|f_k|^2\Big)^{1/2}\Big\|_p \\ &= \Big\|\sum_{k=1}^n P(t_k)|f_k|^2\Big\|_{p/2}^{1/2} \\ &\lesssim \Big\|\sum_{k=1}^n |f_k|^2\Big\|_{p/2}^{1/2} \\ &= \Big\|\Big(\sum_{k=1}^n |f_k|^2\Big)^{1/2}\Big\|_p.\end{aligned}$$

By an approximation argument this estimate extends to arbitrary $f_1, \ldots, f_n \in L^p(\mu)$. Now Proposition 5.5 implies the γ-boundedness of $\{\sqrt{t}D_V P(t) : t > 0\}$.

Taking Laplace transforms and using Proposition 5.3, it follows that $\mathsf{D}_p(L) \subseteq \mathsf{D}_p(D_V)$ and that the collection $\{tD_V(I + t^2L)^{-1} : t > 0\}$ is γ-bounded from $L^p(\mu)$ into $L^p(\mu; \underline{H})$. As in the case $1 < p \le 2$, Lemmas 4.7 and 4.13 imply that $\mathsf{D}_p(L)$ is even a core for $\mathsf{D}_p(D_V)$. □

Proof (Proof of Theorem 4.19, $2 < p < \infty$). Since L has a bounded H^∞-calculus of angle $< \frac{1}{2}\pi$ by Proposition 4.17, the result follows from Theorem 5.47. □

γ-Bisectoriality of the operator Π

Our next aim is to show that the estimates obtained in Section 4.2 imply randomised bisectoriality of the operator Π.

We need a couple of technical results which are necessary for a rigorous L^p-analysis. Readers who are primarily interested in the main ideas are recommended to jump to Theorem 4.32

For the next result we recall that $\mathcal{C} := \mathcal{F}C_{\mathrm{b}}^\infty(E; \mathsf{D}(A))$ is a P-invariant core for $\mathsf{D}_p(L)$. We set $\mathcal{C}^* := \mathcal{F}C_{\mathrm{b}}^\infty(E; \mathsf{D}(A^*))$; this is a P^*-invariant core for $\mathsf{D}_p(L^*)$.

Proposition 4.28. *In L^p we have $L = (D_V^* B)D_V$. More precisely, $f \in \mathsf{D}_p(L)$ if and only if $f \in \mathsf{D}_p(D_V)$ and $D_V f \in \mathsf{D}_p(D_V^* B)$, in which case we have $Lf = (D_V^* B)D_V f$.*

Proof. First note that for all $f, g \in \mathcal{C}$ we have $\langle Lf, g\rangle = \langle D_V f, B^* D_V g\rangle$. Since $\mathcal{C}$ is a core for $\mathsf{D}_p(L)$, and $\mathsf{D}_p(L)$ is core for $\mathsf{D}_p(D_V)$ by the first part of Theorem 4.18, this identity extends to all $f \in \mathsf{D}_p(L)$ and $g \in \mathsf{D}_p(D_V)$. This implies that $D_V f \in \mathsf{D}_p((B^* D_V)^*)$ and $(B^* D_V)^* D_V f = Lf$. Since $(B^* D_V)^* = D_V^* B$, we find that $L \subseteq (D_V^* B)D_V$.

To prove the other inclusion we take $f \in \mathsf{D}_p(D_V)$ such that $D_V f \in \mathsf{D}_p(D_V^* B)$. We have $\langle f, L^* g\rangle = \langle D_V f, B^* D_V g\rangle = \langle (D_V^* B)D_V f, g\rangle$ for all $g \in$

$\mathcal{C}^*$, where the second identity follows from $D_V f \in \mathsf{D}_p(D_V^* B) = \mathsf{D}_p((B^* D_V)^*)$. Since $\mathcal{C}^*$ is a core for $\mathsf{D}_p(L^*)$ this implies that $f \in \mathsf{D}(L)$ and $Lf = (D_V^* B) D_V f$. □

We shall be interested in the restriction $D_V^* B|_{\overline{\mathsf{R}_p(D_V)}}$ of $D_V^* B$ to $\overline{\mathsf{R}_p(D_V)}$. As its domain we take

$$\mathsf{D}_p(D_V^* B|_{\overline{\mathsf{R}_p(D_V)}}) := \{f \in \overline{\mathsf{R}_p(D_V)} : \ Bf \in \mathsf{D}_p(D_V^*)\} = \mathsf{D}_p(D_V^* B) \cap \overline{\mathsf{R}_p(D_V)}.$$

In the middle expression, as before we consider D_V^* as a densely defined operator from $L^p(\mu; \underline{H})$ to $L^p(\mu)$.

Corollary 4.29. *The restriction $D_V^* B|_{\overline{\mathsf{R}_p(D_V)}}$ is closed and densely defined.*

Proof. Let $f \in \mathsf{D}_p(D_V)$. By the first part of Theorem 4.18 there exist functions $f_n \in \mathsf{D}_p(L)$ such that $f_n \to f$ in $\mathsf{D}_p(D_V)$. Proposition 4.28 implies that $D_V f_n \in \mathsf{D}_p(D_V^* B|_{\overline{\mathsf{R}_p(D_V)}})$. This shows that $D_V^* B|_{\overline{\mathsf{R}_p(D_V)}}$ is densely defined on $\overline{\mathsf{R}_p(D_V)}$. Closedness is clear. □

Proposition 4.30. *The domain $\mathsf{D}_p(\underline{L})$ is a core for $\mathsf{D}_p(D_V^* B|_{\overline{\mathsf{R}_p(D_V)}})$. Moreover, for all $t > 0$ the operators $(I + t^2 L)^{-1} D_V^* B|_{\overline{\mathsf{R}_p(D_V)}}$ and $P(t) D_V^* B|_{\overline{\mathsf{R}_p(D_V)}}$ (initially defined on $\mathsf{D}_p(D_V^* B|_{\overline{\mathsf{R}_p(D_V)}})$) extend uniquely to bounded operators from $\overline{\mathsf{R}_p(D_V)}$ to $L^p(\mu)$, and for all $F \in \overline{\mathsf{R}_p(D_V)}$ we have*

$$(I + t^2 L)^{-1} D_V^* B F = D_V^* B (I + t^2 \underline{L})^{-1} F$$

and

$$P(t) D_V^* B F = D_V^* B \underline{P}(t) F.$$

Proof. We split the proof into four steps.

Step 1 - By Proposition 4.28, for all $f \in \mathsf{D}_p(L)$ we have $f \in \mathsf{D}_p(D_V)$ and $D_V f \in \mathsf{D}_p(D_V^* B|_{\overline{\mathsf{R}_p(D_V)}})$, and for all $t > 0$ we have

$$P(t)(D_V^* B) D_V f = P(t) L f = L P(t) f = (D_V^* B) D_V P(t) f = D_V^* B \underline{P}(t) D_V f.$$

By taking Laplace transforms and using the closedness of $D_V^* B$, this gives $(I + t^2 \underline{L})^{-1} D_V f \in \mathsf{D}_p(D_V^* B|_{\overline{\mathsf{R}(D_V)}})$ and

$$(I + t^2 L)^{-1} (D_V^* B) D_V f = D_V^* B (I + t^2 \underline{L})^{-1} D_V f. \tag{4.6}$$

Step 2 - By Theorem 4.18, for all $t > 0$ the operator $T(t) := B^* D_V (I + t^2 L^*)^{-1}$ is bounded from $L^{p'}(\mu)$ into $L^{p'}(\mu; \underline{H})$, $\frac{1}{p} + \frac{1}{p'} = 1$. For all $F \in \mathsf{D}_p(D_V^* B|_{\overline{\mathsf{R}_p(D_V)}})$ and $g \in L^q(\mu)$ we have

$$\langle F, T(t) g \rangle = \langle F, B^* D_V (I + t^2 L^*)^{-1} g \rangle = \langle (I + t^2 L)^{-1} D_V^* B F, g \rangle. \tag{4.7}$$

Now let $F \in \overline{\mathsf{R}_p(D_V)}$ be arbitrary and let $(F_n)_{n\geq 1} \subseteq \mathsf{D}_p(D_V^* B|_{\overline{\mathsf{R}_p(D_V)}})$ be a sequence converging to F in $\overline{\mathsf{R}_p(D_V)}$. By Proposition 4.28 and the fact that $\mathsf{D}_p(L)$ is a core for $\mathsf{D}_p(D_V)$ we may take the F_n of the form $D_V f_n$ with $f_n \in \mathsf{D}_p(L)$. Then $(I + t^2\underline{L})^{-1}F_n \to (I + t^2\underline{L})^{-1}F$, and from (4.6) we obtain

$$D_V^* B(I + t^2\underline{L})^{-1}F_n = (I + t^2 L)^{-1}D_V^* BF_n = T^*(t)F_n \to T^*(t)F.$$

The closedness of $D_V^* B$ implies that $(I + t^2\underline{L})^{-1}F \in \mathsf{D}_p(D_V^* B|_{\overline{\mathsf{R}_p(D_V)}})$. This proves the domain inclusion $\mathsf{D}_p(\underline{L}) \subseteq \mathsf{D}_p(D_V^* B|_{\overline{\mathsf{R}_p(D_V)}})$, along with the identity

$$D_V^* B(I + t^2\underline{L})^{-1}F = T^*(t)F, \qquad F \in \overline{\mathsf{R}_p(D_V)}.$$

Note that for $F \in \mathsf{D}_p(D_V^* B)$, from (4.7) we also obtain

$$D_V^* B(I + t^2\underline{L})^{-1}F = T^*(t)F = (I + t^2 L)^{-1}D_V^* BF. \tag{4.8}$$

Step 3 - By Step 2 the operator $D_V^* B(I+t^2\underline{L})^{-1}$ is bounded from $\overline{\mathsf{R}_p(D_V)}$ to $L^p(\mu)$. Therefore, by (4.6), the operator $(I + t^2 L)^{-1}D_V^* B$ (initially defined on the dense domain $\mathsf{D}_p(D_V^* B|_{\overline{\mathsf{R}_p(D_V)}})$) uniquely extends to a bounded operator from $\overline{\mathsf{R}_p(D_V)}$ to $L^p(\mu)$, and for this extension we obtain the identity

$$(I + t^2 L)^{-1}D_V^* B = D_V^* B(I + t^2\underline{L})^{-1}.$$

On $\mathsf{D}_p(D_V^* B|_{\overline{\mathsf{R}_p(D_V)}})$, the identity $D_V^* B\underline{P}(t) = P(t)D_V^* B$ follows from (4.8) by real Laplace inversion (cf. the proof of Lemma 4.11). The existence of a unique bounded extension of $P(t)D_V^* B$ is proved in the same way as before.

Step 4 – It remains to prove that $\mathsf{D}_p(\underline{L})$ is a core for $\mathsf{D}_p(D_V^* B|_{\overline{\mathsf{R}_p(D_V)}})$. Take $F \in \mathsf{D}_p(D_V^* B|_{\overline{\mathsf{R}_p(D_V)}})$. Then $\lim_{t\to 0}(I + t^2\underline{L})^{-1}F = F$ in $\overline{\mathsf{R}_p(D_V)}$ and, by (4.8) $\lim_{t\to 0} D_V^* B(I + t^2\underline{L})^{-1}F = \lim_{t\to 0}(I + t^2 L)^{-1}D_V^* BF = D_V^* BF$ in $L^p(\mu)$. This gives the result. $\square$

Proposition 4.31. *For all $F \in \mathsf{D}_p(\underline{L})$ we have $F \in \mathsf{D}_p(D_V^* B)$, $D_V^* BF \in \mathsf{D}_p(D_V)$, and $D_V(D_V^* B)F = \underline{L}F$.*

Proof. Since $\mathsf{D}_p(L)$ is a core for $\mathsf{D}_p(D_V)$, the set $\mathscr{P} := \{D_V(I + L)^{-1}g : g \in \mathsf{D}_p(D_V)\}$ is a $\underline{P}$-invariant dense subspace of $\overline{\mathsf{R}_p(D_V)}$. To see that $\mathscr{P}$ is contained in $\mathsf{D}_p(\underline{L})$, note that if $g \in \mathsf{D}_p(D_V)$, then $f := (I + L)^{-1}g \in \mathsf{D}_p(L)$ and $D_V f = D_V(1+L)^{-1}g = (1+\underline{L})^{-1}D_V g \in \mathsf{D}_p(\underline{L})$ as claimed. It follows that $\mathscr{P}$ is a core for $\mathsf{D}_p(\underline{L})$, and hence a core for $\mathsf{D}_p(D_V^* B|_{\overline{\mathsf{R}_p(D_V)}})$ by Proposition 4.30. Moreover, $(1 + \underline{L})D_V f = D_V g = D_V(I + L)f$, and therefore $\underline{L}D_V f = D_V Lf$.

For $F \in \mathscr{P}$, say $F = D_V f$ with $f = (I + L)^{-1}g$ for some $g \in \mathsf{D}_p(D_V)$, we then have

$$\begin{aligned}\underline{L}F = \underline{L}D_V f = D_V Lf &= D_V((D_V^* B)D_V)f \\ &= (D_V(D_V^* B))D_V f = D_V(D_V^* B)F.\end{aligned}$$

To see that this above identity extends to arbitrary $F \in \mathsf{D}_p(\underline{L})$, let $F_n \to F$ in $\mathsf{D}_p(\underline{L})$ with all F_n in $\mathcal{P}$. It follows from Proposition 4.30 that $F_n \to F$ in $\mathsf{D}_p(D_V^* B)$. In particular, $D_V^* BF_n \to D_V^* BF$ in $L^p(\mu)$. Since $D_V(D_V^* B)F_n = \underline{L}F_n \to \underline{L}F$ in $\overline{\mathsf{R}_p(D_V)}$, the closedness of D_V then implies that $D_V^* BF \in \mathsf{D}_p(D_V)$ and $D_V(D_V^* B)F = \underline{L}F$. $\square$

In the remainder of this section we consider $D_V^* B$ as a closed and densely defined operator from $\overline{\mathsf{R}_p(D_V)}$ to $L^p(\mu)$ and write $D_V^* B$ instead of using the more precise notation $D_V^* B|_{\overline{\mathsf{R}_p(D_V)}}$.

Theorem 4.32. *Let $1 < p < \infty$. The operator Π is γ-bisectorial on $L^p(\mu) \oplus \overline{\mathsf{R}_p(D_V)}$.*

Proof. First we check that $\mathsf{N}(I - it\Pi) = \{0\}$ for $t \in \mathbb{R} \setminus \{0\}$. For this purpose, suppose that $(I - it\Pi)(f, F) = 0$ for some $f \in L^p(\mu)$ and $F \in \overline{\mathsf{R}_p(D_V)}$. Then $f - itD_V^* BF = 0$ and $F - itD_V f = 0$. Combining these identities we deduce that $f + t^2(D_V^* B)D_V f = 0$. Since $(D_V^* B)D_V = L$ by Proposition 4.28, we find that $f = 0$ by the sectoriality of L. It follows that $F = 0$ as well.

A computation based on the technical lemmata in this section and the resolvent formula below shows that $I - it\Pi$ is surjective as an operator on $L^p(\mu) \oplus \overline{\mathsf{R}_p(D_V)}$. It follows that $i\mathbb{R} \setminus \{0\} \subseteq \rho(\Pi)$ and

$$(I - it\Pi)^{-1} = \begin{bmatrix} (1 + t^2 L)^{-1} & it(I + t^2 L)^{-1} D_V^* B \\ itD_V(I + t^2 L)^{-1} & (I + t^2 \underline{L})^{-1} \end{bmatrix}, \qquad t \in \mathbb{R} \setminus \{0\};$$

the rigorous interpretation of this identity (in particular, the surjectivity of $it - \Pi$) is provided by the above propositions. Note that the off-diagonal entries are well defined and bounded by Theorem 4.18 and Proposition 4.30; the proof of the latter result also shows that $(I + t^2 L)^{-1} D_V^* B$ is the adjoint of $BD_V(I + t^2 L)^{-1}$.

It remains to check the γ-boundedness of the entries of the right-hand side matrix for $t \in \mathbb{R} \setminus \{0\}$. For the upper left and the lower right entry this follows from the γ-sectoriality of L and $\underline{L}$ on $L^p(\mu)$ and $\overline{\mathsf{R}_p(D_V)}$ respectively. Theorem 4.18 ensures the γ-boundedness of the lower left entry, and the γ-boundedness of the upper right entry follows from Proposition 5.4 (applied with B and L replaced by B^* and L^*). $\square$

As a consequence of the bisectoriality of Π, the operator Π^2 is sectorial. Moreover,

$$\Pi^2 = \begin{bmatrix} (D_V^* B)D_V & 0 \\ 0 & D_V(D_V^* B) \end{bmatrix} = \begin{bmatrix} L & 0 \\ 0 & \underline{L} \end{bmatrix}.$$

To justify the latter identity, we appeal to Propositions 4.28 and 4.31 to obtain the inclusion $\begin{bmatrix} L & 0 \\ 0 & \underline{L} \end{bmatrix} \subseteq \Pi^2$. Since both operators are sectorial of angle $< \frac{1}{2}\pi$, they are in fact equal.

Proposition 4.33. *On $L^p(\mu)$ and $\overline{\mathsf{R}_p(D_V)}$ the following identities hold:*

$$\overline{\mathsf{R}_p(L)} = \overline{\mathsf{R}_p(D_V^* B)}, \qquad \mathsf{N}_p(L) = \mathsf{N}_p(D_V),$$
$$\overline{\mathsf{R}_p(\underline{L})} = \overline{\mathsf{R}_p(D_V)}, \qquad \mathsf{N}_p(\underline{L}) = \mathsf{N}_p(D_V^* B) = \{0\}.$$

Moreover, $L^p(\mu) = \overline{\mathsf{R}_p(D_V^* B)} \oplus \mathsf{N}_p(D_V)$.

We recall that $D_V^* B$ is interpreted as a densely defined closed operator from $\overline{\mathsf{R}_p(D_V)}$ to $L^p(\mu)$. In the final section we will show that under the assumptions of Theorem 4.37(c) we have $\overline{\mathsf{R}_p(D_V^* B)} = \overline{\mathsf{R}_p(D_V^*)}$ and that in this situation the space $\overline{\mathsf{R}_p(D_V^* B)}$ does not change if we consider $D_V^* B$ as an unbounded operator from $L^p(\mu; \underline{H})$ to $L^p(\mu)$.

Proof. The bisectoriality of Π on $L^p(\mu) \oplus \overline{\mathsf{R}_p(D_V)}$ implies that

$$\overline{\mathsf{R}_p(\Pi^2)} = \overline{\mathsf{R}_p(\Pi)} \qquad \text{and} \qquad \mathsf{N}_p(\Pi^2) = \mathsf{N}_p(\Pi).$$

The result follows from this by considering both coordinates separately. The fact that $\mathsf{N}_p(D_V^* B) = \{0\}$ follows from the bisectorial decomposition $L^p(\mu) \oplus \overline{\mathsf{R}_p(D_V)} = \overline{\mathsf{R}_p(\Pi)} \oplus \mathsf{N}_p(\Pi)$ and considering the second coordinate. The final identity follows by inspecting the first coordinate of the same decomposition. □

4.3 L^p-Boundedness of the Riesz transform

In this section we will complete the proof of Theorem 4.37, which is a refined version of Theorem 0.1. First we will prove square function estimates for the operator $\underline{L}$ in a slightly more general setting.

Square function estimates for generators of tensor product semigroups

In this section we will leave the Wiener space framework and prove a general result on square function estimates for tensor product semigroups on Hilbert space-valued L^p-spaces.

Let $\underline{H}$ be a Hilbert space and let (M, μ) a σ-finite measure space, and fix $1 < p < \infty$. We consider γ-sectorial operators L and $\underline{A}$ on $L^p(\mu)$ and $\underline{H}$ respectively of angle $\omega_\gamma^+(L), \omega^+(\underline{A}) < \frac{1}{2}\pi$. It follows that $-L$ and $-\underline{A}$ generate γ-bounded analytic C_0-semigroups P and $\underline{S}$ on $L^p(\mu)$ and $\underline{H}$. We denote by $-\underline{L}$ the generator of the tensor product C_0-semigroup $\underline{P} = P \otimes \underline{S}$ on $L^p(\mu; \underline{H})$. It follows that $\underline{P}$ is a bounded analytic C_0-semigroup on Σ_θ^+, where $\theta := \frac{1}{2}\pi - \max\{\omega_R^+(L), \omega^+(\underline{A})\}$. Lemma 5.46 implies that the operator $\underline{L}$ is γ-sectorial of angle θ on $L^p(\mu; \underline{H})$.

We consider the following three square function norms:

$$\|u\|_{\underline{A}} := \Big(\int_0^\infty \|t\underline{A}\,\underline{S}(t)u\|^2\,\frac{dt}{t}\Big)^{1/2}, \qquad u \in \underline{H};$$
$$\|f\|_{p,L} := \Big\|\Big(\int_0^\infty |tLP(t)f|^2\,\frac{dt}{t}\Big)^{1/2}\Big\|_p, \qquad f \in L^p(\mu);$$
$$\|F\|_{p,\underline{L}} := \Big\|\Big(\int_0^\infty \|t\underline{L}\,\underline{P}(t)F\|^2\,\frac{dt}{t}\Big)^{1/2}\Big\|_p, \qquad F \in L^p(\mu;\underline{H}).$$

The main result in this subsection is the following:

Proposition 4.34. *Under the above assumptions we have:*

(1) *If* $\|u\|_{\underline{A}} \lesssim \|u\|$ *for all* $u \in \underline{H}$ *and* $\|f\|_{p,L} \lesssim \|f\|_p$ *for all* $f \in L^p(\mu)$*, then* $\|F\|_{p,\underline{L}} \lesssim \|F\|_p$ *for all* $F \in L^p(\mu;\underline{H})$*.*
(2) *If* $\|u\|_{\underline{A}} \gtrsim \|(I - P_{\mathsf{N}(\underline{A})})u\|$ *for all* $u \in \underline{H}$ *and* $\|f\|_{p,L} \gtrsim \|(I - P_{\mathsf{N}(L)})f\|_p$ *for all* $f \in L^p(\mu)$*, then* $\|F\|_{p,\underline{L}} \gtrsim \|(I - P_{\mathsf{N}(\underline{L})})F\|_p$ *for all* $F \in L^p(\mu;\underline{H})$*.*

As a consequence, if $\underline{A}$ *and* L *have bounded* H^∞*-functional calculi of angles less than* $\frac12\pi$*, then* $\underline{L}$ *has a bounded* H^∞*-functional calculus of angle less than* $\frac12\pi$*.*

Proof. Let us first show that (1) implies (2). It is well known that the assumptions of (2) imply the dual estimates $\|u\|_{\underline{A}^*} \lesssim \|u\|$ and $\|f\|_{q,L^*} \lesssim \|f\|_q$, where $\frac1p + \frac1q = 1$. By (1) we obtain that $\|F\|_{q,\underline{L}^*} \lesssim \|F\|_q$, and by duality we obtain the conclusion of (2).

The final assertion follows by combining (1) and (2) with Theorem 5.40.

It remains to prove (1). We proceed in three steps.

Step 1: We prove that

$$\|t(I\otimes\underline{A})(I\otimes\underline{S}(t))F\|_{\gamma(L^2(\mathbb{R}_+,\frac{dt}{t}),L^p(\mu;\underline{H}))} \lesssim \|F\|_p.$$

For $F \in L^p(\mu;\underline{H})$ we have, for μ-almost all $x \in M$,

$$\Big(\int_0^\infty \|t(I\otimes\underline{A})(I\otimes\underline{S}(t))F(x)\|^2\frac{dt}{t}\Big)^{1/2} \lesssim \|F(x)\|.$$

Integrating this estimate over M yields

$$\Big\|\Big(\int_0^\infty \|t(I\otimes\underline{A})(I\otimes\underline{S}(t))F\|^2\frac{dt}{t}\Big)^{1/2}\Big\|_p \le \|F\|_p.$$

Step 2: We prove that

$$\|t(L\otimes I)(P(t)\otimes I)F\|_{\gamma(L^2(\mathbb{R}_+,\frac{dt}{t}),L^p(\mu;\underline{H}))} \lesssim \|F\|_p.$$

Let $(h_j)_{j=1}^k$ be a finite orthonormal system in $\underline{H}$ and pick $F := \sum_{j=1}^k f_j \otimes h_j \in L^p(\mu;\underline{H})$. For $f \in L^p(\mu)$ let

$$(Uf)(t) := tLP(t)f$$

and notice that U is a bounded operator from $L^p(\mu)$ into $\gamma(L^2(\mathbb{R}_+, \frac{dt}{t}), L^p(\mu))$ by the assumption in (1) and Proposition 5.15.

Let $(r'_j)_{j\ge 1}$ and $(\gamma'_j)_{j\ge 1}$ be a Rademacher and a Gaussian sequence respectively on a probability space $(\Omega', \mathbb{P}')$. Noting the pointwise equality

$$\|t(L\otimes I)(P(t)\otimes I)F\|^2 = \sum_{j=1}^k |Uf_j(t)|^2$$

we have

$$\begin{aligned}
&\Big\|\Big(\int_0^\infty \|t(L\otimes I)(P(t)\otimes I)F\|^2 \frac{dt}{t}\Big)^{1/2}\Big\|_p = \Big\|\Big(\int_0^\infty \sum_{j=1}^k |Uf_j(t)|^2 \frac{dt}{t}\Big)^{1/2}\Big\|_p \\
&= \Big\|\Big(\int_0^\infty \mathbb{E}'\Big|\sum_{j=1}^k r'_j Uf_j(t)\Big|^2 \frac{dt}{t}\Big)^{1/2}\Big\|_p \eqsim \Big\|\sum_{j=1}^k r'_j Uf_j\Big\|_{\gamma(L^2(\mathbb{R}_+\times\Omega', \frac{dt}{t}\otimes\mathbb{P}'), L^p(\mu))} \\
&\overset{(*)}{\eqsim} \Big\|U\sum_{j=1}^k r'_j f_j\Big\|_{\gamma(L^2(\Omega',\mathbb{P}'),\gamma(L^2(\mathbb{R}_+,\frac{dt}{t}),L^p(\mu)))} \overset{(**)}{\lesssim} \Big\|\sum_{j=1}^k r'_j f_j\Big\|_{\gamma(L^2(\Omega',\mathbb{P}'),L^p(\mu))} \\
&\overset{(***)}{=} \Big(\mathbb{E}'\Big\|\sum_{j=1}^k \gamma'_j f_j\Big\|^2_{L^p(\mu)}\Big)^{1/2} \eqsim \Big(\mathbb{E}'\Big\|\sum_{j=1}^k \gamma'_j f_j\Big\|^p_{L^p(\mu)}\Big)^{1/p} \\
&= \Big\|\Big(\mathbb{E}'\Big|\sum_{j=1}^k \gamma'_j f_j\Big|^p\Big)^{1/p}\Big\|_{L^p(\mu)} \eqsim \Big\|\Big(\sum_{j=1}^k |f_j|^2\Big)^{1/2}\Big\|_{L^p(\mu)} \\
&= \|F\|_p.
\end{aligned}$$

In $(*)$ we used Proposition 5.12, in $(**)$ we used the boundedness of U from $L^p(\mu)$ into $\gamma(L^2(\mathbb{R}_+, \frac{dt}{t}), L^p(\mu))$, and in $(***)$ the definition of the radonifying norm of finite rank operators.

Step 3: We combine the previous two estimates. By Lemma 5.46 the family $\{P(t):\ t\ge 0\}$ is γ-bounded on $L^p(\mu)$. Hence by Proposition 5.6 the family $\{P(t)\otimes I:\ t\ge 0\}$ is γ-bounded on $L^p(\mu;\underline{H})$. Also, by a simple application of Fubini's theorem, $\{I\otimes \underline{S}(t):\ t\ge 0\}$ is γ-bounded. Combining these facts with Proposition 5.16, for $F\in L^p(\mu;\underline{H})$ we obtain

$$\begin{aligned}
&\Big\|\Big(\int_0^\infty \|t\underline{L}\,\underline{P}F\|^2\frac{dt}{t}\Big)^{1/2}\Big\|_p \\
&\quad\eqsim \|t\underline{L}\,\underline{P}F\|_{\gamma(L^2(\mathbb{R}_+,\frac{dt}{t}),L^p(\mu;\underline{H}))} \\
&\quad\lesssim \|(I\otimes S(t))t(L\otimes I)(P(t)\otimes I)F\|_{\gamma(L^2(\mathbb{R}_+,\frac{dt}{t}),L^p(\mu;\underline{H}))} \\
&\qquad\quad + \|(P(t)\otimes I)t(I\otimes\underline{A})(I\otimes\underline{S}(t))F\|_{\gamma(L^2(\mathbb{R}_+,\frac{dt}{t}),L^p(\mu;\underline{H}))} \\
&\quad\lesssim \|t(L\otimes I)(P(t)\otimes I)F\|_{\gamma(L^2(\mathbb{R}_+,\frac{dt}{t}),L^p(\mu;\underline{H}))} \\
&\qquad\quad + \|t(I\otimes\underline{A})(I\otimes\underline{S}(t))F\|_{\gamma(L^2(\mathbb{R}_+,\frac{dt}{t}),L^p(\mu;\underline{H}))} \\
&\quad\lesssim \|F\|_p.
\end{aligned}$$

□

Remark 4.35. The final assertion in Proposition 4.34 is due to Lancien, Lancien, and Le Merdy [98, Theorem 1.4] who proved it using operator-valued H^∞-functional calculi.

Proof of the main result

We return to the Wiener space setting and present the proof of Theorem 0.1. We start with an L^p-analogue of Proposition 3.12. Observe that the proof uses the γ-bisectoriality of the operator Π which has been proved in Theorem 4.32.

Proposition 4.36. *Let $1 < p < \infty$. For $\psi \in H_0^\infty(\Sigma_\theta^+)$ where $\theta \in (2\omega_\gamma(\Pi), \pi)$ we have*

$$\|\sqrt{L}f\|_p \eqsim \|\psi(t\underline{L})D_V f\|_{\gamma(L^2(\mathbb{R}_+, \frac{dt}{t}), L^p(\mu, \underline{H}))}, \qquad f \in \mathsf{D}(L).$$

Proof. Take $\widetilde{\varphi} \in H_0^\infty(\Sigma_{2\theta}^+)$ and define $\varphi \in H_0^\infty(\Sigma_\theta)$ by $\varphi(z) := \widetilde{\varphi}(z^2)$. We obtain

$$\begin{aligned}
\|\sqrt{L}u\| &\eqsim \|\psi(tL)\sqrt{L}u\|_{\gamma(L^2(\mathbb{R}_+, \frac{dt}{t}), L^p(\mu))} && \text{(Proposition 3.11)}\\
&= \Big\|\psi(t\Pi^2)\sqrt{\Pi^2}\begin{bmatrix} u\\ 0\end{bmatrix}\Big\|_{\gamma(L^2(\mathbb{R}_+, \frac{dt}{t}), \mathcal{L}^p(\mu))} \\
&\eqsim \Big\|\tilde{\psi}(t\Pi)\sqrt{\Pi^2}\begin{bmatrix} u\\ 0\end{bmatrix}\Big\|_{\gamma(L^2(\mathbb{R}_+, \frac{dt}{t}), \mathcal{L}^p(\mu))} && \text{(Proposition 5.32)}\\
&= \Big\|\operatorname{sgn}(t\Pi)\tilde{\psi}(t\Pi)\Pi\begin{bmatrix} u\\ 0\end{bmatrix}\Big\|_{\gamma(L^2(\mathbb{R}_+, \frac{dt}{t}), \mathcal{L}^p(\mu))} && \text{(Proposition 5.30)}\\
&\eqsim \Big\|\tilde{\psi}(t\Pi)\Pi\begin{bmatrix} u\\ 0\end{bmatrix}\Big\|_{\gamma(L^2(\mathbb{R}_+, \frac{dt}{t}), \mathcal{L}^p(\mu))} && \text{(Corollary 5.39)}\\
&= \Big\|\tilde{\psi}(t\Pi)\begin{bmatrix} 0\\ D_V u\end{bmatrix}\Big\|_{\gamma(L^2(\mathbb{R}_+, \frac{dt}{t}), \mathcal{L}^p(\mu))} \\
&\eqsim \Big\|\psi(t\Pi^2)\begin{bmatrix} 0\\ D_V u\end{bmatrix}\Big\|_{\gamma(L^2(\mathbb{R}_+, \frac{dt}{t}), \mathcal{L}^p(\mu))} && \text{(Proposition 5.32)}\\
&\eqsim \|\psi(t\underline{L})D_V u\|_{\gamma(L^2(\mathbb{R}_+, \frac{dt}{t}), L^p(\mu; \underline{H}))}.
\end{aligned}$$

The extension to arbitrary $\psi \in H_0^\infty(\Sigma_\theta^+)$ follows from Corollary 5.39. □

Now we are ready to prove our main result, which is a comprehensive version of Theorem 0.1 involving one-sided estimates for the Riesz transform associated with L.

Theorem 4.37. *Let $1 < p < \infty$.*

(a) *The following assertions are equivalent:*
  (a1) $\mathsf{D}_p(\sqrt{L}) \subseteq \mathsf{D}_p(D_V)$ *with* $\|D_V f\|_p \lesssim \|\sqrt{L}f\|_p$*;*

(a2) $\underline{L}$ *satisfies a square function estimate on* $\overline{\mathsf{R}_p(D_V)}$*:*

$$\|F\|_p \lesssim \Big\| \Big(\int_0^\infty \|t\underline{L}\,\underline{P}(t)F\|^2 \, \frac{dt}{t} \Big)^{1/2} \Big\|_p ;$$

(a3) $\mathsf{D}(\sqrt{A}) \subseteq \mathsf{D}(V)$ *with* $\|Vh\| \lesssim \|\sqrt{A}h\|$*;*

(a4) $\underline{A}$ *satisfies a square function estimate on* $\overline{\mathsf{R}(V)}$*:*

$$\|u\| \lesssim \Big(\int_0^\infty \|t\underline{A}\,\underline{S}(t)u\|^2 \, \frac{dt}{t} \Big)^{1/2} .$$

(b) *The same result holds with '$\lesssim$' and '$\subseteq$' replaced by '$\gtrsim$' and '$\supseteq$'.*

(c) *The following assertions are equivalent:*

(c1) $\mathsf{D}_p(\sqrt{L}) = \mathsf{D}_p(D_V)$ *with* $\|D_V f\|_p \eqsim \|\sqrt{L}f\|_p$*;*

(c2) $\underline{L}$ *admits a bounded* H^∞*-functional calculus on* $\overline{\mathsf{R}_p(D_V)}$*;*

(c3) $\mathsf{D}(\sqrt{A}) = \mathsf{D}(V)$ *with* $\|Vh\| \eqsim \|\sqrt{A}h\|$*;*

(c4) $\underline{A}$ *admits a bounded* H^∞*-functional calculus on* $\overline{\mathsf{R}(V)}$*.*

Proof. The equivalence of (a1) and (a2) follows from Proposition 4.36, and the equivalence of (a3) and (a4) has been proved in Theorem 3.13. The implication (a1) $\Rightarrow$ (a3) is trivial, given the equivalence of L^p-norms on the first Wiener-Itô chaos (Theorem 1.18) and Theorem 4.3. (The the implication (a2) $\Rightarrow$ (a4) is trivial as well.) The implication (a4) $\Rightarrow$ (a2) is a consequence of Proposition 4.34.

Part (b) follows by the same arguments, and (c) follows by putting together the estimates obtained in (a) and (b) and appealing to Proposition 5.40. □

For the sake of completeness we give an alternative proof of the fact that (a2) implies (a1). It follows the more traditional approach based on square functions and avoids the use of the Hodge-Dirac operator.

Proof (Alternative proof of Theorem 4.37, (a2) $\Rightarrow$ (a1)). Fix $1 < p < \infty$ and let $\frac{1}{p} + \frac{1}{q} = 1$. The proof of is based on a lower bound for the square function associated with the semigroup $\underline{Q}$ generated by $-\sqrt{\underline{L}}$.

Consider the functions $\varphi(z) = ze^{-z}$ and $\psi(z) = \sqrt{z}e^{-\sqrt{z}}$. These functions belong to $H_0^\infty(\Sigma_\theta^+)$ for $\theta < \frac{1}{2}\pi$, and with the substitution $t = s^2$ we obtain

$$\Big\| \Big(\int_0^\infty \|\psi(t\underline{L})F\|^2 \, \frac{dt}{t} \Big)^{1/2} \Big\|_p = \sqrt{2} \Big\| \Big(\int_0^\infty \|s\sqrt{\underline{L}}\underline{Q}(s)F\|^2 \, \frac{ds}{s} \Big)^{1/2} \Big\|_p .$$

Using (a2) and the first part of Theorem 5.40, the identity of Theorem 4.14 (which extends to Q), and Lemma 4.20 and Theorem 4.19, for all $f \in \mathsf{D}_p(L)$ we obtain

$$\begin{aligned}
\|D_V f\|_p &\lesssim \Big\| \Big(\int_0^\infty \|t\underline{L}\,\underline{P}(t) D_V f\|^2 \,\frac{dt}{t} \Big)^{1/2} \Big\|_p \\
&\eqsim \Big\| \Big(\int_0^\infty \|s\sqrt{\underline{L}}\underline{Q}(s) D_V f\|^2 \,\frac{ds}{s} \Big)^{1/2} \Big\|_p \\
&= \Big\| \Big(\int_0^\infty \|s D_V Q(s)\sqrt{L} f\|^2 \,\frac{ds}{s} \Big)^{1/2} \Big\|_p \\
&= \|\mathscr{G}(\sqrt{L}f)\|_p \le \|\mathscr{H}(\sqrt{L}f)\|_p \lesssim \|\sqrt{L}f\|_p.
\end{aligned}$$

Since $\mathsf{D}_p(L)$ is a core for both $\mathsf{D}_p(\sqrt{L})$ and $\mathsf{D}_p(D_V)$, the desired domain inclusion follows and the norm estimate holds for all $f \in \mathsf{D}_p(\sqrt{L})$. □

We finish this section by pointing out two further equivalences to the ones of Theorem 4.37.

Proposition 4.38. *The conditions* (c1)–(c4) *of Theorem 4.37 are equivalent to*

(c5) $\mathsf{D}_p(\sqrt{\underline{L}}) = \mathsf{D}_p(D_V^* B)$ *with* $\|\sqrt{\underline{L}}F\|_p \eqsim \|D_V^* BF\|_p$ *for* $F \in \mathsf{D}_p(\sqrt{\underline{L}})$*;*
(c6) $\mathsf{D}(\sqrt{\underline{A}}) = \mathsf{D}(V^* B)$ *with* $\|\sqrt{\underline{A}}u\|_p \eqsim \|V^* Bu\|_p$ *for* $u \in \mathsf{D}(\sqrt{\underline{A}})$*.*

Proof. To see that (c1) implies (c5), note that for $f \in \mathsf{D}_p(L)$ we have

$$\|(D_V^* B) D_V f\|_p = \|Lf\|_p \eqsim \|D_V \sqrt{L} f\|_p = \|\sqrt{\underline{L}} D_V f\|_p.$$

Since $D_V(\mathsf{D}_p(L))$ is a core for both $\mathsf{D}_p(D_V^* B)$ and $\mathsf{D}_p(\sqrt{\underline{L}})$, (c5) follows. The converse implication that (c5) implies (c1) is proved similarly. The equivalence (c3)⇔(c6) has already been proved in Proposition 3.14. □

It is clear from the proofs that the one-sided versions of these implications hold as well.

4.4 The Hodge decomposition

In this section we will apply Theorem 4.37 to prove the following decomposition theorem.

Theorem 4.39 (Hodge decompositions). *Let* $1 < p < \infty$*. One has the direct sum decomposition*

$$L^p(\mu) = \overline{\mathsf{R}_p(D_V^* B)} \oplus \mathsf{N}_p(D_V),$$

where $D_V^* B$ *is interpreted a closed densely defined operator from* $\overline{\mathsf{R}_p(D_V)}$ *to* $L^p(\mu)$*. If the equivalent conditions of Theorem 0.1 hold, then the above decomposition remains true when* $D_V^* B$ *is interpreted as a closed densely defined operator from* $L^p(\mu; \underline{H})$ *to* $L^p(\mu)$*. In that case one has the direct sum decomposition*

$$L^p(\mu;\underline{H}) = \overline{\mathsf{R}_p(D_V)} \oplus \mathsf{N}_p(D_V^* B),$$

where $D_V^* B$ *is interpreted as a closed densely defined operator from* $L^p(\mu;\underline{H})$ *to* $L^p(\mu)$.

The first part of this result has already been proved in Proposition 4.33. We begin with some preparations for the proof of the second part.

In the remainder of this section we interpret $D_V^* B$ as a closed densely defined operator from $L^p(\mu;\underline{H})$ to $L^p(\mu)$.

Proof (Proof of Theorem 4.39, second part). We shall prove separately that

$$\overline{\mathsf{R}_p(D_V)} + \mathsf{N}_p(D_V^* B) = L^p(\mu;\underline{H}), \tag{4.9}$$

$$\overline{\mathsf{R}_p(D_V)} \cap \mathsf{N}_p(D_V^* B) = \{0\}. \tag{4.10}$$

The proof of (4.9) is more or less standard. The idea behind the proof of (4.10) is to note that for $p = 2$ the Hodge decomposition is obtained as a special case of the Hodge decomposition theorem of Axelsson, Keith, and M$^{\mathrm{c}}$Intosh [12], and to use this fact together with the fact that the L^p-norm and L^2-norm are equivalent on each summand in the Wiener-Itô decomposition.

We begin with the proof of (4.9). By Theorem 0.1(1) the operator $R := D_V/\sqrt{L}$ is well defined on $\mathsf{R}_p(\sqrt{L})$ and bounded. In view of the decomposition $L^p(\mu) = \overline{\mathsf{R}_p(\sqrt{L})} \oplus \mathsf{N}_p(\sqrt{L})$ we may extend R to $L^p(\mu)$ by putting $R|_{\mathsf{N}_p(\sqrt{L})} := 0$. A similar remark applies to the operator $R_* := D_V/\sqrt{L^*}$.

For $F \in L^p(\mu;\underline{H})$ we claim that $RR_*^* F \in \overline{\mathsf{R}_p(D_V)}$, where $R_*^* := (R_*)^*$. Indeed, there exists $f \in \mathsf{N}_p(\sqrt{L})$ and a sequence $f_n \in \mathsf{D}_p(\sqrt{L})$ such that $f + \sqrt{L} f_n \to R_*^* F$ in $L^p(\mu)$. Therefore $RR_*^* F = \lim_{n\to\infty} D_V f_n \in \overline{\mathsf{R}_p(D_V)}$.

Now, for functions $\psi \in \mathsf{D}_p(\sqrt{L})$ and $\phi \in \mathsf{D}_q(\sqrt{L^*})$,

$$\langle D_V \psi, B^* D_V \phi\rangle = \langle L\psi, \phi\rangle = \langle \sqrt{L}\psi, \sqrt{L^*}\phi\rangle.$$

Furthermore, approximating a function $f \in L^p(\mu)$ by a sequence $(f_0 + \sqrt{L} f_n)_{n\geq 1}$ with $f_0 \in \mathsf{N}_p(\sqrt{L})$ and $f_n \in \mathsf{D}_p(\sqrt{L})$ we obtain

$$\begin{aligned}\langle Rf, B^* D_V \phi\rangle &= \lim_{n\to\infty} \langle D_V f_n, B^* D_V \phi\rangle \\ &= \lim_{n\to\infty} \langle \sqrt{L} f_n, \sqrt{L^*}\phi\rangle \\ &= \langle f - f_0, \sqrt{L^*}\phi\rangle \\ &= \langle f, \sqrt{L^*}\phi\rangle.\end{aligned}$$

Hence for the duality between $L^p(\mu)$ and $L^q(\mu)$ we obtain

$$\langle F - RR_*^* BF, B^* D_V \phi\rangle = \langle F, B^* D_V \phi\rangle - \langle F, B^* R_* \sqrt{L^*}\phi\rangle = 0.$$

This shows that $F - RR_*^* BF \in \mathsf{N}_p(D_V^* B)$. This completes the proof of (4.9).

We continue with the proof of (4.10). Assume that $G \in \mathsf{D}_p(D_V^* B)$ satisfies $D_V^* BG = 0$. Then for all $f \in \mathsf{D}_q(D_V)$ we have $\langle B^* D_V f, G\rangle = 0$, where the duality is between $L^q(\mu; \underline{H})$ and $L^p(\mu; \underline{H})$.

Let $I_{p,m}$ and $I_{q,m}$ denote the projections in $L^p(\mu)$ and $L^q(\mu)$ onto the m-th Wiener-Itô chaoses. The ranges of $I_{p,m}$ and $I_{q,m}$ are isomorphic by the equivalence of norms on the Wiener-Itô chaoses. Note that $I_{p,m}^* = I_{q,m}$. Then $I_{p,m} \otimes I$ and $I_{q,m} \otimes I$ are bounded projections in $L^p(\mu; \underline{H})$ and $L^q(\mu; \underline{H})$. Let $j_{p,m}$ denote the induced isomorphism of the range of $I_{p,m} \otimes I$ onto the range of $I_{2,m} \otimes I$.

For cylindrical polynomials $f \in \mathcal{FP}(E; \mathsf{D}(V)) \cap H^{(m)}$ we have the identity $B^* D_V f = (I_{q,m-1} \otimes I) B^* D_V f$ and

$$\begin{aligned}[][j_{p,m-1}(I_{p,m-1} \otimes I)G, B^* D_V f] &= \langle (I_{p,m-1} \otimes I)G, B^* D_V f\rangle \\ &= \langle G, (I_{q,m-1} \otimes I) B^* D_V f\rangle \\ &= \langle G, B^* D_V f\rangle \\ &= 0.\end{aligned} \tag{4.11}$$

In the first term, the duality is the inner product of $L^2(\mu; \underline{H})$.

On the other hand, if $f \in \mathcal{FP}(E; \mathsf{D}(V)) \cap H^{(n)}$ for some $n \neq m$, then $j_{p,m-1}^* = j_{q,m-1}$ implies

$$\begin{aligned}&[j_{p,m-1}(I_{p,m-1} \otimes I)G, B^* D_V f] \\ &\quad = \langle (I_{p,m-1} \otimes I)G, B^* D_V f\rangle \\ &\quad = \langle (I_{p,m-1} \otimes I)G, (I_{q,n-1} \otimes I) B^* D_V f\rangle \\ &\quad = [j_{p,n-1}(I_{p,n-1} \otimes I)(I_{p,m-1} \otimes I)G, B^* D_V f] \\ &\quad = 0,\end{aligned} \tag{4.12}$$

since $D_V f$ is in the $(n-1)$-th chaos; in the last step we used the $L^2(\mu)$-orthogonality of the chaoses.

Since the cylindrical polynomials form a core for $\mathsf{D}(D_V)$ by Lemma 1.35 and B is bounded on $\underline{H}$, we conclude from (4.11) and (4.12) that $j_{p,m-1}(I_{m-1} \otimes I)G$ annihilates $\mathsf{R}(B^* D_V)$ and therefore it belongs to $\mathsf{N}(D_V^* B)$.

Next we claim that if $G \in \overline{\mathsf{R}_p(D_V)}$, then $j_{p,m-1}(I_{p,m-1} \otimes I)G \in \overline{\mathsf{R}(D_V)}$. Indeed, from $G = \lim_{k\to\infty} D_V g_k$ in $L^p(\mu; \underline{H})$ it follows that

$$j_{p,m-1}(I_{p,m-1} \otimes I)G = \lim_{k\to\infty} D_V j_{p,m}(I_{p,m} \otimes I) g_k \in \overline{\mathsf{R}(D_V)}.$$

Combining what we have proved, we see that if $G \in \overline{\mathsf{R}_p(D_V)} \cap \mathsf{N}_p(D_V^* B)$, then $j_{p,m-1}(I_{p,m-1} \otimes I)G \in \overline{\mathsf{R}(D_V)} \cap \mathsf{N}(D_V^* B)$. Hence, $j_{p,m-1}(I_{p,m-1} \otimes I)G = 0$ by the Hodge decomposition of $L^2(\mu; \underline{H})$ [12]. It follows that $(I_{p,m-1} \otimes I)G = 0$ for all $m \geq 1$, and therefore $G = 0$. This concludes the proof of (4.10). $\square$

The next application is included for reasons of completeness.

Corollary 4.40. *If the equivalent conditions of Theorem 4.37(c) hold, then*

$$\overline{\mathsf{R}_p(D_V^* B)} = \overline{\mathsf{R}_p(D_V^*)}.$$

Note that by the second part of Theorem 4.39 it is immaterial whether we view $D_V^* B$ as an unbounded operator from $L^p(\mu; \underline{H})$ to $L^p(\mu)$ or from $\overline{\mathsf{R}_p(D_V^* B)}$ to $L^p(\mu)$.

Proof. By the first part of Theorem 4.39 (first applied to B and then to I) we have the deompositions

$$L^p(\mu) = \mathsf{N}_p(D_V) \oplus \overline{\mathsf{R}_p(D_V^* B)} = \mathsf{N}_p(D_V) \oplus \overline{\mathsf{R}_p(D_V^*)},$$

where both $D_V^* B$ and D_V^* are viewed as closed densely defined operators from $\overline{\mathsf{R}_p(D_V)}$ to $L^p(\mu)$. The corollary will follow if we check that $\overline{\mathsf{R}_p(D_V^* B)} \subseteq \overline{\mathsf{R}_p(D_V^*)}$. This inclusion is trivial if we may interpret $D_V^* B$ and D_V^* as unbounded operators from $L^p(\mu; \underline{H})$ to $L^p(\mu)$. By the preceding remark, we may indeed do so for $D_V^* B$. The proof will be finished by checking that the conditions of Theorem 4.37(c) also hold with B replaced by I, since then we may do the same for D_V^*. But this follows from the fact that VV^*, being selfadjoint on $\overline{\mathsf{R}(V)}$, admits a bounded H^∞-calculus on $\overline{\mathsf{R}(V)}$. $\square$

We already showed in Theorem 4.32 that the operator Π is γ-bisectorial on the space $L^p(\mu) \oplus \overline{\mathsf{R}(D_V)}$. The Hodge decomposition from Theorem 4.39 allows us to prove a stronger result:

Theorem 4.41 (γ-bisectoriality). *Let $1 < p < \infty$. If the equivalent conditions of Theorem 4.37(c) hold, then Π is γ-bisectorial on $L^p(\mu) \oplus L^p(\mu; \underline{H})$.*

Proof (of Theorem 4.41). We use the notation

$$X_1 := L^p(\mu) \oplus \overline{\mathsf{R}_p(D_V)} \qquad \text{and} \qquad X_2 := \mathsf{N}_p(D_V^* B).$$

Fix $t \in \mathbb{R} \setminus \{0\}$. First we show that $it - \Pi$ is injective on $L^p(\mu) \oplus L^p(\mu; \underline{H})$. Theorem 4.39 implies the decomposition

$$L^p(\mu) \oplus L^p(\mu; \underline{H}) = X_1 \oplus X_2. \tag{4.13}$$

Take $x = x^{(1)} + x^{(2)} \in X_1 \oplus X_2$, and suppose that $(it - \Pi)x = 0$. Then $(it - \Pi)x^{(1)} = 0$ and $itx^{(2)} = 0$. Thus $x^{(1)} = x^{(2)} = 0$, since $\Pi|_{X_1}$ in X_1 is bisectorial.

Next we show that $it - \Pi$ is surjective on $L^p(\mu) \oplus L^p(\mu; \underline{H})$. Let $y^{(1)} \in X_1$ and $y^{(2)} \in X_2$. The equation $(it - \Pi)(x^{(1)} + x^{(2)}) = y^{(1)} + y^{(2)}$ is solved by

$$x^{(1)} = (it - \Pi|_{X_1})^{-1} y^{(1)} \qquad \text{and} \qquad x^{(2)} = (it)^{-1} y^{(2)}.$$

This implies that $it - \Pi$ is surjective.

Using (4.13) and the sectoriality of Π on X_1 it follows that

$$\begin{aligned}\|x^{(1)}+x^{(2)}\| &\le \|(it-\Pi|_{X_1})^{-1}\|\|y^{(1)}\|+|t|^{-1}\|y^{(2)}\|\\ &\lesssim t^{-1}\big(\|y^{(1)}\|+\|y^{(2)}\|\big)\\ &\lesssim t^{-1}\|y^{(1)}+y^{(2)}\|,\end{aligned}$$

which is the desired resolvent estimate that shows that Π is bisectorial on $X_1\oplus X_2$.

To show γ-bisectoriality of Π on $L^p(\mu)\oplus L^p(\mu;\underline{H})$ we take $y_j=y_j^{(1)}+y_j^{(2)}\in X_1\oplus X_2$. Let $(r_j)_{j\ge 1}$ be a Rademacher sequence. Using the γ-bisectoriality of $\Pi|_{X_1}$ we obtain

$$\begin{aligned}\mathbb{E}\Big\|\sum_{j=1}^k r_j t_j(it_j-\Pi)^{-1}y_j\Big\|_p &\le \mathbb{E}\Big\|\sum_{j=1}^k r_j t_j(it_j-\Pi|_{X_1})^{-1}y_j^{(1)}\Big\|_p\\ &\qquad+\mathbb{E}\Big\|\sum_{j=1}^k r_j t_j(it_j-\Pi|_{X_2})^{-1}y_j^{(2)}\Big\|_p\\ &\lesssim \mathbb{E}\Big\|\sum_{j=1}^N r_j y_j^{(1)}\Big\|_p+\mathbb{E}\Big\|\sum_{j=1}^k r_j t_j(t_j^{-1}y_j^{(2)})\Big\|_p\\ &\lesssim \mathbb{E}\Big\|\sum_{j=1}^k r_j y_j\Big\|_p.\end{aligned}$$

By an application of the Kahane-Khintchine inequalities we conclude that $\{t(it-\Pi)^{-1}\ :\ t\in\mathbb{R}\setminus\{0\}\}$ is γ-bounded on $L^p(\mu)\oplus L^p(\mu;\underline{H})$. This completes the proof. $\square$

4.5 Domain characterisation

We continue with the proof of the following result, already announced in the introduction of this thesis.

Theorem 4.42 (Domain of L). *Let $1<p<\infty$, and let the equivalent conditions of Theorem 0.1 be satisfied. Then we have equality of domains*

$$\mathsf{D}_p(L)=\mathsf{D}_p(D_V^2)\cap\mathsf{D}_p(D_A)$$

with equivalence of norms

$$\|f\|_p+\|Lf\|_p\eqsim\|f\|_p+\|D_Vf\|_p+\|D_V^2f\|_p+\|D_Af\|_p.$$

In the remainder of this section it will be a standing assumption that the equivalent conditions of Theorem 0.1 are satisfied. As we have already observed in Theorem 3.13, the corresponding equivalences obtained by replacing B with B^* then also hold.

We will use the bounded analytic C_0-semigroups

$$\underline{P}_{(k)}(t) = P(t) \otimes \underline{S}^{\otimes k}(t),$$

which are defined on the spaces

$$\underline{L}^p_{(k)} := L^p(\mu; \underline{H}^{\otimes k}), \qquad k = 1, 2.$$

Note that $\underline{L}^p_{(1)} = L^p(\mu; \underline{H})$ and $\underline{P}_{(1)}$ coincides with $\underline{P}$ on the closed subspace $\overline{\mathsf{R}_p(D_V)}$. The generators of $\underline{P}_{(k)}$ will be denoted by $-\underline{L}_{(k)}$. The semigroups generated by $-\sqrt{I + \underline{L}_{(k)}}$ will be denoted by $\underline{Q}_{(k)}$.

In Propositions 1.36 and 1.37 we considered the closed operators

$$D_V^{(1)} : \mathsf{D}(D_V^{(1)}) \subseteq L^p(\mu; \underline{H}) \to L^p(\mu; \underline{H}^{\otimes 2}),$$
$$D_V^2 : \mathsf{D}(D_V^2) \subseteq \mathsf{D}(D_V) \to L^p(\mu; \underline{H}^{\otimes 2}).$$

In the remainder of this section we will write

$$\underline{D}_V := D_V^{(1)}.$$

We remark that for $t > 0$ the operators

$$\underline{D}_V \underline{P}_{(1)}(t) = (D_V P(t)) \otimes \underline{S}(t), \qquad \underline{D}_V \underline{P}^*_{(1)}(t) = (D_V P^*(t)) \otimes \underline{S}^*(t)$$

are bounded from $\underline{L}^p_{(1)}$ to $\underline{L}^p_{(2)}$ as a consequence of Theorem 4.18.

Proposition 4.43. *Let $1 < p < \infty$.*

(i) *The collections $\{\sqrt{t}\underline{D}_V \underline{P}_{(1)}(t) : t > 0\}$ and $\{\sqrt{t}\underline{D}_V \underline{P}^*_{(1)}(t) : t > 0\}$ are γ-bounded in $\mathcal{L}(\underline{L}^p_{(1)}, \underline{L}^p_{(2)})$.*

(ii) *The following square function estimates hold for $F \in \underline{L}^p_{(1)}$:*

$$\Big\| \Big(\int_0^\infty \|\sqrt{t}\underline{D}_V \underline{P}_{(1)}(t) F\|^2 \,\frac{dt}{t}\Big)^{1/2} \Big\|_p \lesssim \|F\|_p,$$
$$\Big\| \Big(\int_0^\infty \|\sqrt{t}\underline{D}_V \underline{P}^*_{(1)}(t) F\|^2 \,\frac{dt}{t}\Big)^{1/2} \Big\|_p \lesssim \|F\|_p.$$

(iii) *The domain inclusions $\mathsf{D}_p(\sqrt{\underline{L}_{(1)}}) \subseteq \mathsf{D}_p(\underline{D}_V)$ and $\mathsf{D}_p(\sqrt{\underline{L}^*_{(1)}}) \subseteq \mathsf{D}_p(\underline{D}_V)$ hold with norm estimates*

$$\|\underline{D}_V F\|_p \lesssim \|F\|_p + \|\sqrt{\underline{L}_{(1)}} F\|_p \quad \textit{and} \quad \|\underline{D}_V F\|_p \lesssim \|F\|_p + \|\sqrt{\underline{L}^*_{(1)}} F\|_p.$$

Proof. (i): The γ-boundedness is a consequence from (an easy Hilbert space-valued extension of) Proposition 5.6 combined with Theorem 4.18.

(ii): Since $\underline{A}$ has a bounded H^∞-calculus on $\underline{H}$ of angle $< \frac{1}{2}\pi$, the same holds for $\underline{A}^*$. Proposition 4.34 implies that $\underline{L}_{(1)}$ and $\underline{L}^*_{(1)}$ have bounded H^∞-functional calculi on $\underline{L}^p_{(1)}$ of angle $< \frac{1}{2}\pi$. The domain inclusions $\mathsf{D}_p(\underline{L}_{(1)}) \subseteq$

$\mathsf{D}_p(\underline{D}_V)$ and $\mathsf{D}_p(\underline{L}^*_{(1)}) \subseteq \mathsf{D}_p(\underline{D}_V)$ follow from (i) by taking Laplace transforms. By combining (i) and Theorem 5.47 we obtain the desired result.

(iii): Combining the fact that $\sqrt{I+\underline{L}_{(2)}}$ has a bounded H^∞-calculus of angle $< \frac{1}{2}\pi$ with Theorem 5.40, the commutation relation $\underline{D}_V \underline{P}_{(1)}(t) = \underline{P}_{(2)}(t)\underline{D}_V$, the $\underline{H}$-valued analogue of Lemma 4.20, and the first estimate of (ii), for all $F \in \mathsf{D}_p(\underline{L}_{(1)})$ we obtain

$$
\begin{aligned}
\|\underline{D}_V F\|_p &\lesssim \Big\| \Big(\int_0^\infty \|t\sqrt{I+\underline{L}_{(2)}}\underline{Q}_{(2)}(t)\underline{D}_V F\|^2 \,\frac{dt}{t}\Big)^{1/2}\Big\|_p \\
&= \Big\| \Big(\int_0^\infty \|t\underline{D}_V \underline{Q}_{(1)}(t)\sqrt{I+\underline{L}_{(1)}}F\|^2 \,\frac{dt}{t}\Big)^{1/2}\Big\|_p \\
&\le \Big\| \Big(\int_0^\infty \|\sqrt{t}\underline{D}_V e^{-t}\underline{P}_{(1)}(t)\sqrt{I+\underline{L}_{(1)}}F\|^2 \,\frac{dt}{t}\Big)^{1/2}\Big\|_p \\
&\le \Big\| \Big(\int_0^\infty \|\sqrt{t}\underline{D}_V \underline{P}_{(1)}(t)\sqrt{I+\underline{L}_{(1)}}F\|^2 \,\frac{dt}{t}\Big)^{1/2}\Big\|_p \\
&\lesssim \Big\|\sqrt{I+\underline{L}_{(1)}}F\Big\|_p \\
&\eqsim \|F\|_p + \Big\|\sqrt{\underline{L}_{(1)}}F\Big\|_p.
\end{aligned}
$$

This gives the first estimate. Since $\mathsf{D}_p(\underline{L}_{(1)})$ is a core for $\mathsf{D}_p(\sqrt{\underline{L}_{(1)}})$, the domain inclusion follows as well.

To prove the second estimate we put $T := P^* \otimes \underline{S}_* \otimes \underline{S}^*$, where $\underline{S}_*$ is the bounded analytic semigroup generated by $-VV^*B^*$; this notation is as in Theorem 3.13. Note that the negative generator C of T has a bounded H^∞-calculus of angle $< \frac{1}{2}\pi$; this follows from the fact that if Theorem 4.37(c) holds for B, then it also holds for B^* (see Theorem 3.13) and therefore the negative generators of $\underline{S}^*$ and $\underline{S}_*$ both have bounded H^∞-calculi of angle $< \frac{1}{2}\pi$. Let R be the semigroup generated by $-\sqrt{I+C}$. Using the identity

$$\underline{D}_V \underline{P}^*_{(1)}(t)F = T(t)\underline{D}_V F,$$

and arguing as above, for all $F \in \mathsf{D}_p(\underline{L}^*_{(1)})$ we obtain

$$
\begin{aligned}
\|\underline{D}_V F\|_p &\lesssim \Big\| \Big(\int_0^\infty \|t\sqrt{I+C}R(t)\underline{D}_V F\|^2 \,\frac{dt}{t}\Big)^{1/2}\Big\|_p \\
&= \Big\| \Big(\int_0^\infty \|t\underline{D}_V \underline{Q}^*_{(1)}(t)\sqrt{I+\underline{L}^*_{(1)}}F\|^2 \,\frac{dt}{t}\Big)^{1/2}\Big\|_p \\
&\le \Big\| \Big(\int_0^\infty \|\sqrt{t}\underline{D}_V e^{-t}\underline{P}^*_{(1)}(t)\sqrt{I+\underline{L}^*_{(1)}}F\|^2 \,\frac{dt}{t}\Big)^{1/2}\Big\|_p \\
&\le \Big\| \Big(\int_0^\infty \|\sqrt{t}\underline{D}_V \underline{P}^*_{(1)}(t)\sqrt{I+\underline{L}^*_{(1)}}F\|^2 \,\frac{dt}{t}\Big)^{1/2}\Big\|_p \\
&\lesssim \Big\|\sqrt{I+\underline{L}^*_{(1)}}F\Big\|_p \\
&\eqsim \|F\|_p + \Big\|\sqrt{\underline{L}^*_{(1)}}F\Big\|_p.
\end{aligned}
$$

The second domain inclusion now follows from the fact that $\mathsf{D}_p(\underline{L}^*_{(1)})$ is a core for $\mathsf{D}_p(\sqrt{\underline{L}^*_{(1)}})$. $\square$

In the following theorem we give a characterisation of $\mathsf{D}_p(\sqrt{\underline{L}_{(1)}})$. Since $\sqrt{\underline{L}} = \sqrt{\underline{L}_{(1)}}$ on $\overline{\mathsf{R}_p(D_V)}$, this gives a further equivalence of norms for $\sqrt{\underline{L}}$ on $\overline{\mathsf{R}_p(D_V)}$, different from the one in Theorem 0.1. In the proof of Theorem 4.42 we use both equivalences to determine the domain of L.

First we need a simple lemma.

Lemma 4.44. *Let $1 < p < \infty$. The semigroup $\underline{Q}_{(1)}$ restricts to C_0-semigroups on the space $\mathsf{D}_p(\underline{D}_V) \cap \mathsf{D}_p(\sqrt{I \otimes \underline{A}})$.*

Proof. It suffices to prove the result with $\underline{Q}_{(1)}$ replaced by $\underline{P}_{(1)}$; the latter is readily seen to restrict to a C_0-semigroup on $\mathsf{D}_p(\underline{D}_V) \cap \mathsf{D}_p(\sqrt{I \otimes \underline{A}})$ by the identities $\underline{D}_V P_{(1)}(t) = \underline{P}_{(2)}(t)\underline{D}_V$ and $\sqrt{I \otimes \underline{A}}\,\underline{P}_{(1)}(t) = \underline{P}_{(1)}(t)\sqrt{I \otimes \underline{A}}$. $\square$

Theorem 4.45. *Let $1 < p < \infty$. We have equality of domains*

$$\mathsf{D}_p(\sqrt{\underline{L}_{(1)}}) = \mathsf{D}_p(\underline{D}_V) \cap \mathsf{D}_p(\sqrt{I \otimes \underline{A}}),$$

with equivalence of norms

$$\|F\|_p + \big\|\sqrt{\underline{L}_{(1)}}F\big\|_p \eqsim \|F\|_p + \|\underline{D}_V F\|_p + \|\sqrt{I \otimes \underline{A}}F\|_p,$$

Proof. By a result of Kalton and Weis [89, Theorem 6.3], applied to the sums $\underline{L}_{(1)} = L \otimes I + I \otimes \underline{A}$ and $\underline{L}^*_{(1)} = L^* \otimes I + I \otimes \underline{A}^*$, we have the estimates

$$\begin{aligned} \|(I \otimes \underline{A})F\|_p &\lesssim \|F\|_p + \|\underline{L}_{(1)}F\|_p, \qquad && F \in \mathsf{D}_p(\underline{L}_{(1)}) \\ \|(I \otimes \underline{A}^*)F\|_p &\lesssim \|F\|_p + \|\underline{L}^*_{(1)}F\|_p, \qquad && F \in \mathsf{D}_p(\underline{L}^*_{(1)}). \end{aligned}$$

Since the square root domains equal the complex interpolation spaces at exponent $\frac{1}{2}$ for sectorial operators with bounded imaginary powers [78, Theorem 6.6.9], by interpolating the inclusions

$$\mathsf{D}_p(\underline{L}_{(1)}) \hookrightarrow \mathsf{D}_p(I \otimes \underline{A}), \quad \mathsf{D}_p(\underline{L}^*_{(1)}) \hookrightarrow \mathsf{D}_p(I \otimes \underline{A}^*),$$

with the identity operator, we obtain the estimates

$$\begin{aligned} \|\sqrt{I \otimes \underline{A}}F\|_p &\lesssim \|F\|_p + \big\|\sqrt{\underline{L}_{(1)}}F\big\|_p, \qquad && F \in \mathsf{D}_p(\underline{L}_{(1)}) \\ \|\sqrt{I \otimes \underline{A}^*}F\|_p &\lesssim \|F\|_p + \big\|\sqrt{\underline{L}^*_{(1)}}F\big\|_p, \qquad && F \in \mathsf{D}_p(\underline{L}^*_{(1)}). \end{aligned} \tag{4.14}$$

Combining these estimates with Proposition 4.43 we obtain

$$\|F\|_p + \|\underline{D}_V F\|_p + \|\sqrt{I\otimes\underline{A}}F\|_p \lesssim \|F\|_p + \big\|\sqrt{\underline{L}_{(1)}}F\big\|_p, \qquad F\in \mathsf{D}_p(\underline{L}_{(1)})$$

$$\|F\|_p + \|\underline{D}_V F\|_p + \|\sqrt{I\otimes\underline{A}^*}F\|_p \lesssim \|F\|_p + \big\|\sqrt{\underline{L}^*_{(1)}}F\big\|_p, \qquad F\in \mathsf{D}_p(\underline{L}^*_{(1)}).$$

Next we prove the reverse estimates. For $F\in \mathsf{D}_p(L)\otimes \mathsf{D}_p(\underline{A})$ and $G\in \mathsf{D}_q(L^*)\otimes \mathsf{D}_p(\underline{A}^*)$ ($\frac1p+\frac1q=1$) we have $F\in \mathsf{D}_p(\underline{L}_{(1)})$, $G\in \mathsf{D}_q(\underline{L}^*_{(1)})$, and

$$\begin{aligned}
\langle \sqrt{I+\underline{L}_{(1)}}F, G\rangle &= \langle (I+\underline{L}_{(1)})F, 1/\sqrt{\underline{L}^*_{(1)}+I}G\rangle\\
&= \langle F, 1/\sqrt{I+\underline{L}^*_{(1)}}G\rangle + \langle (L\otimes I)F, 1/\sqrt{I+\underline{L}^*_{(1)}}G\rangle\\
&\quad + \langle (I\otimes\underline{A})F, 1/\sqrt{I+\underline{L}^*_{(1)}}G\rangle\\
&= \langle F, 1/\sqrt{I+\underline{L}^*_{(1)}}G\rangle + \langle B\underline{D}_V F, \underline{D}_V/\sqrt{I+\underline{L}^*_{(1)}}G\rangle\\
&\quad + \langle \sqrt{I\otimes\underline{A}}F, \sqrt{I\otimes\underline{A}^*}/\sqrt{I+\underline{L}^*_{(1)}}G\rangle.
\end{aligned}$$

Using the boundedness of the three operators $1/\sqrt{I+\underline{L}^*_{(1)}}$, $\underline{D}_V/\sqrt{I+\underline{L}^*_{(1)}}$ (by Proposition 4.43(iii)), and $\sqrt{I\otimes\underline{A}^*}/\sqrt{I+\underline{L}^*_{(1)}}$ (by the second estimate in (4.14)), we find

$$\begin{aligned}
\big\|\sqrt{I+\underline{L}_{(1)}}F\big\|_p &= \sup_{\|G\|_q\le 1} |\langle \sqrt{I+\underline{L}_{(1)}}F, G\rangle|\\
&\le \sup_{\|G\|_q\le 1} \|F\|_p\,\|1/\sqrt{I+\underline{L}^*_{(1)}}G\|_q\\
&\qquad + \|B\|\,\|\underline{D}_V F\|_p\,\|\underline{D}_V/\sqrt{I+\underline{L}^*_{(1)}}G\|_q\\
&\qquad + \|\sqrt{I\otimes\underline{A}}F\|_p\,\|\sqrt{I\otimes\underline{A}^*}/\sqrt{I+\underline{L}^*_{(1)}}G\|_q\\
&\lesssim \|F\|_p + \|\underline{D}_V F\|_p + \|\sqrt{I\otimes\underline{A}}F\|_p.
\end{aligned}$$

The estimate

$$\big\|\sqrt{I+\underline{L}^*_{(1)}}F\big\|_p \lesssim \|F\|_p + \|\underline{D}_V F\|_p + \|\sqrt{I\otimes\underline{A}^*}F\|_p$$

is proved similarly and will not be needed.

It remains to prove the equality of domains. Since $\mathsf{D}_p(L)\otimes \mathsf{D}(\underline{A})$ is a core for $\mathsf{D}_p(\underline{L}_{(1)})$, it is also a core for $\mathsf{D}_p(\sqrt{\underline{L}_{(1)}})$. Using this, the domain inclusion $\mathsf{D}_p(\sqrt{\underline{L}_{(1)}}) \subseteq \mathsf{D}_p(\underline{D}_V)\cap \mathsf{D}_p(\sqrt{I\otimes\underline{A}})$ follows, and the equivalence of norms extends to all $F\in \mathsf{D}_p(\sqrt{\underline{L}_{(1)}})$.

Again by the equivalence of norms, $\mathsf{D}_p(\sqrt{\underline{L}_{(1)}})$ is closed in $\mathsf{D}_p(\underline{D}_V)\cap \mathsf{D}_p(\sqrt{I\otimes\underline{A}})$. It remains to prove that the inclusion is dense. This follows from Lemma 4.44, since for $F\in \mathsf{D}_p(\underline{D}_V)\cap \mathsf{D}_p(\sqrt{I\otimes\underline{A}})$ and $t>0$ we have $\underline{Q}_{(1)}(t)F\in \mathsf{D}_p(\sqrt{\underline{L}_{(1)}})$ and $\underline{Q}_{(1)}(t)F\to F$ in the norm of $\mathsf{D}_p(\underline{D}_V)\cap \mathsf{D}_p(\sqrt{I\otimes\underline{A}})$ as $t\downarrow 0$. □

Recall that D denotes the Malliavin derivative. Since A is a closed operator, it follows from Theorem 1.33 that the operator D_A, initially defined on $\mathcal{F}C_{\rm b}^1(E;\mathsf{D}(A))$, is closable as an operator from $L^p(\mu)$ into $L^p(\mu;H)$ for $1<p<\infty$. We denote its closure by D_A.

Lemma 4.46. *Let $1<p<\infty$. The semigroup P restricts to a C_0-semigroup on the space $\mathsf{D}_p(D_V^2)\cap\mathsf{D}_p(D_A)$.*

Proof. A straightforward argument shows that $P(t)\mathsf{D}_p(D_V^2)\subseteq\mathsf{D}_p(D_V^2)$ and

$$D_V^2P(t)f := \underline{P}_{(2)}(t)D_V^2f, \qquad f\in\mathsf{D}_p(D_V^2).$$

Similarly, we have $P(t)\mathsf{D}_p(D_A)\subseteq\mathsf{D}_p(D_A)$ and

$$D_AP(t)f = e^{-t}(P(t)\otimes I)D_Af, \qquad f\in\mathsf{D}_p(D_A).$$

These identities easily imply the result. □

Proof (Proof of Theorem 4.42). Using the fact that $\mathsf{D}_p(L)\subseteq\mathsf{D}_p(D_V)$, Proposition 4.28, the domain equality $\mathsf{D}_p(\sqrt{L})=\mathsf{D}_p(D_V^*B)$ (see Proposition 4.38), Theorem 4.45, the domain equality $\mathsf{D}(\sqrt{\underline{A}})=\mathsf{D}(V^*B)$ on $\overline{\mathsf{R}(V)}$ (see Proposition 3.14), and the definition of D_A, for $f\in\mathsf{D}_p(L)$ we obtain

$$\begin{aligned}
\|f\|_p+\|Lf\|_p &\eqsim \|f\|_p+\|D_Vf\|_p+\|Lf\|_p\\
&= \|f\|_p+\|D_Vf\|_p+\|(D_V^*B)D_Vf\|_p\\
&\eqsim \|f\|_p+\|D_Vf\|_p+\|\sqrt{\underline{L}}D_Vf\|_p\\
&\eqsim \|f\|_p+\|D_Vf\|_p+\|D_V^2f\|_p+\|\sqrt{\underline{A}}D_Vf\|_p\\
&\eqsim \|f\|_p+\|D_Vf\|_p+\|D_V^2f\|_p+\|(V^*B)D_Vf\|_p\\
&\eqsim \|f\|_p+\|D_Vf\|_p+\|D_V^2f\|_p+\|D_Af\|_p,
\end{aligned}$$

This proves the equivalence of norms and the domain inclusion $\mathsf{D}_p(L)\subseteq\mathsf{D}_p(D_V^2)\cap\mathsf{D}_p(D_A)$. To obtain equality of domains it remains to show that this inclusion is both closed and dense. Closedness follows easily from the norm estimate and density follows from Lemma 4.46 in the same way as in Theorem 4.45. □

4.6 Notes

An approach to Kolmogorov equations via sectorial forms can be found in the lecture notes by Röckner [148].

The main results (Theorem 4.37 and Theorem 4.42) have a long history. The special case $A=I$ is a fundamental result in Malliavin calculus due to P.-A. Meyer [128]. Various analytic [75, 113, 142, 163] and probabilistic [74] proofs have been given. Perhaps the simplest is the analytic proof by Pisier

[146] which uses the transference principle of Coifman and Weiss [38]. The one-dimensional case had already been proved much earlier by Muckenhoupt [129].

For more general Ornstein-Uhlenbeck operators, the domain characterisation for $p = 2$ is due to Lunardi [102] in finite dimensions and Da Prato and Goldys in infinite dimensions [43]. In the non-symmetric finite dimensional case the L^p-domain has been characterised by Metafune, Prüss, Rhandi, and Schnaubelt [125]. See also [114] for the boundedness of the Riesz transforms and a weak-(1,1) type result.

Boundedness of the Riesz transforms in the symmetric infinite dimensional L^p-setting has been proved by Shigekawa [152] (see also [154]). In this setting Chojnowska-Michalik and Goldys [32] proved two-sided bounds for higher order Riesz transforms. In particular they characterised the domain of symmetric Ornstein-Uhlenbeck operators. The results in the infinite dimensional non-symmetric setting presented here can be found in a joint paper with van Neerven [107].

Gradient estimates in L^p for Ornstein-Uhlenbeck semigroups have been proved earlier (see, e.g., [45, Proposition 10.3.1]). However, the randomised boundedness of these operators (Theorem 4.18) is new.

5

Appendix: Tools from Operator Theory

In this chapter we present a collection of results from operator theory involving the notions of randomised boundedness, radonifying operators, and H^∞-functional calculus. These results are our main tools in the study of elliptic operators on Wiener spaces in Chapter 4.

We work in a general Banach space setting and use the language of radonifying operators, although for the applications in Chapter 4 it would be sufficient to consider square functions in Hilbert spaces and L^p-spaces. Apart from having the advantage of covering the Hilbertian and L^p-setting at the same time, we believe that this allows to present the proofs in the most transparent way. We do not attempt to state all results in the greatest possible generality, but instead prefer to give formulations which are flexible enough for applications as in Chapter 4.

Throughout this chapter we will use the following notation:

- X, Y and Z are Banach spaces,
- $\mathscr{H}, H$ and H_n are Hilbert spaces,
- $(M, \mu), (M_n, \mu_n)$ are σ-finite measure spaces,
- $\varphi := (\varphi_j)_{j\geq 1}$ is a sequence of i.i.d. random variables, being either $\mathcal{R}$ or γ, where
  - $\mathcal{R} := (r_j)_{j\geq 1}$ is a sequence of independent *Rademacher variables*, i.e., $\mathbb{P}(r_j = 1) = \mathbb{P}(r_j = -1) = \frac{1}{2}$ for each $j \geq 1$.
  - $\gamma := (\gamma_j)_{j\geq 1}$ is a sequence of independent standard *Gaussian variables*, i.e., $\mathbb{P}(\gamma_j \leq \lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\lambda} \exp(-\frac{1}{2}\xi^2)\, d\xi$, for each $\lambda \in \mathbb{R}$ and $j \geq 1$.

  Such sequences will be called Rademacher sequences and Gaussian sequences.

5.1 Randomised boundedness

In this chapter we will study various operator theoretic notions involving moments of Banach space-valued random sums. Second moments in Hilbert spaces are easy to compute. Indeed, the orthogonality of φ in $L^2(\mathbb{P})$ implies that

$$\mathbb{E}\Big\| \sum_{j=1}^{k} \varphi_j h_j \Big\|^2 = \sum_{j=1}^{k} \|h_j\|^2 \tag{5.1}$$

for all $h_1, \ldots, h_k \in H$. Consequently, if $\mathcal{T}$ is a uniformly bounded collection of operators on H, then

$$\Big(\mathbb{E}\Big\| \sum_{j=1}^{k} \varphi_j T_j x_j \Big\|^2\Big)^{1/2} \le \sup_{T\in\mathcal{T}} \|T\| \Big(\mathbb{E}\Big\| \sum_{j=1}^{k} \varphi_j x_j \Big\|^2\Big)^{1/2}. \tag{5.2}$$

In Banach spaces this randomised boundedness property is no longer automatic. This motivates the following definition.

Definition 5.1. *A collection of bounded linear operators $\mathcal{T} \subseteq \mathcal{L}(X,Y)$ is said to be* φ-bounded *if there exists $C \ge 0$ such that for all $k = 1, 2, \ldots$ and all choices of $x_1, \ldots, x_k \in X$ and $T_1, \ldots, T_k \in \mathcal{T}$, we have*

$$\mathbb{E}\Big\| \sum_{j=1}^{k} \varphi_j T_j x_j \Big\|^2 \le C^2 \mathbb{E}\Big\| \sum_{j=1}^{k} \varphi_j x_j \Big\|^2.$$

The infimum over all $C \ge 0$ for which the estimate holds is denoted by $\varphi(\mathcal{T})$.

Although Rademacher sums and Gaussian sums behave similar in many respects (see Remark 5.2(vi) below), both classes have their advantages in particular situations; Rademachers are especially powerful when dealing with unconditionally convergent series, whereas Gaussians are more natural in the presence of stochastic integrals, radonifying operators and Malliavin calculus. In the sequel we exploit the nice features of both.

We collect some basic properties of φ-boundedness:

Remark 5.2. (i) Every φ-bounded set is uniformly bounded.

(ii) Kahane's contraction principle (see, e.g., [52, Theorem 12.2]) asserts that for $1 \le p < \infty$, for $a_j \in \mathbb{C}$ with $|a_j| \le 1$, and $x_j \in X$,

$$\Big(\mathbb{E}\Big\| \sum_{j=1}^{k} a_j \varphi_j x_j \Big\|^p\Big)^{1/p} \le 2\Big(\mathbb{E}\Big\| \sum_{j=1}^{k} \varphi_j x_j \Big\|^p\Big)^{1/p}.$$

In other words, the collection $\{aI_X : |a| \le 1\}$ is φ-bounded.

(iii) If $\mathcal{S}, \mathcal{T} \subseteq \mathcal{L}(X,Y)$ are φ-bounded, then $\mathcal{S}\mathcal{T} := \{ST : S \in \mathcal{S}, T \in \mathcal{T}\}$ and $\mathcal{S} \cup \mathcal{T}$ are φ-bounded as well. In particular, since singletons are φ-bounded, all finite sets of operators are φ-bounded.

(iv) We have already seen in (5.2) that every uniformly bounded subset of operators on a Hilbert space is φ-bounded. It follows from a famous result by Kwapień [95] that the following converse holds: if every uniformly bounded collection of operators on X is φ-bounded, then X is isomorphic to a Hilbert space.

(v) The Kahane-Khintchine inequalities (see, e.g., [52, Theorem 11.1]) say that for any $x_1, \dots, x_k \in X$ and $1 \le p < \infty$ one has

$$\Big(\mathbb{E}\Big\|\sum_{j=1}^{k} \varphi_j x_j\Big\|^2\Big)^{1/2} \eqsim \Big(\mathbb{E}\Big\|\sum_{j=1}^{k} \varphi_j x_j\Big\|^p\Big)^{1/p}, \tag{5.3}$$

with universal constants depending only on p. Consequently, one may replace the exponents 2 in the definition of φ-boundedness by arbitrary $p \in [1, \infty)$; at worst this changes the value of the constant C.

(vi) One can always estimate Rademacher sums by Gaussian sums (see, e.g., [131, Corollary 3.6]): For $1 \le p < \infty$ and $x_1, \dots, x_k \in X$,

$$\Big(\mathbb{E}\Big\|\sum_{j=1}^{k} r_j x_j\Big\|^p\Big)^{1/p} \le \sqrt{\tfrac{\pi}{2}}\Big(\mathbb{E}\Big\|\sum_{j=1}^{k} \gamma_j x_j\Big\|^p\Big)^{1/p}.$$

If X has finite cotype (in particular if X is a closed subspace of $L^p(\mu; \mathscr{H})$), then the reverse estimate holds as well (with a different constant depending on X). Consequently, in these spaces the notions of $\mathcal{R}$-boundedness and γ-boundedness coincide.

(vii) If $\mathcal{T}$ is φ-bounded, then the closure with respect to the strong operator topology of the absolutely convex hull of $\mathcal{T}$ is φ-bounded as well. Moreover, $\varphi(\overline{\mathcal{T}}^{\mathrm{SOT}}) \le 2\varphi(\mathcal{T})$. For a proof we refer to [94, Theorem 2.13].

A useful consequence of the last remark is the following result.

Proposition 5.3. *Let $\mathcal{T} \subseteq \mathcal{L}(X,Y)$ be φ-bounded, and let $f : M \to \mathcal{L}(X,Y)$ be a function with values in $\mathcal{T}$ such that $\xi \mapsto f(\xi)x$ is strongly μ-measurable for all $x \in E$. For $\phi \in L^1(\mu)$ define*

$$T_{\phi,f}x := \int_M \phi(\xi) f(\xi) x \, d\mu(\xi), \qquad x \in X.$$

Then the collection $\{T_{\phi,f} : \|\phi\|_{L^1(\mu)} \le 1\}$ is φ-bounded in $\mathcal{L}(X,Y)$.

Proof. See [94, Corollary 2.14]. □

We need the following duality result for φ-bounded families. According to a celebrated result in Banach space theory by Pisier [145], X is K-convex (see also Remark 11.7) if and only if X has nontrivial type. Examples of K-convex spaces include the spaces $L^p(\mu; \mathscr{H})$ for $1 < p < \infty$.

Proposition 5.4. *If X and Y are K-convex Banach spaces, then a family $\mathcal{T} \subseteq \mathcal{L}(X,Y)$ is φ-bounded if and only if the adjoint family $\mathcal{T}^* \subseteq \mathcal{L}(Y^*, X^*)$ is φ-bounded.*

Proof. See [87, Proposition 3.5]. □

Randomised boundedness in L^p-spaces

In L^p-spaces, randomised boundedness is closely related to square function estimates in the spirit of harmonic analysis:

Proposition 5.5. *Let $1 \le p < \infty$. A family $\mathcal{T}$ of operators from $L^p(\mu_1; H_1)$ to $L^p(\mu_2; H_2)$ is φ-bounded if and only if there exists a constant $C \ge 0$ such that for all $k \ge 1$ and all choices of $T_1, \dots, T_k \in \mathcal{T}$ and $F_1, \dots F_k \in L^p(\mu_1; H_1)$,*

$$\Big\| \Big(\sum_{j=1}^{k} \|T_j F_j\|_{H_2}^2 \Big)^{1/2} \Big\|_{L^p(\mu_2)} \le C \Big\| \Big(\sum_{j=1}^{k} \|F_j\|_{H_1}^2 \Big)^{1/2} \Big\|_{L^p(\mu_1)}.$$

Proof. See [94, Remark 2.9] for the case $H_1 = H_2 = I$. The Hilbert space version is an easy extension. $\square$

The next result may be known to specialists, but since we could not find a reference for it we include a proof.

Proposition 5.6. *Let $1 \le p < \infty$. If $\mathcal{T} \subseteq \mathcal{L}(L^p(\mu))$ is φ-bounded and $\mathcal{S} \subseteq \mathcal{L}(H)$ is bounded, then $\mathcal{T} \otimes \mathcal{S} \subseteq \mathcal{L}(L^p(\mu; H))$ is φ-bounded.*

Proof. Since $T \otimes S = (T \otimes I)(I \otimes S)$, and $I \otimes \mathcal{S}$ is φ-bounded by Fubini's theorem, it suffices to show that $\mathcal{T} \otimes I$ is φ-bounded.

Let $(h_i)_{i=1}^n$ be an orthonormal system in H and let $F_1, \dots, F_k$ be functions in $L^p(\mu; H)$ of the form $F_j := \sum_{i=1}^n f_{ij} \otimes h_i$. Note that functions of this form are dense in $L^p(\mu; H)$. Let $(\varphi_i)_{i \ge 1}$ and $(\widetilde{\varphi}_i)_{i \ge 1}$ be independent φ-sequences. Then, putting $g_i := \sum_{j=1}^k \varphi_j T_j f_{ij}$, and using Fubini's theorem,

$$\begin{aligned}
\mathbb{E}\Big\| \sum_{j=1}^{k} \varphi_j (T_j \otimes I) F_j \Big\|_p^p &= \mathbb{E}\Big\| \sum_{j=1}^{k} \varphi_j \sum_{i=1}^{n} T_j f_{ij} \otimes h_i \Big\|_p^p \\
&= \mathbb{E} \int_M \Big\| \sum_{j=1}^{k} \varphi_j \sum_{i=1}^{n} T_j f_{ij} \otimes h_i \Big\|^p \, d\mu = \mathbb{E} \int_M \Big\| \sum_{i=1}^{n} g_i \otimes h_i \Big\|^p \, d\mu \\
&\eqsim \mathbb{E} \int_M \widetilde{\mathbb{E}} \Big| \sum_{i=1}^{n} \widetilde{\varphi}_i g_i \Big|^p \, d\mu = \widetilde{\mathbb{E}}\mathbb{E} \Big\| \sum_{j=1}^{k} \varphi_j T_j \Big(\sum_{i=1}^{n} \widetilde{\varphi}_i f_{ij} \Big) \Big\|_p^p \\
&\lesssim \widetilde{\mathbb{E}}\mathbb{E} \Big\| \sum_{j=1}^{k} \varphi_j \Big(\sum_{i=1}^{n} \widetilde{\varphi}_i f_{ij} \Big) \Big\|_p^p \eqsim \mathbb{E} \Big\| \sum_{j=1}^{k} \varphi_j F_j \Big\|_p^p.
\end{aligned}$$

The last step follows by performing the computation in reverse order. The result follows from the Kahane-Khintchine inequalities (5.3). $\square$

5.2 Radonifying operators

In this section we will study a class of operators which play the role of L^p-square functions in a general Banach space setting.

For $h \in H$ and $x \in X$ we let $h \otimes x$ denote the rank-1 operator from H to X defined by

$$h \otimes x : g \mapsto [g, h]\, x, \quad g \in H.$$

Definition 5.7. *We denote by $\gamma(H, X)$ the completion of the finite rank operators from H to X with respect to the norm*

$$\Big\| \sum_{j=1}^{k} h_j \otimes x_j \Big\|_{\gamma(H,X)} := \Big(\mathbb{E} \Big\| \sum_{j=1}^{k} \gamma_j x_j \Big\|^2 \Big)^{1/2},$$

where it is assumed that the vectors $h_1, \dots, h_k$ are orthonormal in H. Operators in $\mathcal{L}(H, X)$ belonging to $\gamma(H, X)$ are called radonifying.

Remark 5.8. (i) It is not difficult to check that $\|T\|_{\gamma(H,X)}$ does not depend on the choice of the orthonormal system. Moreover, since $\|T\|_{\mathcal{L}(H,X)} \le \|T\|_{\gamma(H,X)}$ for all finite rank operators $T \in \mathcal{L}(H, X)$, it follows that $\gamma(H, X) \hookrightarrow \mathcal{L}(H, X)$.

(ii) If H is separable and $T \in \gamma(H, X)$, then

$$\|T\|_{\gamma(H,X)} = \Big(\mathbb{E} \Big\| \sum_{j=1}^{\infty} \gamma_j T h_j \Big\|^2 \Big)^{1/2},$$

where $(h_j)_{j \ge 1}$ is an arbitrary orthonormal basis of H (see, e.g., [131, Theorem 5.15]).

(iii) The terminology is explained by the fact that an operator $T \in \mathcal{L}(H, X)$ is radonifying if and only if there exists a Gaussian Radon measure on X whose covariance operator equals TT^* (see, e.g., [131, Theorem 5.16]).

The norm of a radonifying operator is easy to compute if the Banach space is a Hilbert or L^p-space:

Proposition 5.9. *(i) We have*

$$\|T\|_{\gamma(H,\mathscr{H})} = \|T\|_{\mathcal{L}_2(H,\mathscr{H})},$$

where $\| \cdot \|_{\mathcal{L}_2(H,\mathscr{H})}$ denotes the Hilbert-Schmidt norm.

(ii) Let $1 \le p < \infty$. If H is separable and $(h_j)_{j \ge 1}$ is an orthonormal basis of H, then

$$\|T\|_{\gamma(H,L^p(\mu;\mathscr{H}))} \eqsim \Big\| \Big(\sum_{j=1}^{\infty} \|T h_j\|_H^2 \Big)^{1/2} \Big\|_{L^p(\mu)},$$

with constants depending only on p.

Proof. See [131, Theorems 5.19 & 5.20]. □

The following result is a convenient reformulation of the right ideal property of $\gamma(H,X)$.

Lemma 5.10. *Let $T \in \mathcal{L}(H)$. Then the operator $T^{\otimes}$ defined by*

$$T^{\otimes} : h \otimes x \mapsto (Th) \otimes x, \quad h \in H, x \in X,$$

uniquely extends to a bounded operator on $\gamma(H,X)$ of norm

$$\|T^{\otimes}\|_{\mathcal{L}(\gamma(H,X))} \leq \|T\|_{\mathcal{L}(H)}.$$

Proof. See [131, Proposition 5.11]. □

This result admits a useful generalisation to collections of operators if the Banach space has the following geometric property introduced by Pisier [144]. Let $(r_k)_{k\geq 1}$ and $(\overline{r}_k)_{k\geq 1}$ be independent Rademacher sequences. We say that X has *property* (α) if there exists a constant $C > 0$ depending only on X such that for any $x_{jk} \in X$ and $\alpha_{jk} \in \{-1, 1\}$,

$$\Big(\mathbb{E}\Big\| \sum_{j,k=1}^{n} \alpha_{ij} r_j \overline{r}_k x_{jk} \Big\|^2\Big)^{1/2} \leq C\Big(\mathbb{E}\Big\| \sum_{j,k=1}^{n} r_j \overline{r}_k x_{jk} \Big\|^2\Big)^{1/2}.$$

Proposition 5.11. *If $\mathcal{T} \subseteq \mathcal{L}(H)$ is uniformly bounded and X has property (α), then $\{T^{\otimes} : T \in \mathcal{T}\}$ is φ-bounded in $\mathcal{L}(\gamma(H,X))$.*

Proof. See [76, Theorem 3.18]. □

We continue with an observation about iterated radonifying norms which follows from the Kahane-Khintchine inequalities and Fubini's theorem.

Proposition 5.12. *Suppose that X has property (α). The mapping*

$$h_1 \otimes (h_2 \otimes x) \mapsto (h_1 \otimes h_2) \otimes x, \qquad h_1 \in H_1, h_2 \in H_2, x \in X,$$

extends uniquely to an isomorphism of Banach spaces

$$\gamma(H_1, \gamma(H_2, X)) \simeq \gamma(H_1 \otimes H_2, X).$$

Proof. See [90] or [136, Corollary 3.5]. □

The following trace duality result will be useful.

Proposition 5.13. *If $T \in \gamma(H,X)$ and $S \in \gamma(H,X^*)$, then*

$$\operatorname{tr}(T^* S) \leq \|T\|_{\gamma(H,X)} \|S\|_{\gamma(H,X^*)}.$$

Proof. See [90, Proposition 5.2]. □

The case $H = L^2(\sigma)$ and representation by functions

- In the remainder of this section we let (S, σ) be a σ-finite measure space.

In this section we will study the space $\gamma(L^2(\sigma), X)$.

Our first goal is to study integral operators in $\gamma(L^2(\sigma), X)$, i.e. operators which are formally given by

$$Tf := \int_S f(s)F(s)\,d\sigma(s), \qquad f \in L^2(S, \sigma),$$

for a suitable function $F : S \to X$. This intuition is made precise in the following definition:

Definition 5.14. *Let the strongly σ-measurable function $F : S \to X$ be scalarly-L^2, i.e., for all $x^* \in X^*$ the function $s \mapsto \langle F(s), x^*\rangle$ is square integrable. We say that F represents an operator $T \in \gamma(L^2(\sigma), X)$ if*

$$\langle Tf, x^*\rangle = \int_S f(s)\langle F(s), x^*\rangle\,d\sigma(s), \quad f \in L^2(\sigma),\ x^* \in X^*.$$

In this case, with a slight abuse of notation, we simply write $F \in \gamma(L^2(\sigma), X)$.

The next result [20, 134] shows that radonifying norms reduce to square functions in the special case that $X = L^p(\mu; \mathscr{H})$.

Proposition 5.15. *Let $1 \le p < \infty$ and let $F : S \to L^p(\mu; \mathscr{H})$ be strongly measurable and scalarly-L^2. Then the function F represents an operator in $\gamma(L^2(\sigma), L^p(\mu; \mathscr{H}))$ if and only if*

$$\Big(\int_S \|F(s)\|_{\mathscr{H}}^2\,d\sigma(s)\Big)^{1/2} \in L^p(\mu).$$

In this situation we have an equivalence of norms

$$\|F\|_{\gamma(L^2(\sigma), L^p(\mu;\mathscr{H}))} \eqsim \Big\|\Big(\int_S \|F(s)\|_{\mathscr{H}}^2\,d\sigma(s)\Big)^{1/2}\Big\|_{L^p(\mu)}.$$

The concepts of γ-boundedness and γ-radonifying operators are connected by the following multiplier result from Kalton and Weis [90]. We use the formulation from [131].

Proposition 5.16. *Let (S, σ) be a σ-finite measure space, and let $K : S \to \mathcal{L}(X, Y)$ be a function such that $K(\cdot)x$ is strongly σ-measurable for all $x \in X$. If the set $\mathcal{T}_K = \{K(s) :\ s \in S\}$ is γ-bounded, then the mapping*

$$T_K : f(\cdot) \otimes x \mapsto f(\cdot) \otimes K(\cdot)x, \qquad f \in L^2(\sigma), x \in X,$$

extends uniquely to a bounded operator T_K from $\gamma(L^2(\sigma), X)$ to $\gamma(L^2(\sigma), Y)$ of norm $\|T_K\| \le \gamma(\mathcal{T}_K)$.

Proof. See [90, Proposition 4.11] or [131, Theorem 9.14]. □

We close this section by formulating a result from the previous section in the L^2-setting of the current section.

Proposition 5.17. *Suppose that $F : S \to X$ and $G : S \to X^*$ represent operators in $\gamma(L^2(\sigma), X)$ and $\gamma(L^2(\sigma), X^*)$ respectively. Then*

$$\int_S \langle F(s), G(s)\rangle \, d\sigma(s) \le \|F\|_{\gamma(L^2(\sigma),X)} \|G\|_{\gamma(L^2(\sigma),X^*)}.$$

Proof. This follows from Proposition 5.13. □

5.3 The H^∞-calculus for bisectorial operators

In this section we consider the functional calculus for bisectorial operators. The results in this section are all known, but in the literature they are often formulated for sectorial operators and (for convenience) under an additional injective assumption. For a detailed presentation of the functional calculus for sectorial operators we refer to the monograph by Haase [78].

For $\theta \in (0, \pi)$, let

$$\Sigma_\theta^+ := \{z \in \mathbb{C} \setminus \{0\} : |\arg(z)| < \theta\}$$

denote the open sector around $\mathbb{R}^+$ of angle θ. We set $\Sigma_\theta^- := -\Sigma_\theta^+$, and for $\theta \in (0, \frac{1}{2}\pi)$ we let

$$\Sigma_\theta := \Sigma_\theta^+ \cup \Sigma_\theta^-$$

be the open bisector of angle θ.

Bisectorial operators

Let us now introduce the class of operators that will be studied.

Definition 5.18. *An operator $A : \mathsf{D}(A) \subseteq X \to X$ is said to be bisectorial of angle $\omega \in (0, \frac{1}{2}\pi)$ if $\sigma(A)$ is contained in $\overline{\Sigma_\omega}$ and the set $\{z(z-A)^{-1} : z \notin \overline{\Sigma_{\omega'}}\}$ is uniformly bounded for each $\omega' \in (\omega, \frac{1}{2}\pi)$. The infimum over all such angles ω is denoted by$\omega(A)$. An operator A is said to be bisectorial if it is bisectorial for some $\omega \in (0, \frac{1}{2}\pi)$.*

The notion of sectoriality is defined similarly, replacing all bisectors Σ_θ by bisectors Σ_θ^+. In this case the infimum over all possible angles is denoted by $\omega^+(A) \in [0, \pi)$. Later on we will use the fact that the results in this section remain valid for sectorial operators with obvious modifications.

A first consequence of the definition is the following useful lemma.

Lemma 5.19. *Let A be a bisectorial operator on X. The following equivalences hold for $x \in X$:*

(i) $x \in \overline{\mathsf{D}(A)} \Leftrightarrow \lim_{t\to\infty} it(it+A)^{-1}x = x \Leftrightarrow \lim_{t\to\infty} A(it+A)^{-1}x = 0$.
(ii) $x \in \overline{\mathsf{R}(A)} \Leftrightarrow \lim_{t\to 0} A(it+A)^{-1}x = x \Leftrightarrow \lim_{t\to 0} it(it+A)^{-1}x = 0$.

Proof. (i): For $x \in \mathsf{D}(A)$ we have $x = it(it+A)^{-1}x + \frac{1}{it}it(it+A)^{-1}Ax$. This identity implies that $\lim_{t\to\infty} it(it+A)^{-1}x = x$. By the uniform boundedness of $\{it(it+A)^{-1} : t > 0\}$ the latter identity extends to $x \in \overline{\mathsf{D}(A)}$. The remaining statements are now obvious.

(ii): is proved similarly using the identity $x = A(it+A)^{-1}x + ity + t^2(it+A)^{-1}y$ for $y \in \mathsf{D}(A)$ and $x := Ay$. □

Proposition 5.20. *A bisectorial operator A on a reflexive Banach space X is densely defined and induces a direct sum decomposition*

$$X = \mathsf{N}(A) \oplus \overline{\mathsf{R}(A)}.$$

Moreover, the part of A in $\overline{\mathsf{R}(A)}$ is an injective and sectorial operator on $\overline{\mathsf{R}(A)}$.

Proof. First we show that $\overline{\mathsf{D}(A)} = X$. Since $\rho(A) \neq \varnothing$, the operator A is closed. It is even weakly closed, since weak and strong closures of linear subspaces coincide. Take $x \in X$. Since $(in(in+A)^{-1}x)_{n\geq 1}$ is bounded, it has a subsequence $in_k(in_k+A)^{-1}x$ converging weakly to some $y \in X$. This implies that $A(in_k+A)^{-1}x \rightharpoonup x - y$. Since $(in_k+A)^{-1}x \to 0$ and A is weakly closed, it follows that $x - y = 0$, hence $x = y \in \overline{\mathsf{D}(A)}^w = \overline{\mathsf{D}(A)}$.

Lemma 5.19 implies that $\mathsf{N}(A) \cap \overline{\mathsf{R}(A)} = \{0\}$. For $x \in X$ there exists a sequence $t_n \downarrow 0$ such that $it_n(it_n+A)^{-1}x$ converges weakly to some $z \in X$. Since $it_nA(it_n+A)^{-1}x \to 0$ and A is weakly closed, we find that $z \in \mathsf{N}(A)$. The weak convergence $A(it_n+A)^{-1} \rightharpoonup x - z$ implies that $x - z \in \overline{\mathsf{R}(A)}^w = \overline{\mathsf{R}(A)}$, hence $x \in \mathsf{N}(A) + \overline{\mathsf{R}(A)}$.

The straightforward proof of the final assertion is left to the reader. □

Holomorphic functional calculus

For $\theta \in (0, \frac{1}{2}\pi)$ let $H_0^\infty(\Sigma_\theta)$ be the Dunford-Riesz class consisting of all bounded holomorphic functions $\psi : \Sigma_\theta \to \mathbb{C}$ which satisfy an estimate

$$|\psi(z)| \leq C\frac{|z|^\alpha}{|i+z|^{2\alpha}}, \qquad z \in \Sigma_\theta,$$

for some $\alpha > 0$ and $C \geq 0$.

Definition 5.21. *Let A be a bisectorial operator on X and let ψ be a function in $H_0^\infty(\Sigma_\theta)$ with $0 \leq \omega(A) < \gamma < \theta < \frac{1}{2}\pi$. The operator $\psi(A) \in \mathcal{L}(X)$ is defined by*

$$\psi(A)x := \frac{1}{2\pi i}\int_{\partial\Sigma_\gamma} \psi(z)(z-A)^{-1}x\,dz, \qquad x \in X,$$

where $\partial\Sigma_\gamma$ is parametrised counter-clockwise (see the figure).

Note that in this definition the decay of ψ guarantees that the integral exists as a Bochner integral even if $0 \in \sigma(A)$. By Cauchy's theorem its value does not depend on the choice of $\gamma \in (\omega(A), \theta)$.

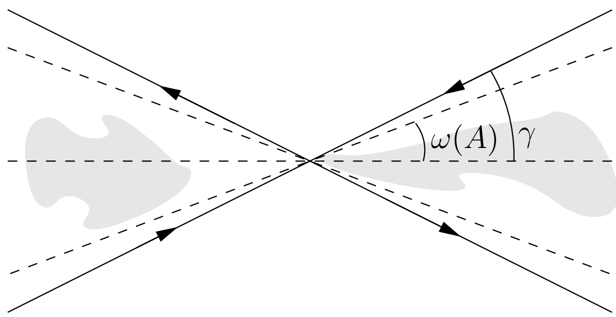

Fig. 5.1. The spectrum of a bisectorial operator is contained in $\overline{\Sigma_\omega(A)}$. The contour $\partial\Sigma_\gamma$ is parametrised counter-clockwise.

In the next result we collect some elementary properties of this functional calculus.

Proposition 5.22. *Let A be a bisectorial operator on X, and $\theta \in (\omega(A), \frac{1}{2}\pi)$.*

(i) The mapping $[\psi \in H_0^\infty(\Sigma_\theta)] \mapsto [\psi(A) \in \mathcal{L}(X)]$ is an algebra homomorphism.
(ii) The adjoint operator A^* is bisectorial on X^* and $\omega(A^*) = \omega(A)$. For $\psi \in H_0^\infty(\Sigma_\theta)$ we have $\psi(A^*) = \psi(A)^*$.
(iii) For $\psi \in H_0^\infty(\Sigma_\theta)$ and $t \in \mathbb{R} \setminus \{0\}$ we have $\psi(tA) = \psi(t\cdot)(A)$.
(iv) For $\psi \in H_0^\infty(\Sigma_\theta)$ and $x \in \mathsf{N}(A)$ we have $\psi(A)x = 0$.
(v) If $\overline{\mathsf{D}(A)} = X$ (in particular if X is reflexive), then $\overline{\mathsf{R}(A)} = \overline{\mathsf{R}(A(i+A)^{-2})}$.
(vi) (Calderón's reproducing formula) For all functions $\psi \in H_0^\infty(\Sigma_\theta)$ satisfying $\int_0^\infty \psi(\pm t)\,\frac{dt}{t} = 1$ and $\varphi \in H_0^\infty(\Sigma_\theta)$ we have

$$\int_0^\infty \psi(tA)x\,\frac{dt}{t} = x, \qquad x \in \mathsf{R}(\varphi(A)).$$

Proof. (i): The non-trivial part of the proof consists of showing that $(\varphi\psi)(A) = \varphi(A)\psi(A)$ for $\varphi, \psi \in H_0^\infty(\Sigma_\theta)$. This identity follows from a computation based on the resolvent identity.

(ii): follows from $\rho(A^*) = \rho(A)$ and $R(z, A^*) = R(z, A)^*$ for $z \in \rho(A)$.

(iii): This is straightforward.

(iv): Since $(z - A)^{-1}x = z^{-1}x$ for $x \in \mathsf{N}(A)$ and $z \in \rho(A) \setminus \{0\}$, we have $\psi(A)x = \frac{1}{2\pi i}\big(\int_{\Gamma_\gamma} \psi(z)\frac{dz}{z}\big)x$, which vanishes by Cauchy's formula.

(v): We will show that $\overline{\mathsf{R}(A)} \subseteq \overline{\mathsf{R}(A(i+A)^{-2})}$, the other inclusion being trivial. Let $x \in \mathsf{R}(A)$. Then $x^t := it(it+A)^{-1}x \in \mathsf{R}(A) \cap \mathsf{D}(A)$ and $x^t \to x$ as $t \to \infty$ by Lemma 5.19. This shows that $\mathsf{R}(A) \subseteq \overline{\mathsf{R}(A) \cap \mathsf{D}(A)}$.

Now take $x \in \mathsf{R}(A) \cap \mathsf{D}(A)$. For some $y \in \mathsf{D}(A^2)$ we have $x = Ay = A(i+A)^{-2}(i+A)^2 y$, hence $x \in \mathsf{R}(A(i+A)^{-2})$. Combining these observations, the result follows.

(vi): The dilation invariance of $\frac{dt}{t}$ implies that $\int_0^\infty \psi(tz)\,\frac{dt}{t} = 1$ for all $z \in \mathbb{R}\setminus\{0\}$, and by the principle of analytic continuation this identity extends to $z \in \Sigma_\theta$. Let $y \in X$ and put $x := \varphi(A)y$. For $\gamma \in (\omega(A), \theta)$,

$$\begin{aligned}\int_0^\infty \psi(tA)x\,\frac{dt}{t} &= \int_0^\infty (\psi(t\cdot)\varphi(\cdot))(A)y\,\frac{dt}{t}\\ &= \int_0^\infty \int_{\partial\Sigma_\gamma} \psi(tz)\varphi(z)(z-A)^{-1}y\,dz\,\frac{dt}{t}\\ &= \int_{\partial\Sigma_\gamma} \Bigl(\int_0^\infty \psi(tz)\,\frac{dt}{t}\Bigr)\varphi(z)(z-A)^{-1}y\,dz\\ &= \int_{\partial\Sigma_\gamma} \varphi(z)(z-A)^{-1}y\,dz\\ &= \varphi(A)y = x.\end{aligned}$$

□

For $\theta \in (0, \frac{1}{2}\pi)$ consider the extended Dunford-Riesz class

$$\mathcal{E}(\Sigma_\theta) := H_0^\infty(\Sigma_\theta) \oplus \langle (i+z)^{-1}\rangle \oplus \langle \mathbf{1}\rangle.$$

The identity $\frac{1}{(i+z)^2} = \frac{iz}{(i+z)^2} - \frac{i}{i+z}$ implies that $\mathcal{E}(\Sigma_\theta)$ is an algebra.

For $f \in \mathcal{E}(\Sigma_\theta)$ of the form $f = \psi + \alpha(i+z)^{-1} + \beta\mathbf{1}$ with $\alpha, \beta \in \mathbb{C}$, we define

$$f(A) := \psi(A) + \alpha(i+A)^{-1} + \beta I \in \mathcal{L}(X).$$

The following result extends Proposition 5.22(i).

Lemma 5.23. *Let A be a bisectorial operator on X. For $\theta \in (\omega(A), \frac{1}{2}\pi)$ the mapping $\psi \mapsto \psi(A)$ is an algebra homomorphism from $\mathcal{E}(\Sigma_\theta)$ into $\mathcal{L}(X)$.*

Proof. See [78, Theorem 2.3.3]. □

A first consequence of this lemma is the following identity:

Lemma 5.24. *Let A be a bisectorial operator on X. Then $(\frac{z}{(i+z)^2})(A) = A(i+A)^{-2}$.*

Proof. Since $\frac{z}{(i+z)^2} = \frac{1}{i+z} - \frac{i}{(i+z)^2}$, Lemma 5.23 implies that

$$\begin{aligned}(\tfrac{z}{(i+z)^2})(A) &= (\tfrac{1}{i+z})(A) - i(\tfrac{1}{(i+z)^2})(A)\\ &= (i+A)^{-1} - i\bigl((\tfrac{1}{i+z})(A)\bigr)^2\\ &= (i+A)^{-1} - i(i+A)^{-2}\\ &= A(i+A)^{-2}.\end{aligned}$$

□

Extension of the holomorphic functional calculus

In the rest of this chapter we will make the simplifying assumption that

- X is a reflexive Banach space.

We will now consider sectorial operators, but it is easily seen that the results extend to the bisectorial case with obvious modifications. As we do not assume that the operators are injective, we consider functions defined on the union $\Sigma_\theta^+ \cup \{0\}$ for some $\theta \in (0, \pi)$.

For $\theta \in (0, \pi)$ we consider the following spaces of functions:

- $H^\infty(\Sigma_\theta^+)$ consists of all bounded functions $f : \Sigma_\theta^+ \cup \{0\} \to \mathbb{C}$ which are holomorphic on Σ_θ^+.
- $\mathcal{F}(\Sigma_\theta^+)$ is the collection of all functions $f : \Sigma_\theta^+ \cup \{0\} \to \mathbb{C}$ which are holomorphic on Σ_θ^+ and obey an estimate
$$|f(z)| \leq C(|z|^\alpha + |z|^{-\alpha}), \qquad z \in \Sigma_\theta^+, \tag{5.4}$$
for some $C \geq 0$ and $\alpha > 0$.

Clearly, $H^\infty(\Sigma_\theta^+) \subsetneq \mathcal{F}(\Sigma_\theta^+)$. We will regard $\mathcal{E}(\Sigma_\theta^+)$ as a subspace of $H^\infty(\Sigma_\theta^+)$ by defining
$$f(0) := \alpha + \beta$$
for functions $f \in \mathcal{E}(\Sigma_\theta^+)$ of the form
$$f := \psi + \alpha(1+z)^{-1} + \beta\mathbf{1}$$
with $\psi \in H_0^\infty(\Sigma_\theta^+)$, and $\alpha, \beta \in \mathbb{C}$.

For our purposes the most important functions f contained in $\mathcal{F}(\Sigma_\theta^+)$ are the fractional powers $f(z) := z^\alpha$ for $\mathrm{Re}\,\alpha > 0$. A very useful function in the bisectorial case is the function $\mathrm{sgn} \in H^\infty(\Sigma_\theta)$ defined by
$$\mathrm{sgn}(z) := \begin{cases} \pm 1, & z \in \Sigma_\theta^\pm, \\ 0, & z = 0. \end{cases} \tag{5.5}$$

Let A be a sectorial operator on X. So far we defined $\psi(A) \in \mathcal{L}(X)$ for $\psi \in \mathcal{E}(\Sigma_\theta^+)$ with $\theta \in (\omega^+(A), \pi)$. (Actually, we gave the definition in the bisectorial case; the sectorial case is analogous.) Our next aim is to define $f(A)$ as a closed operator for $f \in \mathcal{F}(\Sigma_\theta^+)$.

Let A_0 be the part of A in $\overline{\mathsf{R}(A)}$. In view of Proposition 5.20, A_0 is an injective sectorial operator on $\overline{\mathsf{R}(A)}$. For $f \in \mathcal{F}(\Sigma_\theta^+)$ we define
$$\mathrm{Reg}(f) := \{e \in H_0^\infty(\Sigma_\theta^+) : e(A_0) \text{ is injective, } ef \in H_0^\infty(\Sigma_\theta^+)\}.$$

If $e \in \mathrm{Reg}(f)$ we say that e is a *regulariser* for f.

Lemma 5.25. *Let A be a sectorial operator on X and let $\theta \in (\omega^+(A), \pi)$. For each $f \in \mathcal{F}(\Sigma_\theta^+)$ we have Reg$(f) \neq \varnothing$.*

Proof. Take $f \in \mathcal{F}(\Sigma_\theta^+)$ and $\alpha > 0$ such that (5.4) holds, and pick an integer $n > \alpha$. We will show that the functions $\psi_n(z) := \frac{z^n}{(1+z)^{2n}}$ are contained in $\mathrm{Reg}(f)$. Clearly, $\psi_n f \in H_0^\infty(\Sigma_\theta^+)$. Lemma 5.24 and Lemma 5.22(i) imply that $\psi_n(A_0) = A_0^n(I + A_0)^{-2n}x$. To show that $\psi_n(A_0)$ is injective, suppose that $A_0(I + A_0)^{-2}x = 0$ for some $x \in \overline{\mathsf{R}(A)}$. Then $A_0(I + A_0)^{-1}x = 0$ hence $x = (I + \tilde{A})^{-1}x$. This implies that $x \in \mathsf{D}(A_0)$ and $A_0x = 0$, hence $x = 0$ since A_0 is injective. The injectivity of $\psi_n(A_0)$ follows by induction. $\square$

Now we are in a position to define $f(A)$. This will be done in two steps.

1. We define $f(A_0)$ for $f \in \mathcal{F}(\Sigma_\theta^+)$ by a regularisation procedure: Fix $e \in \mathrm{Reg}(f)$ and define
$$f(A_0)x := e(A_0)^{-1}(ef)(A_0)x, \qquad x \in \mathsf{D}(f(A_0)), \tag{5.6}$$
where
$$\mathsf{D}(f(A_0)) := \{x \in \overline{\mathsf{R}(A)} : (ef)(A)x \in \mathsf{R}(e(A_0))\}. \tag{5.7}$$
This definition does not depend on the choice of $e \in \mathrm{Reg}(f)$. Indeed, if $\tilde{e}$ is another regulariser for f, we use the fact that $e(A_0)^{-1}\tilde{e}(A_0)^{-1} = \tilde{e}(A_0)^{-1}e(A_0)^{-1}$ to obtain
$$\begin{aligned}\tilde{e}(A_0)^{-1}(\tilde{e}f)(A_0) &= \tilde{e}(A_0)^{-1}e(A_0)^{-1}e(A_0)(\tilde{e}f)(A_0)\\ &= e(A_0)^{-1}\tilde{e}(A_0)^{-1}(e\tilde{e}f)(A_0)\\ &= e(A_0)^{-1}\tilde{e}(A_0)^{-1}\tilde{e}(A_0)(ef)(A_0)\\ &= e(A_0)^{-1}(ef)(A_0),\end{aligned}$$
which proves the claim.
2. We define $f(A)x$ by
$$f(A)x := f(0)x^0 + f(A_0)x^1, \qquad x := x^0 + x^1 \in \mathsf{N}(A) \oplus \mathsf{D}(f(A_0)).$$

Proposition 5.26. *Let A be a sectorial operator on X, let $\theta \in (\omega^+(A), \pi)$, and let $f \in \mathcal{F}(\Sigma_\theta^+)$. Then $f(A)$ is a closed operator on X.*

Proof. In view of Proposition 5.20 it suffices to show that $f(A_0)$ is closed on $\overline{\mathsf{R}(A)}$. Let e be a regulariser for f. Take $x_n \in \mathsf{D}(f(A_0))$ and $x, y \in X$ such that $x_n \to x$ and $f(A_0)x_n \to y$. This implies that $e(A_0)f(A_0)x_n \to e(A_0)y$. Since $f(A_0)x_n = e(A_0)^{-1}(ef)(A_0)x_n$, it follows that $e(A_0)y = (ef)(A_0)x$, hence $x \in \mathsf{D}(f(A_0))$ and $f(A_0)x = y$. $\square$

Lemma 5.27. *Let A be a sectorial operator on X and let $\theta \in (\omega^+(A), \pi)$. For $f_1, f_2 \in \mathcal{F}(\Sigma_\theta)$ we have $f_1(A)f_2(A) \subseteq (f_1f_2)(A)$.*

Proof. It suffices to prove the result for A_0.

Let e_1 and e_2 be regularisers for f_1 and f_2 respectively. Then e_1e_2 is a regulariser for f_1f_2. Let $x \in \mathsf{D}(f_1(A_0)f_2(A_0))$ and set $z := f_2(A_0)x$ and $w := f_1(A_0)z$. By definition we have $(e_2f_2)(A_0)x = e_2(A_0)z$ and $(e_1f_1)(A_0)z = e_1(A_0)w$, and therefore

$$\begin{aligned}(e_1e_2f_1f_2)(A_0)x &= (e_1f_1)(A_0)(e_2f_2)(A_0)x = (e_1f_1)(A_0)e_2(A_0)z \\ &= e_2(A_0)(e_1f_1)(A_0)z = e_2(A_0)e_1(A_0)w = (e_1e_2)(A_0)w.\end{aligned}$$

This means that $x \in \mathsf{D}((f_1f_2)(A_0))$ and $(f_1f_2)(A_0)x = w$. □

Squares, square roots, and signs

Proposition 5.28. *Let A be a sectorial operator on X, let $\theta \in (\omega^+(A), \pi)$, and let $\alpha \in (0, \frac{\pi}{\theta})$.*

(i) $A^\alpha := (z^\alpha)(A)$ is sectorial with $\omega^+(A^\alpha) = \alpha\omega^+(A)$.
(ii) $\overline{\mathsf{R}(A^\alpha)} = \overline{\mathsf{R}(A)}$ and $\mathsf{N}(A^\alpha) = \mathsf{N}(A)$.
(iii) For $f \in H_0^\infty(\Sigma_\theta^+)$ we have $\psi_\alpha(A) = \psi(A^\alpha)$, where $\psi_\alpha \in H_0^\infty(\Sigma_{\theta/\alpha}^+)$ is defined by $\psi_\alpha(z) := \psi(z^\alpha)$.

Proof. See [78, Propositions 3.1.1, 3.1.2 & 3.1.4] □

The bisectorial counterpart is as follows.

Proposition 5.29. *Let A be a bisectorial operator on X and $\theta \in (\omega(A), \frac{1}{2}\pi)$.*

(i) The operator A^2 is sectorial with $\omega^+(A^2) = 2\omega(A)$.
(ii) $\overline{\mathsf{R}(A^2)} = \overline{\mathsf{R}(A)}$ and $\mathsf{N}(A^2) = \mathsf{N}(A)$.
(iii) For $f \in \mathcal{F}(\Sigma_{2\theta}^+)$ we have $\widehat{f}(A) = f(A^2)$, where $\widehat{f} \in \mathcal{F}(\Sigma_\theta)$ is defined by $\widehat{f}(z) := f(z^2)$.

Proof. (i): It is easily checked that $z \in \rho(A)$ implies that $z^2 \in \rho(A^2)$ and

$$(z^2 - A^2)^{-1} = -(z - A)^{-1}(-z - A)^{-1}. \tag{5.8}$$

The resolvent bounded follows from this identity, hence A^2 is sectorial.

(ii): For $x \in \overline{\mathsf{R}(A)}$ Lemma 5.19 implies that

$$x = \lim_{s\to 0} A(is + A)^{-1}x = \lim_{s\to 0}\lim_{t\to 0} A^2(it + A)^{-1}(is + A)^{-1}x,$$

hence $\overline{\mathsf{R}(A)} \subseteq \overline{\mathsf{R}(A^2)}$. The reverse inclusion is trivial.

For the second statement it suffices to show that $Ax = 0$ whenever $A^2x = 0$. For $x \in \mathsf{N}(A^2)$ we have, again by Lemma 5.19,

$$Ax = \lim_{t\to 0}(it + A)^{-1}A^2x = 0,$$

which completes the proof.

(iii): Since $\mathsf{N}(A) = \mathsf{N}(A^2)$ by (ii), it suffices to prove the result for A_0. By a direct computation involving (5.8) and the resolvent identity we have $\widehat{\psi}(A_0) = \psi(A_0^2)$ for any $\psi \in H_0^\infty(\Sigma_{2\theta}^+)$. Note that $\widehat{e}$ is a regulariser for $\widehat{f}$ whenever e is a regulariser for f. Since $\widehat{e}(A_0) = e(A_0^2)$ and $(ef)(A_0) = (\widehat{e}\widehat{f})(A_0)$, it follows that $\mathsf{D}(\widehat{f}(A_0)) = \mathsf{D}(f(A_0^2))$ and $\widehat{f}(A_0)x = f(A_0^2)x$ for $x \in \mathsf{D}(f(A_0^2))$. □

Applying these results to the sgn-function, we obtain the following result.

Proposition 5.30. *Let A be a bisectorial operator on X. For $\theta \in (\omega(A), \pi)$ define $\psi \in H_0^\infty(\Sigma_{2\theta}^+)$ by $\psi(z) := z(1+z)^{-2}$, and $\widehat{\psi} \in H_0^\infty(\Sigma_\theta)$ by $\widehat{\psi}(z) := \psi(z^2)$. Then $\mathsf{R}(\psi(A)) \in \mathsf{D}(\sqrt{A^2}) \cap \mathsf{D}(\operatorname{sgn}(A)A)$ and*

$$\sqrt{A^2}y = \operatorname{sgn}(A)Ay, \qquad y \in \mathsf{R}(\widehat{\psi}(A)).$$

Proof. Take $x \in X$ and set $y := \widehat{\psi}(A)x$. Proposition 5.29(iii) implies that $y = A^2(i + A^2)^{-2}x$, hence $y \in \mathsf{D}(A)$. Moreover, by Lemma 5.27,

$$\begin{aligned}
\big(\operatorname{sgn}(z)\widehat{\psi}(z)\big)(A)Ay &= \big(\operatorname{sgn}(z)\widehat{\psi}(z)\big)(A)A\widehat{\psi}(A)x \\
&= \big(\operatorname{sgn}(z)\widehat{\psi}(z)\big)(A)\big(z\widehat{\psi}(z)\big)(A)x \\
&= \big(z\operatorname{sgn}(z)\widehat{\psi}^2(z)\big)(A)x \\
&= \widehat{\psi}(A)\big(z\operatorname{sgn}(z)\widehat{\psi}(z)\big)(A)x,
\end{aligned}$$

hence, by (5.6) and (5.7) we find that $Ay \in \mathsf{D}(A\operatorname{sgn}(A))$ and $\operatorname{sgn}(A)Ay = \big(z\operatorname{sgn}(z)\widehat{\psi}(z)\big)(A)x$.

On the other hand, by Proposition 5.29(iii),

$$\big(\sqrt{z}\psi(z)\big)(A^2)y = \big(\sqrt{z}\psi(z)\big)(A^2)\psi(A^2)x = \psi(A^2)\big(\sqrt{z}\psi(z)\big)(A^2)x,$$

which implies that $y \in \mathsf{D}(\sqrt{A^2})$ and $\sqrt{A^2}y = \big(\sqrt{z}\psi(z)\big)(A^2)x$. By another application of Proposition 5.29(iii) we infer that

$$\sqrt{A^2}y = \big(\sqrt{z^2}\psi(z^2)\big)(A)x = (z\operatorname{sgn}(z)\widehat{\psi}(z))(A)x,$$

which completes the proof.

□

Square functions

Proposition 5.31. *Let A be a sectorial operator on X. For $\theta \in (\omega^+(A), \pi)$, $\psi \in H_0^\infty(\Sigma_\theta^+)$, and $\alpha \in (0, \frac{\pi}{\theta})$ we have*

$$\|\psi(tA^\alpha)x\|_{\gamma(L^2(\mathbb{R}_+, \frac{dt}{t}), X)} = \sqrt{\alpha}\|\psi_\alpha(tA)x\|_{\gamma(L^2(\mathbb{R}_+, \frac{dt}{t}), X)}, \qquad x \in X,$$

where $\psi_\alpha \in H_0^\infty(\Sigma_{\theta/\alpha}^+)$ is defined by $\psi_\alpha(z) := \psi(z^\alpha)$.

Proof. By Proposition 5.28(iii) we have $\psi(tA^\alpha) = \psi_\alpha(t^{1/\alpha}A)$. Therefore, for any orthonormal basis $(e_k)_{k\ge1}$ of $L^2(\mathbb{R}_+, \frac{dt}{t})$,

$$\begin{aligned}
\|\psi(tA^\alpha)x\|^2_{\gamma(L^2(\mathbb{R}_+,\frac{dt}{t}),X)} &= \|\psi_\alpha(t^{1/\alpha}A)x\|^2_{\gamma(L^2(\mathbb{R}_+,\frac{dt}{t}),X)} \\
&= \mathbb{E}\Big\|\sum_{k=1}^\infty \gamma_k \int_0^\infty e_k(t)\psi_\alpha(t^{1/\alpha}A)x\,\frac{dt}{t}\Big\|^2 \\
&= \mathbb{E}\Big\|\sum_{k=1}^\infty \gamma_k \int_0^\infty \alpha e_k(s^\alpha)\psi_\alpha(sA)x\,\frac{ds}{s}\Big\|^2 \\
&= \alpha\|\psi_\alpha(sA)x\|^2_{\gamma(L^2(\mathbb{R}_+,\frac{ds}{s}),X)},
\end{aligned}$$

where we used that $(\sqrt{\alpha}e_k(s^\alpha))$ is an orthonormal basis of $L^2(\mathbb{R}_+, \frac{ds}{s})$. □

The bisectorial counterpart reads as follows.

Proposition 5.32. *Let A be a bisectorial operator on a Banach space X. For $\psi \in H_0^\infty(\Sigma_{2\theta}^+)$ with $\theta \in (\omega(A), \frac{1}{2}\pi)$, we have*

$$\|\psi(tA^2)x\|_{\gamma(L^2(\mathbb{R}_+,\frac{dt}{t}),X)} = \sqrt{2}\|\widehat{\psi}(tA)x\|_{\gamma(L^2(\mathbb{R}_+,\frac{dt}{t}),X)}, \qquad x \in X,$$

where $\widehat{\psi} \in H_0^\infty(\Sigma_\theta)$ is defined by $\widehat{\psi}(z) := \psi(z^2)$.

Proof. Proposition 5.29(iii) implies that $\widehat{\psi}(A) = \psi(A^2)$. Arguing as in Proposition 5.31 we obtain

$$\begin{aligned}
\|\psi(tA^2)x\|_{\gamma(L^2(\mathbb{R}_+,\frac{dt}{t}),X)} &= \|\widehat{\psi}(\sqrt{t}A)x\|_{\gamma(L^2(\mathbb{R}_+,\frac{dt}{t}),X)} \\
&= \sqrt{2}\|\widehat{\psi}(tA)x\|_{\gamma(L^2(\mathbb{R}_+,\frac{dt}{t}),X)}.
\end{aligned}$$

□

Boundedness of the H^∞-calculus

Definition 5.33. *Let A be a sectorial operator on X and let $\theta \in (\omega^+(A), \pi)$. We say that A has a* bounded $H^\infty(\Sigma_\theta^+)$-functional calculus *if there exists a constant $C \ge 0$ such that for all $\psi \in H_0^\infty(\Sigma_\theta^+)$ and all $x \in X$ we have*

$$\|\psi(A)x\| \le C\|\psi\|_{H^\infty(\Sigma_\theta^+)}\|x\|.$$

The infimum over all possible angles θ for which this estimate holds is denoted by $\omega^+_{H^\infty}(A)$. We say that A has a bounded H^∞-functional calculus *if it has a bounded $H^\infty(\Sigma_\theta^+)$-functional calculus for some $\theta \in (\omega^+(A), \pi)$.*

The following convergence lemma is crucial.

Lemma 5.34. *(Convergence lemma) Let A be a sectorial operator, let $\theta \in (\omega^+(A), \pi)$, and suppose that $f_n, f \in H^\infty(\Sigma_\theta^+)$ satisfy*

- $\sup_{n\ge 1}\|f_n\|_\infty < \infty$ *and* $f_n \to f$ *uniformly on compacta;*
- $f_n(A) \in \mathcal{L}(X)$ *and* $\sup_{n\ge 1}\|f_n(A)\|_{\mathcal{L}(X)} < \infty$.

Then $f(A) \in \mathcal{L}(X)$*,* $f_n(A)x \to f(A)x$ *for all* $x \in X$*, and*

$$\|f(A)x\| \le \sup_{n\ge 1}\|f_n(A)\|\,\|x\|, \qquad x \in \overline{\mathsf{R}(A)}.$$

Proof. See [2, Theorem D]. □

A first application of the convergence lemma is the following result:

Proposition 5.35. *Let A be a sectorial operator with a bounded $H^\infty(\Sigma_\theta^+)$-calculus for some $\theta \in (\omega^+(A), \pi)$. For all $f \in H^\infty(\Sigma_\theta^+)$ we have $f(A) \in \mathcal{L}(X)$ and $\|f(A)\|_{\mathcal{L}(X)} \lesssim \|f\|_{H^\infty(\Sigma_\theta^+)}$.*

Proof. Set $\rho_n(z) := \frac{z}{n^{-1}+z}$ and $f_n(z) := f(z)\rho_n(z)$. Since A has a bounded $H^\infty(\Sigma_\theta^+)$-calculus, Lemma 5.34 implies that $f(A) \in \mathcal{L}(X)$. Moreover, using Lemma 5.34 and the boundedness of the $H^\infty(\Sigma_\theta^+)$-calculus again, we obtain for $x := x^0 + x^1 \in \mathsf{N}(A) \oplus \overline{\mathsf{R}(A)}$,

$$\begin{aligned}
\|f(A)x\| &\le \|f(A)x^0\| + \|f(A)x^1\| \\
&\le \|f(0)x^0\| + \sup_{n\ge 1}\|f_n(A)\|\,\|x^1\| \\
&\lesssim \|f(0)x^0\| + \sup_{n\ge 1}\|f_n\|_\infty\,\|x^1\| \\
&\lesssim \|f\|_\infty\Big(\|x^0\| + \|x^1\|\Big) \\
&\lesssim \|f\|_\infty\|x\|.
\end{aligned}$$

□

5.4 Quadratic estimates and boundedness of the H^∞-calculus

In this section we will use square functions to characterise the boundedness of the H^∞-functional calculus. In order to obtain such results in Banach spaces, it is necessary to strengthen the notion of bisectoriality and combine it with the randomised boundedness from Section 5.1:

Definition 5.36. *A bisectorial operator A is said to be γ-bisectorial if for some $\theta \in (\omega(A), \frac{1}{2}\pi)$ the collection $\{z(z-A)^{-1} : z \notin \overline{\Sigma_\theta}\}$ is γ-bounded. The infimum over such θ is denoted by $\omega_\gamma(A)$.*

The notions of γ-sectoriality and $\mathcal{R}$-(bi)sectoriality are defined similarly.

The boundedness of the H^∞-calculus is closely related to square function estimates. Of fundamental importance is the following result, proved by Le

Merdy [99, Theorem 1.1]) in L^p-spaces, and generalised to Banach spaces by Kalton and Weis [90, Proposition 7.7]. See also [87, Proposition 4.6] for a version with Rademacher sums.

Theorem 5.37. *Let A be a γ-bisectorial operator on X, let $\varphi, \psi \in H_0^\infty(\Sigma_\theta)$, and let $f \in H^\infty(\Sigma_\theta)$ for some $\theta \in (\omega_\gamma(A), \frac{1}{2}\pi)$. Then*

$$\|f(A)\varphi(tA)x\|_{\gamma(L^2(\mathbb{R}_+,\frac{dt}{t}),X)} \lesssim \|f\|_\infty \|\psi(tA)x\|_{\gamma(L^2(\mathbb{R}_+,\frac{dt}{t}),X)}.$$

In the proof we use the following lemma, which is a variation of [87, Lemma 4.7].

Lemma 5.38. *Let $\xi \in L^1(\mathbb{R}_+, \frac{dt}{t})$. The operator S defined for $u \in L^2(\mathbb{R}_+, \frac{dt}{t})$ and $x \in X$ by*

$$S(u \otimes x)(s) := \left(\int_0^\infty \xi(st)u(t)\,\frac{dt}{t} \right) \otimes x, \qquad s > 0, \tag{5.9}$$

extends uniquely to a bounded operator on $\gamma(L^2(\mathbb{R}_+, \frac{dt}{t}), X)$ of norm $\le \|\xi\|_{L^1(\mathbb{R}_+,\frac{dt}{t})}$.

Proof. By Lemma 5.10 it suffices to show that $u \mapsto \int_0^\infty \xi(t\cdot)u(t)\,\frac{dt}{t}$ defines a bounded operator on $L^2(\mathbb{R}_+, \frac{dt}{t})$. Using Jensen's inequality and Fubini's theorem we obtain for $u \in L^2(\mathbb{R}_+, \frac{dt}{t})$,

$$\begin{aligned}\int_0^\infty \Big| \int_0^\infty \xi(st)u(t)\,\frac{dt}{t}\Big|^2 \frac{ds}{s} &\le \|\xi\|_{L^1(\mathbb{R}_+,\frac{dt}{t})} \int_0^\infty \int_0^\infty |\xi(st)|\,|u(t)|^2\,\frac{dt}{t}\,\frac{ds}{s} \\ &= \|\xi\|^2_{L^1(\mathbb{R}_+,\frac{dt}{t})} \int_0^\infty |u(t)|^2\,\frac{dt}{t},\end{aligned}$$

which gives the desired result. □

Proof (of Theorem 5.37). In this proof we will write $L^2(\Omega_t) := L^2(\mathbb{R}_+, \frac{dt}{t})$ and $L^2(\Omega_z) := L^2(\partial\Sigma_\gamma, |\frac{dz}{z}|)$ for brevity.

For $x = x^0 + x^1 \in \mathsf{N}(A) \oplus \overline{\mathsf{R}(A)}$ we have $f(A)\varphi(tA)x = f(A)\varphi(tA)x^1$ and $\psi(tA)x = \psi(tA)x^1$. Therefore we may assume that $x \in \overline{\mathsf{R}(A)}$.

By an approximation argument based on Lemma 5.34 we may assume that $f \in H_0^\infty(\Sigma_\theta)$. We take $\gamma \in (\omega_\gamma(A), \theta)$ and introduce auxiliary functions $\xi, \eta \in H_0^\infty(\Sigma_\theta)$ satisfying $\int_0^\infty \xi(\pm t)\eta(\pm t)\psi(\pm t)\,\frac{dt}{t} = 1$. The dilation invariance of $\frac{dt}{t}$ implies that

$$\int_0^\infty \xi(tz)\eta(tz)\psi(tz)\,\frac{dt}{t} = 1$$

for all $z \in \mathbb{R} \setminus \{0\}$, and by the principle of analytic continuation this identity extends to $z \in \Sigma_\theta$. We define

$$u(z) := \int_0^\infty \xi(tz)f(A)\eta(tA)\psi(tA)x\,\frac{dt}{t}, \qquad z \in \Sigma_\theta,$$

and claim that

$$f(A)\varphi(sA)x = \frac{1}{2\pi i}\int_{\partial\Sigma_\gamma} \varphi(sz)R(z,A)u(z)\,dz. \tag{5.10}$$

To show this, we use Fubini's theorem and Proposition 5.22(i) to write

$$\begin{aligned} f(A) &= \frac{1}{2\pi i}\int_{\partial\Sigma_\gamma} f(z)R(z,A)\,dz \\ &= \frac{1}{2\pi i}\int_0^\infty \int_{\partial\Sigma_\gamma} f(z)\xi(tz)\eta(tz)\psi(tz)R(z,A)\,dz\,\frac{dt}{t} \\ &= \int_0^\infty f(A)\xi(tA)\eta(tA)\psi(tA)\,\frac{dt}{t}. \end{aligned}$$

Using Fubini's theorem once more we obtain

$$\begin{aligned} f(A)\varphi(sA)x &= \int_0^\infty f(A)\xi(tA)\eta(tA)\psi(tA)\varphi(sA)x\,\frac{dt}{t} \\ &= \int_0^\infty \Big(\frac{1}{2\pi i}\int_{\partial\Sigma_\gamma} \xi(tz)\varphi(sz)R(z,A)\,dz\Big) f(A)\eta(tA)\psi(tA)x\,\frac{dt}{t} \\ &= \frac{1}{2\pi i}\int_{\partial\Sigma_\gamma} \varphi(sz)R(z,A)\Big(\int_0^\infty \xi(tz)f(A)\eta(tA)\psi(tA)x\,\frac{dt}{t}\Big)\,dz \\ &= \frac{1}{2\pi i}\int_{\partial\Sigma_\gamma} \varphi(sz)R(z,A)u(z)\,dz, \end{aligned}$$

which proves the claim.

Using (5.10) and Lemma 5.38, Propostition 5.16 and the γ-bisectoriality of A, and Lemma 5.38 once more, we obtain

$$\begin{aligned} \|f(A)\varphi(tA)x\|_{\gamma(L^2(\Omega_t),X)} &\lesssim \|zR(z,A)u(z)\|_{\gamma(L^2(\Omega_z),X)} \\ &\lesssim \|u(z)\|_{\gamma(L^2(\Omega_z),X)} \\ &\lesssim \|f(A)\eta(tA)\psi(tA)x\|_{\gamma(L^2(\Omega_t),X)}. \end{aligned}$$

In view of Proposition 5.16 the only thing that remains to be shown is that $\{f(A)\eta(tA) : t > 0\}$ is γ-bounded. To show this we write

$$f(A)\eta(tA) = \frac{1}{2\pi i}\int_{\partial\Sigma_\gamma} f(z)\eta(tz)zR(z,A)\,\frac{dz}{z}.$$

The desired γ-boundedness follows from an application of Proposition 5.3, taking into account the estimate

$$\sup_{t>0}\int_{\partial\Sigma_\gamma} |f(z)\eta(tz)|\,\Big|\frac{dz}{z}\Big| \le \|f\|_\infty \int_{\partial\Sigma_\gamma} |\eta(z)|\,\Big|\frac{dz}{z}\Big| < \infty,$$

and the γ-boundedness of $\{zR(z,A) : z \in \partial\Sigma_\gamma\}$. $\square$

Corollary 5.39. *Let A be a γ-bisectorial operator on X and take $\theta \in (\omega_\gamma(A), \pi)$. For all $\varphi, \psi \in H_0^\infty(\Sigma_\theta)$ we have equivalence of norms*

$$\|\varphi(tA)x\|_{\gamma(L^2(\mathbb{R}_+, \frac{dt}{t}), X)} \eqsim \|\psi(tA)x\|_{\gamma(L^2(\mathbb{R}_+, \frac{dt}{t}), X)}, \qquad x \in X.$$

Proof. Just take $f \equiv 1$ in Theorem 5.37. □

Now we can state the second main result of this section. In Hilbert spaces it is due to McIntosh [122], in L^p-spaces it is due to Cowling, Doust, McIntosh and Yagi [41] and Le Merdy [100]. The general case is due to Kalton and Weis [90] (see also [87]).

Here we do not state the result under the weakest possible assumptions on the Banach space (for which we refer the interested reader to [87, 90]), but the present formulation is sufficiently general for the applications in Chapter 4. The most important equivalence for our purposes is (i)⇔(v).

Theorem 5.40. *Let A be a γ-sectorial operator on a Banach space X and assume that X has non-trivial type. Fix $\theta \in (\omega_\gamma^+(A), \pi)$ and take $\psi \in H_0^\infty(\Sigma_\theta^+)$. The following assertions are equivalent.*

(i) A has a bounded $H^\infty(\Sigma_\theta^+)$-calculus.
(ii) The collection

$$\Big\{ \sum_{k=n_1}^{n_2} \varepsilon_k \psi(2^k tA) : -\infty < n_1 < n_2 < +\infty, t > 0, \varepsilon_k = \pm 1 \Big\}$$

is uniformly bounded in $\mathcal{L}(X)$.
(iii) Let $(\gamma_k)_{k\in\mathbb{Z}}$ be a Gaussian sequence. For all $x \in X$ and $x^* \in X^*$ we have

$$\mathbb{E}\Big\| \sum_{k=-\infty}^{+\infty} \gamma_k \psi(2^k A)x \Big\| \lesssim \|x\|, \qquad \mathbb{E}\Big\| \sum_{k=-\infty}^{+\infty} \gamma_k \psi(2^k A^*)x^* \Big\| \lesssim \|x^*\|.$$

(iv) For all $x \in X$ and $x^* \in X^*$ we have

$$\|\psi(tA)x\|_{\gamma(L^2(\mathbb{R}_+, \frac{dt}{t}), X)} \lesssim \|x\|, \qquad \|\psi(tA^*)x^*\|_{\gamma(L^2(\mathbb{R}_+, \frac{dt}{t}), X^*)} \lesssim \|x^*\|.$$

(v) For $x \in \overline{\mathsf{R}(A)}$ we have $\|\psi(tA)x\|_{\gamma(L^2(\mathbb{R}_+, \frac{dt}{t}), X)} \eqsim \|x\|$.

In the proof we will use the fact that if X is a Banach space with non-trivial type, then both X and X^* have finite cotype. We will also use the following result by Hoffmann-Jørgensen and Kwapień.

Theorem 5.41. *Let $1 \le p < \infty$ and suppose that the Banach space X does not contain a closed subspace isomorphic to c_0. Let $(\xi_n)_{n\ge 1}$ be a sequence of independent symmetric X-valued random variables satisfying*

$$\sup_{n\geq 1} \mathbb{E}\Big\| \sum_{k=1}^{n} \xi_k \Big\| < \infty.$$

Then there exists an X-valued random variable S such that

$$\mathbb{E}\|S - \sum_{k=1}^{n} \xi_k\|^p \to 0.$$

Proof. See [164, Theorem V.6.1.]. □

Proof (of Theorem 5.40).
(i) ⇒ (ii): For $\psi \in H_0^\infty(\Sigma_\theta^+)$, $\varepsilon_k = \pm 1$, $t > 0$, and $n_1 < n_2$, we consider the function $\psi_{\varepsilon,n_1,n_2,t} \in H^\infty(\Sigma_\theta^+)$ defined by

$$\psi_{\varepsilon,n_1,n_2,t}(z) := \sum_{k=n_1}^{n_2} \varepsilon_k \psi(2^k t z), \qquad z \in \Sigma_\theta^+.$$

By Proposition 5.35 it suffices to show that

$$\sup_{\varepsilon,n_1,n_2,t} \|\psi_{\varepsilon,n_1,n_2,t}\|_{H^\infty(\Sigma_\theta^+)} < \infty.$$

Using that $|\psi(z)| \lesssim \frac{|z|^\alpha}{|1+z|^{2\alpha}}$ for some $\alpha > 0$, we obtain

$$\begin{aligned}
\sup_{\varepsilon,n_1,n_2,t} \sup_{z\in\Sigma_\theta^+} |\psi_{\varepsilon,n_1,n_2,t}(z)| &\leq \sup_{z\in\Sigma_\theta^+} \sum_{k\in\mathbb{Z}} |\psi(2^k z)| \leq \sup_{z\in\Sigma_\theta^+} \sum_{k\in\mathbb{Z}} \frac{(2^k|z|)^\alpha}{|1+2^k z|^{2\alpha}} \\
&\lesssim \sup_{z\in\Sigma_\theta^+} \sum_{k\in\mathbb{Z}} \frac{(2^k|z|)^\alpha}{(1+2^k|z|)^{2\alpha}} = \sup_{t>0} \sum_{k\in\mathbb{Z}} \frac{(2^k t)^\alpha}{(1+2^k t)^{2\alpha}} \\
&= \sup_{t\in[1,2)} \sum_{k\in\mathbb{Z}} \frac{(2^k t)^\alpha}{(1+2^k t)^{2\alpha}} < \infty.
\end{aligned}$$

(ii) ⇒ (iii): Let $(r_k)_{k\geq 1}$ be a Rademacher sequence. Since X has finite cotype we can combine (ii) with Remark 5.2(vi) to obtain

$$\begin{aligned}
\sup_{N\geq 1} \mathbb{E}\Big\| \sum_{k=-N}^{N} \gamma_k \psi(2^k A)x \Big\| &\lesssim \sup_{N\geq 1} \mathbb{E}\Big\| \sum_{k=-N}^{N} r_k \psi(2^k A)x \Big\| \\
&\leq \sup_{N\geq 1} \sup_{\varepsilon_k=\pm 1} \Big\| \sum_{k=-N}^{N} \varepsilon_k \psi(2^k A)x \Big\| \lesssim \|x\|.
\end{aligned}$$

The desired estimate follows from Theorem 5.41. The other estimate follows in the same way, using that X^* has finite cotype and

$$\Big\| \sum_{k=-N}^{N} \varepsilon_k \psi(2^k A) \Big\|_{\mathcal{L}(X)} = \Big\| \sum_{k=-N}^{N} \varepsilon_k \psi(2^k A^*) \Big\|_{\mathcal{L}(X^*)}.$$

(iii) $\Rightarrow$ (iv): By Corollary 5.39 it suffices to show the result for one particular ψ. We choose $\psi(z) := \frac{z^{1/2}}{1+z}$. We observe that

$$\psi(tA) = \sum_{k=-\infty}^{+\infty} \mathbf{1}_{[2^k,2^{k+1})}(t)\frac{t^{1/2}}{2^{k/2}}K(t)\psi(2^k A), \qquad t > 0,$$

where

$$K(t) := \sum_{k=-\infty}^{+\infty} \mathbf{1}_{[2^k,2^{k+1})}(t)\Big[I + \frac{2^k - t}{t}tA(I+tA)^{-1}\Big], \qquad t > 0.$$

Since $\{tA(I+tA)^{-1} : t > 0\}$ is γ-bounded and $\big|\frac{2^k-t}{t}\big| \le 1$ for $t \in [2^k, 2^{k+1})$, it follows from 5.2(iii) that the collection $\{K(t) : t > 0\}$ is γ-bounded as well. Combining this fact with Proposition 5.16, and using the fact that the functions $t \mapsto \mathbf{1}_{[2^k,2^{k+1})}(t)\frac{t^{1/2}}{2^{k/2}}$ form an orthonormal system in $L^2(\mathbb{R}_+, \frac{dt}{t})$ we obtain, for all $x \in X$,

$$\begin{aligned}
&\|\psi(tA)x\|_{\gamma(L^2(\mathbb{R}_+,\frac{dt}{t}),X)} \\
&\quad = \Big\| \sum_{k=-\infty}^{+\infty} \mathbf{1}_{[2^k,2^{k+1})}(t)\frac{t^{1/2}}{2^{k/2}}K(t)\psi(2^k A)x\Big\|_{\gamma(L^2(\mathbb{R}_+,\frac{dt}{t}),X)} \\
&\quad \lesssim \Big\| \sum_{k=-\infty}^{+\infty} \mathbf{1}_{[2^k,2^{k+1})}(t)\frac{t^{1/2}}{2^{k/2}}\psi(2^k A)x\Big\|_{\gamma(L^2(\mathbb{R}_+,\frac{dt}{t}),X)} \\
&\quad = \Big(\mathbb{E}\Big\|\sum_{k=-\infty}^{+\infty}\gamma_k\psi(2^kA)x\Big\|^2\Big)^{1/2} \\
&\quad \lesssim \|x\|.
\end{aligned}$$

The other estimate is proved in the same way, taking into account that the γ-boundedness of adjoint operators is guaranteed by the fact that X has nontrivial type and Proposition 5.4.

(iv) $\Rightarrow$ (v): Take $x \in \mathsf{R}(A(I+A)^{-2})$ and $\psi \in H_0^\infty(\Sigma_\theta^+)$ satisfying $\int_0^\infty \psi^2(t)\,\frac{dt}{t} = 1$. By Calderón's reproducing formula from Proposition 5.22(vi) we have $\int_0^\infty \psi^2(tA)x\,\frac{dt}{t} = x$, hence $\langle x, x^*\rangle = \int_0^\infty \langle \psi(tA)x, \psi(tA^*)x^*\rangle\,\frac{dt}{t}$ for $x^* \in X^*$. Using Proposition 5.17 we obtain

$$\begin{aligned}
\langle x, x^*\rangle &\le \|\psi(tA)x\|_{\gamma(L^2(\mathbb{R}_+,\frac{dt}{t}),X)}\|\psi(tA^*)x^*\|_{\gamma(L^2(\mathbb{R}_+,\frac{dt}{t}),X^*)} \\
&\lesssim \|\psi(tA)x\|_{\gamma(L^2(\mathbb{R}_+,\frac{dt}{t}),X)}\|x^*\|.
\end{aligned}$$

Taking the supremum over x^* in the unit ball of X^*, we obtain the desired estimate $\|x\| \lesssim \|\psi(tA)x\|_{\gamma(L^2(\mathbb{R}_+,\frac{dt}{t}),X)}$, which extends to $x \in \overline{\mathsf{R}(A)}$ by Proposition 5.22(v).

(v) $\Rightarrow$ (i): Using Theorem 5.37 we obtain, for $x \in \overline{\mathsf{R}(A)}$ and $f \in H_0^\infty(\Sigma_\theta^+)$,

$$\begin{aligned}\|f(A)x\| &\eqsim \|f(A)\psi(tA)x\|_{\gamma(L^2(\mathbb{R}_+,\frac{dt}{t}),X)}\\ &\lesssim \|f\|_\infty \|\psi(tA)x\|_{\gamma(L^2(\mathbb{R}_+,\frac{dt}{t}),X)} \eqsim \|f\|_\infty\|x\|.\end{aligned}$$

□

The previous theorem gave a characterisation of the boundedness of the functional calculus under a randomised boundedness condition on the resolvent. The next result from Kalton and Weis [89, Theorem 5.3] shows that in “reasonable” Banach spaces, this condition is always satisfied if the operator under consideration has a bounded H^∞-calculus.

Let $(r_k)_{k\geq 1}$ and $(\overline{r}_k)_{k\geq 1}$ be independent Rademacher sequences. We say that X has *property* (Δ) if there exists a constant $C > 0$ depending only on X such that for any $x_{jk} \in X$,

$$\Big(\mathbb{E}\Big\|\sum_{j=1}^n\sum_{k=1}^j r_j\overline{r}_k x_{jk}\Big\|^2\Big)^{1/2} \leq C\Big(\mathbb{E}\Big\|\sum_{j=1}^n\sum_{k=1}^n r_j\overline{r}_k x_{jk}\Big\|^2\Big)^{1/2}.$$

The infimum over all possible $C \geq 0$ will be denoted by $\delta(X)$. All UMD spaces (in particular Hilbert spaces and L^p-spaces) have property (Δ).

Theorem 5.42. *Let X be a Banach space with non-trivial type and property (Δ). If A is a sectorial operator having a bounded $H^\infty(\Sigma_\theta^+)$-calculus for some $\theta \in (\omega^+(A), \pi)$, then A is γ-sectorial with $\omega_\gamma^+(A) = \omega_{H^\infty}^+(A)$.*

The proof uses the following lemma.

Lemma 5.43. *Let $0 < \nu < \theta$, and let $f : \Sigma_\theta^+ \to \mathcal{L}(X)$ be a bounded holomorphic function. Suppose that*

$$\sup_{t>0}\mathcal{R}(\{f(a^k te^{\pm i\nu}) : k \in \mathbb{Z}\}) < \infty$$

for some $a > 1$. Then the collection $\{f(z) : z \in \Sigma_\omega^+\}$ is $\mathcal{R}$-bounded for each $0 < \omega < \nu$.

Proof. See [170, Lemma 3.8]. □

Proof (of Theorem 5.42). Since X is assumed to have finite cotype we may switch between Rademacher and Gaussian sums by virtue of Remark 5.2(vi). We proceed in several steps.

Step 1. For $i = 1, 2$, let $(U_k)_{k\geq 1}, (V_k)_{k\geq 1} \subseteq \mathcal{L}(X)$ satisfy

$$\sup_{n\geq 1}\sup_{\varepsilon_k=\pm 1}\Big\|\sum_{k=1}^n \varepsilon_k U_k\Big\| \leq M, \qquad \sup_{n\geq 1}\sup_{\varepsilon_k=\pm 1}\Big\|\sum_{k=1}^n \varepsilon_k V_k\Big\| \leq M,$$

for some $M \geq 0$. We claim that the collection $\mathcal{C} := \{\sum_{k=1}^n U_kV_k : n \geq 1\}$ is $\mathcal{R}$-bounded with $\mathcal{R}(\mathcal{C}) < \delta(X)M^2$.

Let $(r_k)_{k\geq 1}$ and $(\overline{r}_k)_{k\geq 1}$ be independent Rademacher sequences. For $y_k \in X$ we have

$$\begin{aligned}\Big\|\sum_{k=1}^n U_k y_k\Big\| &= \Big\|\mathbb{E}\sum_{j=1}^n r_j U_j\Big(\sum_{k=1}^n r_k y_k\Big)\Big\| \\ &\leq \mathbb{E}\Big\|\sum_{k=1}^n r_k U_k\Big\|\,\Big\|\sum_{k=1}^n r_k y_k\Big\| \\ &\leq M\Big(\mathbb{E}\Big\|\sum_{k=1}^n r_k y_k\Big\|^2\Big)^{1/2}.\end{aligned}$$

Now we take $x_k \in X$ and apply this estimate to $y_k = V_k \sum_{j=k}^n \overline{r}_j x_j$ to obtain

$$\begin{aligned}\mathbb{E}\Big(\Big\|\sum_{j=1}^n\sum_{k=1}^j \overline{r}_j U_k V_k x_j\Big\|^2\Big)^{1/2} &\leq M\Big(\mathbb{E}\Big\|\sum_{j=1}^n\sum_{k=1}^j \overline{r}_j r_k V_k x_j\Big\|^2\Big)^{1/2} \\ &\leq \delta(X)M\Big(\mathbb{E}\Big\|\sum_{j=1}^n\sum_{k=1}^n \overline{r}_j r_k V_k x_j\Big\|^2\Big)^{1/2} \\ &\leq \delta(X)M^2\Big(\mathbb{E}\Big\|\sum_{j=1}^n \overline{r}_j x_j\Big\|^2\Big)^{1/2}.\end{aligned}$$

Since the operators in the definition of $\mathcal{R}$-boundedness may be chosen to be mutually distinct [36, Lemma 3.3], it follows that $\mathcal{C}$ is $\mathcal{R}$-bounded.

Step 2. Fix $\nu \in (\theta, \pi)$. We will prove that

$$\sup_{t>0} \mathcal{R}(\{2^n t(2^n t e^{i\nu} - A)^{-1} : n \in \mathbb{Z}\}) < \infty.$$

Consider the function $\varphi \in H_0^\infty(\Sigma_\theta^+)$ defined by $\varphi(z) := z(z - e^{i\nu})^{-1}(2z - e^{i\nu})^{-1}$ and set $\psi(z) := \sqrt{\varphi(z)}$. By the implication (i)⇒(ii) in Theorem 5.40 (for which the γ-sectoriality assumption is not needed!) we have

$$\sup_{n_1,n_2,t,\varepsilon} \Big\|\sum_{k=n_1}^{n_2} \varepsilon_k \psi(2^k tA)\Big\|_{\mathcal{L}(X)} < \infty.$$

Therefore, Step 1 applied with $U_k = V_k = \psi(2^k tA)$ implies that

$$\sup_{t>0} \mathcal{R}\Big(\Big\{\sum_{k=n_1}^{n_2} \varphi(2^{-k}t^{-1}A) : n_1 < n_2\Big\}\Big) < \infty.$$

Using the resolvent identity we obtain

$$\begin{aligned}\sum_{k=n_1}^{n_2} \varphi(2^{-k}t^{-1}A) &= \sum_{k=n_1}^{n_2} Ae^{-i\nu}\big((2^{k-1}te^{i\nu} - A)^{-1} - (2^k te^{i\nu} - A)^{-1}\big) \\ &= 2^{n_1-1}t(2^{n_1-1}te^{i\nu} - A)^{-1} - 2^{n_2}t(2^{n_2}te^{i\nu} - A)^{-1}.\end{aligned}$$

Lemma 5.19 implies that $2^{n_2}t(2^{n_2}te^{i\nu} - A)^{-1}$ converges strongly to $P_{\overline{\mathsf{R}(A)}}$ as $n_2 \to \infty$. Therefore Remark 5.2(vii) implies that

$$\sup_{t>0} \mathcal{R}\big(\big\{2^n t(2^n te^{i\nu} - A)^{-1} : n \in \mathbb{Z}\big\}\big) < \infty.$$

Step 3. We claim that A is γ-sectorial with $\omega_\gamma^+(A) \le \omega_{H^\infty}^+(A)$.
This follows immediately by combining Step 2 with Lemma 5.43.
Step 4. It remains to show that $\omega_\gamma^+(A) \ge \omega_{H^\infty}^+(A)$.

Suppose that A has a bounded $H^\infty(\Sigma_\theta^+)$-calculus for *some* $\theta \in (\omega_\gamma^+(A), \pi)$ and take $\psi \in H_0^\infty(\Sigma_\theta^+)$. The implication (i) $\Rightarrow$ (v) in Theorem 5.40 implies that $\|\psi(tA)x\|_{\gamma(L^2(\mathbb{R}_+, \frac{dt}{t}), X)} \eqsim \|x\|$ for $x \in \overline{\mathsf{R}(A)}$. Obviously, the restriction of ψ belongs to $H_0^\infty(\Sigma_{\theta'}^+)$ for any $\theta' \in (\omega_\gamma^+(A), \theta]$. Applying the opposite implication (v) $\Rightarrow$ (i), we find that A has a $H^\infty(\Sigma_{\theta'}^+)$-calculus, hence $\omega_\gamma^+(A) \ge \omega_{H^\infty}^+(A)$. $\square$

5.5 Analytic semigroups

Let A be a sectorial operator on X with $\omega^+(A) \in [0, \frac{1}{2}\pi)$. For $\lambda \in \Sigma^+_{\frac{1}{2}\pi - \omega^+(A)}$ the functions $e_\lambda(z) := e^{-\lambda z}$ are contained in $\mathcal{E}(\Sigma_\theta^+)$ whenever

$$\theta \in (\omega^+(A), \frac{1}{2}\pi - |\arg \lambda|).$$

This allows us to define

$$e^{-\lambda A} := e_\lambda(A), \qquad \lambda \in \Sigma^+_{\frac{1}{2}\pi - \omega^+(A)}.$$

These operators form a bounded analytic C_0-semigroup in the sense of the following definition. For details we refer to [78, Section 3.4].

Definition 5.44. *A collection of operators $(T(z))_{z\in\Sigma_\theta\cup\{0\}} \subseteq \mathcal{L}(X)$ with $\theta \in (0, \frac{1}{2}\pi]$ is called a bounded analytic C_0-semigroup (of angle θ) if the following conditions hold:*

(i) $T(0) = I$ *and* $T(z_1 + z_2) = T(z_1)T(z_2)$ *for all* $z_1, z_2 \in \Sigma_\theta$;
(ii) The map $z \mapsto T(z)$ *is analytic in* Σ_θ *and strongly continuous in* $\Sigma_{\theta'} \cup \{0\}$ *for each* $0 < \theta' < \theta$;
(iii) $\sup\{\|T(z)\| : z \in \Sigma_{\theta'}\}$ *is bounded for all* $0 < \theta' < \theta$.

If the suprema in (iii) are at most 1, then we say that T is an analytic contraction semigroup (of angle θ).

Remark 5.45. There exists C_0-semigroups $T(t)_{t\ge0}$ which are contractive on $[0,\infty)$ and which can be analytically extend to a sector, without being contractive on any sector of positive angle [64]. We emphasise that in the definition above $T(z)$ is required to be bounded (resp. contractive) on a sector.

The following characterisation of γ-sectoriality in terms of the semigroup will be useful.

Lemma 5.46. *Let A be a sectorial operator on X with $\omega^+(A) < \frac{1}{2}\pi$ and let S be the bounded analytic C_0-semigroup generated by $-A$. The following assertions are equivalent:*

(1) The operator A is γ-sectorial of angle θ for some $\theta \in (\omega^+(A), \frac{1}{2}\pi)$;
(2) The family $\{S(z) : \ z \in \Sigma_\nu^+\}$ is γ-bounded for some $\nu \in (0, \frac{1}{2}\pi)$.

If these equivalent conditions hold, then

$$\sup\{\nu \in (0, \tfrac{1}{2}\pi) : (2) \text{ holds}\} = \frac{1}{2}\pi - \omega_\gamma^+(A).$$

Proof. See [94, Theorem 2.20]. □

Randomised admissibility

The next proposition has been proved under slightly less general assumptions by Haak and Kunstmann in [76] (see also the PhD-thesis [77]). It generalises the L^p-result from Le Merdy [100, Theorem 3.5, Remark 3.6]. The first condition is a randomised analogue of the notion of admissibility in mathematical systems theory.

Theorem 5.47. *Suppose that the Banach space Y has property (α). Let A be γ-sectorial on X of angle $\omega_\gamma^+(A) < \frac{1}{2}\pi$, and let S be the bounded analytic C_0-semigroup on X generated by $-A$. Let $U : \mathsf{D}(A) \to Y$ be a linear operator, bounded with respect to the graph norm of $\mathsf{D}(A)$. Consider the following statements.*

(1) $\|\sqrt{t}US(t)x\|_{\gamma(L^2(\mathbb{R}_+,\frac{dt}{t}),Y)} \lesssim \|x\|, \qquad x \in \mathsf{D}_p(A)$;
(2) *The family* $\{\sqrt{t}US(t) : \ t > 0\}$ *is γ-bounded in $\mathcal{L}(X, Y)$.*

Then (1) *implies* (2)*. If A satisfies the "square function estimate"*

$$\|tAS(t)x\|_{\gamma(L^2(R_+,\frac{dt}{t}),X)} \lesssim \|x\|,$$

then (2) *implies* (1)*.*

Proof. See [76, Theorem 4.2]. □

Remark 5.48. In [100] and other works in the mathematical systems theory literature, condition (2) is replaced by the following equivalent condition:

(2′) The family $\{tU(I + t^2L)^{-1} : \ t > 0\}$ is γ-bounded.

That (2) implies (2′) follows by taking Laplace transforms and the opposite direction is observed in [100, (3.12)]. Since the computations in Chapter 4 involve semigroups rather then resolvents we prefer to use (2).

Analytic contraction semigroups on Hilbert spaces

We will show that generators of C_0-semigroups of contractions on Hilbert spaces have a bounded functional calculus on some sector Σ_θ^+ for some $\theta < \pi$. This is a consequence of the following well-known Sz.-Nagy dilation theorem.

Theorem 5.49. *Let $(T(t))_{t\geq 0}$ be a C_0-contraction semigroup on a Hilbert space H. Then there exists a Hilbert space K, an isometric embedding $J : H \hookrightarrow K$ and a unitary C_0-group $(U(t))_{t\in\mathbb{R}}$ on K such that*

$$T(t) = J^*U(t)J, \qquad t \geq 0.$$

Proof. See [48, Chapter 6, Section 4]. □

Theorem 5.50. *Let $-A$ be the generator of a C_0-contraction semigroup on a Hilbert space H. Then A has a bounded $H^\infty(\Sigma_\theta^+)$-calculus for all $\theta \in (\omega^+(A), \pi)$.*

Proof. By Theorem 5.42 it suffices to prove the result for some $\theta \in (\frac{1}{2}\pi, \pi)$.

Let $(T(t))_{t\geq 0}$ be the semigroup generated by $-A$, and take K, J and $(U(t))_{t\in\mathbb{R}}$ as in Theorem 5.49. By Stone's theorem (see, e.g., [57, Theorem 3.24]) the generator of U can be written as $-iB$ for some selfadjoint operator B on K. We claim that $J^*R(z, iB)J = R(z, A)$ for all Re $z < 0$. Indeed, taking Laplace transforms we obtain for $h \in H$,

$$J^*R(z, iB)Jh = \int_0^\infty e^{tz}J^*U(t)Jh\,dt = \int_0^\infty e^{tz}T(t)h\,dt = R(z, A)h.$$

It follows that $J^*\psi(iB)J = \psi(A)$ for all $\psi \in H_0^\infty(\Sigma_\theta^+)$. Using the functional calculus for selfadjoint operators we obtain, for $h \in H$,

$$\begin{aligned}\|\psi(A)h\|_H &= \|J^*\psi(iB)Jh\|_H \\ &\leq \|\psi(iB)Jh\|_K \leq \|\psi\|_\infty\|Jh\|_K = \|\psi\|_\infty\|h\|_H.\end{aligned}$$

□

A representation formula for generators of analytic contraction semigroups

Let us now turn to the situation where $-A$ be the generator of an analytic contraction C_0-semigroup on a complex Hilbert space H. (In case that H is real we complexify all objects.) It is well known [141, Theorem 1.57, Theorem 1.58] and the remarks following these results that A is associated with a sesquilinear form

$$a : \mathsf{D}(a) \times \mathsf{D}(a) \subseteq H \times H \to \mathbb{C},$$

which is
- *densely defined*, i.e., $\overline{\mathsf{D}(a)} = H$;
- *closed*, i.e., $\mathsf{D}(a)$ endowed with the norm $\sqrt{\|h\|_H^2 + a(h,h)}$ is complete;
- *sectorial*, i.e., there exists a constant $C \geq 0$ such that

$$|\mathrm{Im}\, a(h,h)| \leq C \mathrm{Re}\, a(h,h), \qquad h \in \mathsf{D}(a).$$

We will give a representation formula for A which shows that it is always possible to work in the Hodge-Dirac framework of Chapter 3. The result is well-known, but we could not find an explicit reference.

Theorem 5.51. *There exists a Hilbert space $\underline{H}$, a closed operator $V : \mathsf{D}(V) \subseteq H \to \underline{H}$ with dense domain $\mathsf{D}(V) = \mathsf{D}(a)$ and dense range, and a bounded coercive operator $B \in \mathcal{L}(\underline{H})$ such that*

$$A = V^* B V.$$

Equivalently, we have $a(g,h) = [BVg, Vh]$ for all $g, h \in \mathsf{D}(V)$.

Proof. Writing $a(h) := a(h,h)$ by [141, Proposition 1.8] we have

$$|a(g,h)| \lesssim (\mathrm{Re}\, a(g))^{1/2} (\mathrm{Re}\, a(h))^{1/2}, \qquad g, h \in \mathsf{D}(a).$$

We claim that $N := \{h \in \mathsf{D}(a) : \mathrm{Re}\, a(h) = 0\}$ is a closed subspace of $\mathsf{D}(a)$. Indeed, if $h_n \to h$ in $\mathsf{D}(a)$ and $\mathrm{Re}\, a(h_n) = 0$, then

$$\begin{aligned}|\mathrm{Re}\, a(h)| \leq |a(h) - a(h_n)| &\leq (\mathrm{Re}\, a(h))^{1/2} (\mathrm{Re}\, a(h - h_n))^{1/2} \\ &\quad + (\mathrm{Re}\, a(h - h_n))^{1/2} (\mathrm{Re}\, a(h_n))^{1/2},\end{aligned}$$

which becomes arbitrary small as $n \to \infty$.

On the quotient $\mathsf{D}(a)/N$ we define a sesquilinear form

$$\big[[g],[h]\big] := \frac{1}{2}(a(g,h) + \overline{a(h,g)}), \qquad g, h \in \mathsf{D}(a).$$

This form is well defined, since for $n, n' \in N$ we have

$$\begin{aligned}|a(g+n, h+n') - a(g,h)| &\leq (\mathrm{Re}\, a(n))^{1/2} (\mathrm{Re}\, a(h))^{1/2} \\ &\quad + (\mathrm{Re}\, a(g))^{1/2} (\mathrm{Re}\, a(n'))^{1/2} \\ &\quad + (\mathrm{Re}\, a(n))^{1/2} (\mathrm{Re}\, a(n'))^{1/2} \\ &= 0.\end{aligned}$$

Since $\mathrm{Re}\, a(h) = 0$ implies $[h] = [0]$, the form $[\cdot,\cdot]$ is an inner product on $\mathsf{D}(a)/N$. We put

$$\underline{H} := \overline{\mathsf{D}(a)/N},$$

where the completion is taken with respect to the norm induced by $[\cdot,\cdot]$.

Let V be the canonical mapping $h \mapsto [h]$ from $\mathsf{D}(a)$ onto $\mathsf{D}(a)/N$. We interpret V as a linear operator from H into $\underline{H}$ with dense domain $\mathsf{D}(V) = \mathsf{D}(a)$ and dense range. To show that V is closed, we take a sequence $(h_n)_{n\geq 1}$ in $\mathsf{D}(a)$ such that $h_n \to h$ in H and $Vh_n \to u$ in $\underline{H}$. Since $\mathrm{Re}\, a(h_n - h_m) = \|V(h_n - h_m)\|^2 \to 0$ as $m, n \to \infty$, the sequence $(h_n)_{n\geq 1}$ is Cauchy in $\mathsf{D}(a)$. The closedness of a implies that $(h_n)_{n\geq 1}$ has a limit in $D(a)$, which is h since $h_n \to h$ in H. Consequently, $\|Vh_n - Vh\|^2 = \mathrm{Re}\, a(h_n - h) \to 0$. We conclude that V is closed.

Now we define a sesquilinear form b on $\mathsf{R}(V)$ by

$$b(Vg, Vh) := a(g, h).$$

This is well defined, since $Vg = V\widetilde{g}$ and $Vh = V\widetilde{h}$ imply that

$$\begin{aligned}
|a(g,h) - a(\widetilde{g},\widetilde{h})| &\leq |a(g-\widetilde{g},h)| + |a(\widetilde{g},h-\widetilde{h}))| \\
&\leq (\mathrm{Re}\, a(g-\widetilde{g})\,\mathrm{Re}\, a(h))^{1/2} + (\mathrm{Re}\, a(\widetilde{g})\,\mathrm{Re}\, a(h-\widetilde{h}))^{1/2} \\
&= \|V(g-\widetilde{g})\|\,\|Vh\| + \|V\widetilde{g}\|\,\|V(h-\widetilde{h})\| = 0.
\end{aligned}$$

Moreover, the associated operator B extends to a bounded operator on $\underline{H}$, since

$$|b(Vg,Vh)| = |a(g,h)| \lesssim (\mathrm{Re}\, a(g))^{1/2}(\mathrm{Re}\, a(h))^{1/2} = \|Vg\|\,\|Vh\|.$$

We conclude that $a(g,h) = [BVg, Vh]$. By the identity

$$\|Vh\|^2 = \mathrm{Re}\, a(h) = \mathrm{Re}\, [BVh, Vh]$$

and the boundedness of B we infer that $\|u\|^2 = \mathrm{Re}\, [Bu, u]$ for all $u \in \underline{H}$, and the coercivity of B follows. $\square$

Although the triple $(\underline{H}, V, B)$ is not unique, the next result implies that the statements in Theorem 4.37 do not depend on the choice of $(\underline{H}, V, B)$.

Proposition 5.52. *Let $-A$ be the generator of an analytic C_0-contraction semigroup on H. Let $(\underline{H}, V, B)$ and $(\widetilde{\underline{H}}, \widetilde{V}, \widetilde{B})$ be triples with the properties as stated in Theorem 5.51. Then:*

(i) *The coercivity constants κ and $\widetilde{\kappa}$ of B and $\widetilde{B}$ coincide;*
(ii) $\mathsf{D}(V) = \mathsf{D}(\widetilde{V})$ *with* $\|Vh\| \eqsim \|\widetilde{V}h\|$.

If in addition to the above assumptions (E, H, μ) is an abstract Wiener space, then for $1 \leq p < \infty$ we have

(iii) $\mathsf{D}_p(D_V) = \mathsf{D}_p(D_{\widetilde{V}})$ *with* $\|D_V f\|_p \eqsim \|D_{\widetilde{V}} f\|_p$.

Proof. (i): This follows from the identity $[BVh, Vh] = a(h,h) = [\widetilde{B}\widetilde{V}h, \widetilde{V}h]$ for $h \in \mathsf{D}(a)$ and the fact that V and $\widetilde{V}$ have dense range.

(ii): For $h \in \mathsf{D}(A)$ we have

$$\kappa\|Vh\|^2 \le \mathrm{Re}\,[BVh, Vh] = \mathrm{Re}\,[Ah, h] = \mathrm{Re}\,[\widetilde{B}\widetilde{V}h, \widetilde{V}h] \le \|\widetilde{B}\|\,\|\widetilde{V}h\|^2.$$

Since $\mathsf{D}(A)$ is a core for both $\mathsf{D}(V)$ and $\mathsf{D}(\widetilde{V})$ the result follows.

(iii): Let D denote the Malliavin derivative. For $f \in \mathcal{F}C_{\mathrm{b}}^1(E; \mathsf{D}(V))$ we have, by (ii),

$$\|D_V f\|_p^p = \int_E \|VDf\|^p \, d\mu \eqsim \int_E \|\widetilde{V}Df\|^p \, d\mu = \|D_{\widetilde{V}} f\|_p^p.$$

The claim follows from this since $\mathcal{F}C_{\mathrm{b}}^1(E; \mathsf{D}(V))$ is a core for $\mathsf{D}_p(D_V)$ and $\mathsf{D}_p(D_{\widetilde{V}})$. $\square$

Analytic contraction semigroups on L^p-spaces

In this section we collect some of the good properties that analytic contraction semigroups on L^p-spaces enjoy.

We start with a result by Kalton and Weis which generalises results by Duong [56] and Hieber and Prüss [79].

Theorem 5.53. *Let $1 < p < \infty$ and let (M, μ) be a σ-finite measure space. Let $-A$ be the generator of a C_0-semigroup $(T(t))_{t\ge 0}$ of positive contractions on $L^p(\mu)$, and assume that T extends to a bounded analytic semigroup on a sector Σ_θ^+ for some $\theta \in (0, \frac12\pi)$. Then A has a bounded H^∞-calculus and $\omega_{H^\infty}^+(A) = \omega_\gamma^+(A) < \frac12\pi$.*

Proof. See [89, Corollary 5.2]. $\square$

In the proofs of Theorem 4.18 (for $p > 2$) and Theorem 4.19 (for $1 \le 2$) we use the fact that the maximal function associated with the semigroup P is L^p-bounded. This follows from the following simple extension of a well-known result of Cowling [40, Theorem 7] (see also [160]). For the convenience of the reader we include a sketch of the proof.

Proposition 5.54. *Let (M, μ) be a σ-finite measure space and let $(T(t))_{t\ge 0}$ be a bounded analytic C_0-semigroup on $L^2(\mu)$ satisfying*

(i) $\|T(t)f\|_p \le \|f\|_p$ for all $f \in L^2(\mu) \cap L^p$, $1 \le p \le \infty$, and $t \ge 0$,
(ii) $T(t)f \ge 0$ for all $f \ge 0$ and $t \ge 0$.

For $f \in L^p(\mu)$ consider the maximal function

$$T_\star f := \sup_{t>0} |T(t)f|.$$

Then, for $1 < p < \infty$ we have

$$\|T_\star f\|_p \lesssim \|f\|_p, \qquad f \in L^p.$$

Proof. Using Lemma 1.27 it follows that T extends to a bounded analytic semigroup on $L^p(\mu)$. Let $-L$ denote the generator of T on $L^p(\mu)$ and note that L has a bounded H^∞-calculus of angle $\omega < \frac{1}{2}\pi$ by Theorem 5.53. The key idea of the proof is to write

$$\begin{aligned} T(t)f &= \frac{1}{t}\int_0^t T(s)f\,ds + \Big(T(t)f - \frac{1}{t}\int_0^t T(s)f\,ds\Big) \\ &= \frac{1}{t}\int_0^t T(s)f\,ds + m(tL)f, \end{aligned}$$

where $m(z) := e^{-z} - \int_0^1 e^{-sz}\,ds$. By the Hopf-Dunford-Schwartz ergodic theorem [92, Theorem 6.12] we have

$$\Big\| \sup_{t>0} \Big| \frac{1}{t}\int_0^t T(s)f\,ds \Big| \Big\|_p \lesssim \|f\|_p,$$

so that it remains to prove that $\big\| \sup_{t>0} |m(tL)f| \big\|_p \lesssim \|f\|_p$.

Let $n := m \circ \exp$ and let $\widehat{n}$ be its Fourier transform. Using the identities

$$m(z) = \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{n}(u) z^{iu}\,du, \qquad \widehat{n}(u) = (1 - (1+iu)^{-1})\Gamma(iu),$$

and the estimate $|\widehat{n}(u)| \le Ce^{-\frac{1}{2}\pi|u|}$ (see [40]) we obtain

$$\sup_{t>0} |m(tL)f| \le \sup_{t>0} \frac{1}{2\pi}\int_{\mathbb{R}} |\widehat{n}(u)|\,|(tL)^{iu}f|\,du \lesssim \frac{1}{2\pi}\int_{\mathbb{R}} e^{-\frac{1}{2}\pi|u|}|L^{iu}f|\,du.$$

From the H^∞-calculus of L we infer that $\|L^{iu}f\|_p \lesssim e^{\omega|u|}\|f\|_p$. Taking L^p-norms we obtain

$$\begin{aligned} \big\| \sup_{t>0} |m(tL)f| \big\|_p &\lesssim \frac{1}{2\pi}\int_{\mathbb{R}} e^{-\frac{1}{2}\pi|u|}\|L^{iu}f\|_p\,du \\ &\lesssim \frac{1}{2\pi}\int_{\mathbb{R}} e^{(\omega-\frac{1}{2}\pi)|u|}\|f\|_p\,du \lesssim \|f\|_p. \end{aligned}$$

□

5.6 Notes

The notion of $\mathcal{R}$-boundedness was implicit in the work of Bourgain in the eighties [17] and has been studied systematically by Berkson and Gillespie [13], and Clément, de Pagter, Sukochev, and Witvliet [36]. It plays a crucial role in Weis's operator valued Mihlin multiplier theorem, which provided a solution to the long-standing problem of maximal L^p-regularity [169, 170] (see [88] for a complementary result). A nice account of these developments can be found in the lecture notes by Denk, Hieber and Prüss [51] and Kunstmann and Weis [94].

γ-Radonifying operators originate in the work of Gross on abstract Wiener spaces [72]. Kalton and Weis [90] used them to extend L^p-results in harmonic analysis involving square functions to more general Banach spaces. In the theory of vector-valued stochastic integration γ-radonifying operators have been used in [19, 134, 135].

The theory of H^∞-functional calculus has been introduced by McIntosh [122] and further developed by McIntosh and Yagi [123] in a Hilbert space setting. Later the main results have been extended to the L^p-setting by Cowling, Doust, McIntosh, and Yagi [41] with important contributions by Le Merdy [100]. The extension to general Banach spaces presented here is due to Kalton and Weis [90, 89].

An alternative proof of Theorem 5.50 due to Franks can be found in the lecture notes [2].

Part II

Wasserstein Theory for Infinite Dimensional Diffusions

6

Wasserstein Spaces

In this preliminary chapter we study a class of Wasserstein metrics for probability measures on Banach spaces, induced by Hilbertian subspaces. This setup turns out to be appropriate for applications to certain Fokker-Planck equations associated with stochastic equations, which will be considered in Chapter 10.

6.1 Probability measures on metric spaces

In this section we collect some background results on probability measures on metric spaces.

- Let X, Y be a separable metric spaces.

We denote by $\mathscr{P}(X)$ the collection of Borel probability measures on X. The *weak topology on* $\mathscr{P}(X)$ is the topology generated by the base consisting of all sets of the form

$$B(\mu; \delta, f_1, \ldots, f_n) := \left\{ \nu \in \mathscr{P}(X) : \max_{1 \leq k \leq n} \left| \int_X f_k \, d\nu - \int_X f_k \, d\mu \right| < \delta \right\},$$

for $\mu \in \mathscr{P}(X)$, $\delta > 0$, $n \geq 1$, and $f_1, \ldots, f_n \in C_{\mathrm{b}}(X)$. It turns out that $\mathscr{P}(X)$ endowed with the weak topology is a separable and metrisable space (see [158, Theorem 3.1.5]), which is complete whenever X is complete.

The following theorem provides a characterisation of weak compactness. A subset $\mathcal{M} \subseteq \mathscr{P}(X)$ is called *tight* if for each $\varepsilon > 0$ there exists a compact set $K \subseteq X$ such that $\mu(K) > 1 - \varepsilon$ for every $\mu \in \mathcal{M}$.

Theorem 6.1 (Prokhorov). *Suppose that X is complete. A subset $\mathcal{M} \subseteq \mathscr{P}(X)$ is relatively weakly compact if and only if $\mathcal{M}$ is tight.*

Proof. See [158, Theorem 3.1.9]. □

Clearly, a sequence $(\mu_n)_{n\geq 1} \subseteq \mathscr{P}(X)$ converges weakly to $\mu \in \mathscr{P}(X)$ if and only if

$$\int_X f \, d\mu_n \to \int_X f \, d\mu, \qquad f \in C_{\mathrm{b}}(X),$$

in which case we use the notation $\mu_n \rightharpoonup \mu$. The following characterisation will be useful.

Lemma 6.2. *A sequence $(\mu_n)_{n\geq 1} \subseteq \mathscr{P}(X)$ converges weakly to $\mu \in \mathscr{P}(X)$ if and only if*

$$\int_X f \, d\mu \leq \varliminf_{n\to\infty} \int_X f \, d\mu_n$$

for every lower semicontinuous (lsc) function $f : X \to \mathbb{R} \cup \{\infty\}$ which is bounded from below.

Proof. See [158, Theorem 3.1.5]. □

We introduce some notation which will be used throughout this chapter. Let $X_1, \ldots, X_n$ be separable metric spaces, and let $\mathscr{B}(X)$ be the Borel σ-algebra. We consider the canonical mappings defined for $i = 1, \ldots, n$ by

$$\pi^i : X_1 \times \cdots \times X_n \to X_i, \qquad (x_1, \ldots, x_n) \mapsto x_i.$$

Similarly, for $i, j = 1, \ldots, n$ we define

$$\pi^{i,j} : X_1 \times \cdots \times X_n \to X_i \times X_j, \qquad (x_1, \ldots, x_n) \mapsto (x_i, x_j).$$

It will also be useful to interpolate these maps, for instance, for $t \in \mathbb{R}$ we set

$$\pi_t^{i,j\to k} : X^n \to X^2, \qquad (x_1, \ldots, x_n) \mapsto (x_i, (1-t)x_j + tx_k).$$

For $\mu \in \mathscr{P}(X)$ and a Borel mapping $T : X \to Y$ we let $T_\# \mu \in \mathscr{P}(Y)$ denote the *push-forward measure* defined by

$$T_\# \mu(A) := \mu(T^{-1}(A)), \qquad A \in \mathscr{B}(Y).$$

For $\mu \in \mathscr{P}(X)$ and $\nu \in \mathscr{P}(Y)$ we let $\Gamma(\mu, \nu)$ denote the set of all *transport plans*, i.e., all $\Sigma \in \mathscr{P}(X \times Y)$ satisfying $\pi^1_\# \Sigma = \mu$ and $\pi^2_\# \Sigma = \nu$. We remark that $\Gamma(\mu, \nu)$ is non-empty, since it contains the product measure $\mu \otimes \nu$. Similar notation will be used for 3-fold product spaces.

The following result will be used frequently.

Theorem 6.3 (Disintegration). *Suppose that X, Y are complete. Take $\nu \in \mathscr{P}(Y)$, let $\pi : Y \to X$ be a Borel map, and set $\mu := \pi_\# \nu \in \mathscr{P}(X)$. There exists a μ-a.e. uniquely determined family of probability measures $(\nu_x)_{x\in X} \subseteq \mathscr{P}(Y)$ such that*

(i) $\nu_x(Y \setminus \pi^{-1}(\{x\})) = 0$ *for* μ*-a.e.* $x \in X$*;*
(ii) $x \mapsto \nu_x(B) : X \to \mathbb{R}$ *is Borel measurable for every* $B \in \mathscr{B}(Y)$*;*
(iii) For every Borel map $f : Y \to [0, \infty]$ *we have*

$$\int_Y f(y)\, d\nu(y) = \int_X \left(\int_{\pi^{-1}(\{x\})} f(y)\, d\nu_x(y) \right) d\mu(x).$$

Proof. See [55, Section 10.2]. □

In the situation described in the theorem we will write

$$\nu = \int_X \nu_x \, d\mu(x).$$

A useful consequence of this result is the so-called gluing lemma.

Corollary 6.4 (Gluing Lemma). *Suppose that* X_1, X_2, X_3 *are complete. Let* $\Sigma_{12} \in \mathscr{P}(X_1 \times X_2)$ *and* $\Sigma_{13} \in \mathscr{P}(X_1 \times X_3)$ *be such that* $\pi^1_\# \Sigma_{12} = \pi^1_\# \Sigma_{13}$*. Then there exists a probability measure* $\Xi \in \Gamma(X_1 \times X_2 \times X_3)$ *satisfying*

$$\pi^{1,2}_\# \Xi = \Sigma_{12}, \qquad \pi^{1,3}_\# \Xi = \Sigma_{13}.$$

Proof. See [165, Lemma 7.6]. □

6.2 The setup

In this section we introduce the setup in which we will work throughout this chapter:

- E is a real separable Banach space,
- H is a real separable Hilbert space,
- $i : H \hookrightarrow E$ is a continuous embedding.

We will write

$$Q := ii^* \in \mathcal{L}(E^*, E).$$

Frequently we omit the embedding i and identify H with its image under i. For instance, we set

$$|x|_H := \begin{cases} |h|_H, & x = ih,\ h \in H, \\ \infty, & x \in E \setminus iH. \end{cases}$$

A special case of this framework is the Wiener space setting, which has been studied in [59]. This setting is obtained when the operator Q is the covariance operator of a Gaussian measure on E. Here we do *not* make this assumption.

Lemma 6.5. *The mapping $x \mapsto |x|_H$ is lower semicontinuous on E.*

Proof. Let $(x_n)_{n\geq 1} \subseteq E$ be a sequence satisfying $|x_n - x|_E \to 0$ and $|x_n|_H \to \alpha$ for some $\alpha \geq 0$. We have to show that $|x|_H \leq \alpha$. Without loss of generality we may assume that $|x_n|_H < \infty$ for every $n \geq 1$, hence $x_n = ih_n$ for some $h_n \in H$. Since H is reflexive there exists a subsequence, again denoted by $(h_n)_{n\geq 1}$, converging weakly to some $h \in H$. For every $x^* \in E^*$ we have

$$\langle x, x^* \rangle = \lim_{n\to\infty} \langle x_n, x^* \rangle = \lim_{n\to\infty} [h_n, i^* x^*]_H = [h, i^* x^*]_H = \langle ih, x^* \rangle,$$

which implies that $x = ih$. Since

$$|h|_H^2 = \lim_{n\to\infty} [h_n, h]_H \leq \lim_{n\to\infty} |h_n|_H \, |h|_H = \alpha |h|_H,$$

we infer that $|x|_H = |h|_H \leq \alpha$. □

For $1 \leq p < \infty$ and $\mu, \nu \in \mathscr{P}(E)$ we consider the *p-Wasserstein distance* defined by

$$W_{p,H}(\mu,\nu) := \inf \left\{ \left(\int_{E\times E} |x-y|_H^p \, d\Sigma(x,y) \right)^{1/p} : \Sigma \in \Gamma(\mu,\nu) \right\} \in [0,\infty]. \tag{6.1}$$

The collection of all $\Sigma \in \Gamma(\mu,\nu)$ for which the infimum in (6.1) is attained will be denoted by $\Gamma_o(\mu,\nu)$. Elements of $\Gamma_o(\mu,\nu)$ are called *optimal transport plans*. Clearly, $\Gamma_o(\mu,\nu)$ depends on p, but we will suppress this dependence in the notation, since it will always be clear from the context which value of p is under consideration. The next proposition shows that $\Gamma_o(\mu,\nu)$ is always non-empty.

Proposition 6.6. *Let $1 \leq p < \infty$. For all $\mu, \nu \in \mathscr{P}(E)$ we have $\Gamma_o(\mu,\nu) \neq \varnothing$.*

Proof. If $W_{p,H}(\mu,\nu) = \infty$, then trivially $\mu \otimes \nu \in \Gamma_o(\mu,\nu)$.

If $W_{p,H}(\mu,\nu) < \infty$, it follows from Lemma 6.5 combined with Lemma 6.2 that the function

$$\Phi : \Gamma(\mu,\nu) \to [0,\infty], \qquad \Sigma \mapsto \int_{E\times E} |x-y|_H^p \, d\Sigma$$

is lsc with respect to weak convergence in $\mathscr{P}(E \times E)$. Since $\Gamma(\mu,\nu)$ is weakly closed and tight, hence weakly compact, Φ attains its minimum on $\Gamma(\mu,\nu)$. □

The following result is a special case of the Kantorovich duality theorem.

Theorem 6.7. *Let $1 \leq p < \infty$. For $\mu, \nu \in \mathscr{P}(E)$ we have the duality*

$$W_{p,H}^p(\mu,\nu) = \sup \left(\int_E \phi(y) \, d\nu(y) - \int_E \psi(x) \, d\mu(x) \right),$$

where the supremum ranges over all pairs $(\psi, \phi) \in L^1(\mu) \times L^1(\nu)$ *satisfying*

$$\phi(y) - \psi(x) \leq |x - y|_H^p$$

for all $x, y \in E$. *The result remains true if we replace to* $L^1(\mu) \times L^1(\nu)$ *by* $C_{\mathrm{b}}(E) \times C_{\mathrm{b}}(E)$.

Proof. See [166, Theorem 5.10]. □

For $\mu \in \mathscr{P}(E)$ we set

$$\mathscr{P}_{p,H,\mu}(E) := \{\nu \in \mathscr{P}(E) : W_{p,H}(\mu, \nu) < \infty\}.$$

If $p = 2$ we simplify the notation and write

$$W_H := W_{2,H}, \qquad \mathscr{P}_{H,\mu}(E) := \mathscr{P}_{2,H,\mu}(E).$$

If $H = E$ is finite dimensional, W_H defines a (finite) metric on the set of Borel probability measures with finite second moment. In the infinite dimensional setting, as has been pointed out in [59], only considering measures satisfying $\int_E |x|_H^2 \, d\mu(x) < \infty$ would be severely restrictive. Therefore we will work on the full space $\mathscr{P}(E)$ and have to live with the fact that W_H "often" attains the value infinity.

In the following proposition we show that $(\mathscr{P}(E), W_{p,H})$ is a complete pseudo-metric space (in the sense that the distance may attain the value ∞) for $1 \leq p < \infty$.

Proposition 6.8. *Let* $1 \leq p < \infty$. *The following assertions hold.*

(i) If $\mu, \nu \in \mathscr{P}(E)$ *satisfy* $W_{p,H}(\mu, \nu) = 0$, *then* $\mu = \nu$.
(ii) $W_H(\mu^1, \mu^3) \leq W_H(\mu^1, \mu^2) + W_H(\mu^2, \mu^3)$ *for all* $\mu^j \in \mathscr{P}(E)$, $j = 1, 2, 3$.
(iii) The space $(\mathscr{P}(E), W_{p,H})$ *is complete.*

Proof. (i): Take $\Sigma \in \Gamma_o(\mu, \nu)$. Since $\int_{E \times E} |x - y|^p \, d\Sigma(x, y) = 0$, we have $\Sigma(\{(x, x) : x \in A\}) = 1$, hence $\mu(A) = \Sigma(\{(x, x) : x \in E\}) = \nu(A)$ for any $A \in \mathscr{B}(E)$.

(ii): To avoid trivialities we assume that the right-hand side is finite. By the gluing lemma (Corollary 6.4) there exists $\Sigma \in \Gamma(\mu^1, \mu^2, \mu^3)$ such that $\pi_\#^{1,2}\Sigma \in \Gamma_o(\mu^1, \mu^2)$ and $\pi_\#^{2,3}\Sigma \in \Gamma_o(\mu^2, \mu^3)$. Therefore

$$\begin{aligned} W_H(\mu^1, \mu^3) &\leq \|\pi^1 - \pi^3\|_{L^p(\Sigma; H)} \\ &\leq \|\pi^1 - \pi^2\|_{L^p(\Sigma; H)} + \|\pi^2 - \pi^3\|_{L^p(\Sigma; H)} \\ &= W_H(\mu^1, \mu^2) + W_H(\mu^2, \mu^3). \end{aligned}$$

(iii): We follow [4, Proposition 7.1.5] and suppose that $(\mu^n)_{n \geq 1}$ is a Cauchy sequence with respect to $W_{p,H}$. We may assume that

$$\sum_{n=1}^{\infty} W_{p,H}(\mu^n, \mu^{n+1}) < \infty.$$

By the countable version of the gluing lemma [4, Lemma 5.3.4] there exists $\Sigma \in \mathscr{P}(E^{\mathbb{N}})$ satisfying $\pi^{n,n+1}_{\#}\Sigma \in \Gamma_o(\mu^n, \mu^{n+1})$ for any $n \geq 1$. Therefore

$$\sum_{n=1}^{\infty} \|\pi^n - \pi^{n+1}\|_{L^p(\Sigma;H)} = \sum_{n=1}^{\infty} W_{p,H}(\mu^n, \mu^{n+1}) < \infty,$$

from which we infer that $(\pi^n)_{n\geq 1}$ is a Cauchy sequence in $L^p(\Sigma; H)$ converging to a limit π^∞. Writing $\mu^\infty := \pi^\infty_{\#}\Sigma$ we obtain

$$W_{p,H}(\mu^n, \mu^\infty) \leq \|\pi^n - \pi^\infty\|_{L^p(\Sigma;H)},$$

which converges to 0. □

6.3 Topological properties

In the next result we let $W_{p,E}$ denote the Wasserstein distance defined by replacing the norm of H by that of E in (6.1).

Lemma 6.9. *Let $1 \leq p < \infty$ and let $(\mu^n)_{n\geq 1}$ be a sequence in $\mathscr{P}(E)$ which is Cauchy with respect to $W_{p,E}$. Then $(\mu^n)_{n\geq 1}$ is tight.*

Proof. See [166, Lemma 6.14] for a proof under the additional condition that all measures have finite p^{th} moment. A close inspection of the proof shows that the same argument works without this assumption. □

Proposition 6.10. *Let $1 \leq p < \infty$ and suppose that $\mu, \mu^n \in \mathscr{P}(E)$ satisfy*

$$W_{p,H}(\mu^n, \mu) \to 0.$$

Then μ^n converges weakly to μ.

Proof. We may assume that $W_{p,H}(\mu^n, \mu) < \infty$ for all $n \geq 1$. Consider an arbitrary subsequence of $(\mu^n)_{n\geq 1}$, which we denote again by $(\mu^n)_{n\geq 1}$. To prove the result, it suffices to prove that this subsequence contains a subsequence converging weakly to μ.

Since

$$W_{p,E}(\mu^n, \mu) \leq \|i\|_{\mathcal{L}(H,E)} W_{p,H}(\mu^n, \mu),$$

Lemma 6.9 implies that there exists a subsequence $(\mu^{n_k})_{k\geq 1}$ converging weakly to some $\tilde{\mu} \in \mathscr{P}(E)$. Take $\Sigma_{n_k} \in \Gamma_o(\mu^{n_k}, \mu)$. The tightness of $(\mu^n)_{n\geq 1}$ implies that $(\Sigma^{n_k})_{k\geq 1}$ is tight, hence there exists a subsequence $\Sigma^{n_{k_l}}$ converging weakly to some $\Sigma \in \mathscr{P}(E \times E)$. We claim that $\Sigma \in \Gamma(\tilde{\mu}, \mu)$. Indeed, for $f \in C_{\mathrm{b}}(E)$,

$$\begin{aligned}\int_{E\times E} f(x)\,d\Sigma(x,y) &= \lim_{l\to\infty}\int_{E\times E} f(x)\,d\Sigma^{n_{k_l}}(x,y)\\ &= \lim_{l\to\infty}\int_{E\times E} f(x)\,d\mu^{n_{k_l}}(x) = \int_E f(x)\,d\tilde{\mu}(x),\end{aligned}$$

which shows that $\pi^1_\# \Sigma = \tilde{\mu}$. Similarly, $\pi^2_\# \Sigma = \mu$. Lemma 6.2 implies that

$$\begin{aligned}W^p_{p,H}(\tilde{\mu},\mu) &\leq \int_{E\times E} |x-y|^p_H\,d\Sigma(x,y)\\ &\leq \varliminf_{l\to\infty}\int_{E\times E} |x-y|^p_H\,d\Sigma^{n_{k_l}}(x,y) = \varliminf_{l\to\infty} W^p_{p,H}(\mu^{n_{k_l}},\mu) = 0,\end{aligned}$$

hence $\tilde{\mu} = \mu$ by Proposition 6.8. This completes the proof. □

Before we stating the next proposition, we introduce some notation which will be used throughout Part II. Since E is separable, there exists a separating sequence of functionals $(y_k^*)_{k\geq 1} \subseteq E^*$. Applying the Gram-Schmidt procedure in H to the sequence $(i^* y_k^*)_{k\geq 1}$ we obtain an orthonormal basis

$$(i^* x_k^*)_{k\geq 1} \tag{6.2}$$

of H which we will keep fixed from now on. We let $\mathsf{P}_n \in \mathcal{L}(H)$ denote the corresponding orthogonal projections given by

$$\mathsf{P}_n h := \sum_{k=1}^n [h, i^* x_k^*]_H\, i^* x_k^*, \qquad h \in H.$$

Each of these operators is the part in H of a bounded operator on E, denoted by the same symbol, and given by

$$\mathsf{P}_n x := \sum_{k=1}^n \langle x, x_k^*\rangle Q x_k^*, \qquad x \in E. \tag{6.3}$$

The next result is an easy generalisation of a well-known result in the theory of Wasserstein spaces (see, e.g., [165, Theorem 7.12]), where usually ν is taken to be a Dirac measure.

Proposition 6.11. *Let $\mu^n, \mu \in \mathscr{P}(E)$ satisfy $W_H(\mu^n,\mu) \to 0$, and take $\nu \in \mathscr{P}_{H,\mu}(E)$. For every weakly convergent sequence $\Sigma_n \in \Gamma_o(\mu^n,\nu)$ we have*

$$\lim_{M\to\infty}\varlimsup_{n\to\infty}\int_{\{|x_1-x_2|_H\geq M\}} |x_1-x_2|^2_H\,d\Sigma^n(x_1,x_2) = 0.$$

Proof. We will denote the weak limit of $(\Sigma^n)_{n\geq 1}$ by Σ. To simplify notation, set $d(x_1,x_2) := |x_1-x_2|_H$ and $d_M(x_1,x_2) := |\mathsf{P}_{\lfloor M\rfloor}(x_1-x_2)|_H \wedge M$. Since $\Sigma^n \rightharpoonup \Sigma$ and $d_M \in C_{\mathrm{b}}(E\times E)$,

$$\lim_{n\to\infty}\int_{E\times E} d_M^2(x_1,x_2)\,d\Sigma^n = \int_{E\times E} d_M^2(x_1,x_2)\,d\Sigma. \tag{6.4}$$

It is clear that $\Sigma \in \Gamma(\mu,\nu)$. We claim that $\Sigma \in \Gamma_o(\mu,\nu)$. Indeed, using Lemma 6.2 we obtain

$$\begin{aligned} W_H^2(\mu,\nu) &\le \int_{E\times E} d^2(x_1,x_2)\,d\Sigma \le \varliminf_{n\to\infty}\int_{E\times E} d^2(x_1,x_2)\,d\Sigma^n \\ &= \varliminf_{n\to\infty} W_H^2(\mu^n,\nu) = W_H^2(\mu,\nu). \end{aligned}$$

It follows that the first inequality above is actually an equality, which proves the claim. Since $W_H(\mu^n,\nu)\to W_H(\mu,\nu)$, we have

$$\lim_{n\to\infty}\int_{E\times E} d^2(x_1,x_2)\,d\Sigma^n = \int_{E\times E} d^2(x_1,x_2)\,d\Sigma. \tag{6.5}$$

Combining (6.4) and (6.5) we arrive at

$$\lim_{n\to\infty}\int_{E\times E} d^2(x_1,x_2) - d_M^2(x_1,x_2)\,d\Sigma^n = \int_{E\times E} d^2(x_1,x_2) - d_M^2(x_1,x_2)\,d\Sigma.$$

The monotone convergence theorem implies that

$$\lim_{M\to\infty}\lim_{n\to\infty}\int_{E\times E} d^2(x_1,x_2) - d_M^2(x_1,x_2)\,d\Sigma^n = 0.$$

Using the fact that for $x_1,x_2\in E$ with $d(x_1,x_2)\ge 2M$,

$$d^2(x_1,x_2) - d_M^2(x_1,x_2) \ge d^2(x_1,x_2) - M^2 \ge \frac{3}{4}d^2(x_1,x_2),$$

it follows that

$$\lim_{M\to\infty}\varlimsup_{n\to\infty}\int_{\{d(x_1,x_2)\ge 2M\}} d^2(x_1,x_2)\,d\Sigma^n = 0,$$

which proves the result. □

In some results we will impose the following additional assumption:

(H) The embedding $i : H \hookrightarrow E$ is compact.

This assumption is automatically fulfilled if $Q = ii^*$ is the covariance of a Gaussian measure. It allows us to prove the following compactness result.

Proposition 6.12. *Assume (H). For each $\mu \in \mathscr{P}(E)$ and $R \ge 0$ the set*

$$\mathscr{B}_R(\mu) := \{\nu \in \mathscr{P}(E) : W_H(\mu,\nu) \le R\}$$

is weakly compact. In particular, the set

$$\{\nu \in \mathscr{P}(E) : \int_E |x|_H^2\,d\nu(x) \le R^2\}$$

is weakly compact.

Proof. Let $(\nu_n)_{n\geq 1} \subseteq \mathscr{B}_R(\mu)$ and let $\Sigma_n \in \Gamma_o(\mu, \nu_n)$. Let $\varepsilon > 0$ and choose a compact set $K \subseteq E$ such that $\mu(E \setminus K) < \varepsilon$. By assumption, the set $B_r(0) := \{x \in E : |x|_H \leq r\}$ is compact for each $r > 0$. Since the Minkowski sum of compact sets is again compact, the set $K_r := K + B_r(0)$ is compact as well. We obtain

$$
\begin{aligned}
\nu_n(E \setminus K_r) &= \int_{E\times E} \mathbf{1}_{E\setminus K_r}(y)\, d\Sigma_n(x,y) \\
&= \int_{E\times E} \mathbf{1}_K(x)\mathbf{1}_{E\setminus K_r}(y)\, d\Sigma_n(x,y) \\
&\qquad + \int_{E\times E} \mathbf{1}_{E\setminus K}(x)\mathbf{1}_{E\setminus K_r}(y)\, d\Sigma_n(x,y) \\
&\leq \int_{E\times E} \mathbf{1}_K(x)\mathbf{1}_{E\setminus K_r}(y) \frac{|x-y|_H^2}{r^2}\, d\Sigma_n(x,y) + \mu(E \setminus K) \\
&\leq \frac{1}{r^2} W_H^2(\mu, \nu_n) + \varepsilon \\
&\leq 2\varepsilon,
\end{aligned}
$$

whenever r is large enough. This shows the tightness of $\mathscr{B}_R(\mu)$. It follows that Σ_n is tight as well.

By passing to a subsequence we may assume Σ_n converges weakly to some $\Sigma \in \Gamma(\mu, \nu)$. In view of Lemma 6.2 we obtain

$$
\begin{aligned}
W_H^2(\mu, \nu) &\leq \int_{E\times E} |x-y|_H^2\, d\Sigma(x,y) \\
&\leq \varliminf_{n\to\infty} \int_{E\times E} |x-y|_H^2\, d\Sigma_n(x,y) = \varliminf_{n\to\infty} W_H^2(\mu, \nu_n) \leq R,
\end{aligned}
$$

which shows that $\nu \in \mathscr{B}_R(\mu)$. We conclude that $\mathscr{B}_R(\mu)$ is weakly compact.

The final assertion follows by observing that the set under consideration can be written as $\mathscr{B}_R(\delta_0)$. □

6.4 Convergence of the inner product

In this section we prove a technical result concerning the limit behaviour of expressions of the form $\int_{E\times E} [x_1, x_2]_H \, d\Sigma_n$ for some weakly convergent sequence of measures $(\Sigma_n)_{n\geq 1}$. This is not straightforward, since the function $(x_1, x_2) \mapsto [x_1, x_2]_H$ is not lower semicontinuous on $E \times E$.

In this expression, for $x_1, x_2 \in E$ we set $[x_1, x_2]_H := \infty$ whenever at least one of the elements x_1, x_2 is not contained in H. In most occurrences below this convention is actually irrelevant, as the inner product appears under an integral with respect to a measure $\Sigma \in \mathscr{P}(E \times E)$ satisfying $\Sigma(H \times H) = 1$. First we need a lemma.

Lemma 6.13. *Let $(\mu_n)_{n\geq 1} \subseteq \mathscr{P}(E)$, let $f, r : [0,\infty] \to [0,\infty]$, and suppose that $\lim_{t\to\infty} r(t) = 0$. Then $g(t) := r(t)f(t)$ satisfies*

$$C := \sup_{n\geq 1} \int_E f(|x|_H)\, d\mu_n < \infty \quad \Rightarrow \quad \lim_{m\to\infty} \sup_{n\geq 1} \int_{\{|x|_H \geq m\}} g(|x|_H)\, d\mu_n = 0.$$

In particular, for every $0 < p' < p$,

$$\sup_{n\geq 1} \int_E |x|_H^p \, d\mu_n < \infty \qquad \Rightarrow \qquad \lim_{m\to\infty} \sup_{n\geq 1} \int_{\{|x|_H \geq m\}} |x|_H^{p'}\, d\mu_n = 0$$

Proof. Clearly,

$$\int_{\{|x|_H \geq m\}} g(|x|_H)\, d\mu_n \leq \Big(\sup_{|x|\geq m} r(|x|_H) \Big) \int_{\{|x|_H\geq m\}} f(|x|_H)\, d\mu_n,$$

from which we obtain

$$\sup_{n\geq 1} \int_{\{|x|_H \geq m\}} g(|x|_H)\, d\mu_n \leq C \sup_{t\geq m} r(t).$$

The first part of the result follows by passing to the limit $m \to \infty$.

The second part follows by taking $f(t) := t^p$ and $r(t) = t^{p'-p}$. □

A variant of the next result is proved in the Wiener space setting in [150, Lemma 3.9]. See also [4, Lemma 5.2.4] for a Hilbert space version.

Proposition 6.14. *Let $\Sigma, \Sigma_n \in \mathscr{P}(E\times E)$ and suppose that Σ_n converges weakly to Σ. If*

$$\begin{aligned} \lim_{n\to\infty} \int_{E\times E} |x_1|_H^2 \, d\Sigma_n &= \int_{E\times E} |x_1|_H^2\, d\Sigma < \infty, \\ \lim_{R\to\infty} \overline{\lim_{n\to\infty}} \int_{\{|x_1|_H \geq R\}} |x_1|_H^2 \, d\Sigma_n &= 0, \end{aligned} \tag{6.6}$$

and

$$\sup_{n\geq 1} \int_E |x_2|_H^2 \, d\Sigma_n < \infty, \tag{6.7}$$

then

$$\lim_{n\to\infty} \int_{E\times E} [x_1, x_2]_H \, d\Sigma_n = \int_{E\times E} [x_1,x_2]_H\, d\Sigma.$$

Proof. We set

$$\mu := \pi^1_\# \Sigma, \qquad \nu := \pi^2_\# \Sigma, \qquad \mu_n := \pi^1_\# \Sigma_n, \qquad \nu_n := \pi^2_\# \Sigma_n,$$

and divide the proof into three steps.

Step 1: We show that the function $g(x_1, x_2) := |x_1|_H |x_2|_H$ satisfies

$$\lim_{R\to\infty} \varlimsup_{n\to\infty} \int_{\{g\geq R\}} g \, d\Sigma_n = 0. \tag{6.8}$$

Set $C^2 := \sup_{n\geq 1} \int_E |x|_H^2 \, d\nu_n < \infty$ by (6.7). For $n \geq 1$ we have

$$\int_A g \, d\Sigma_n \leq C \bigg(\int_A |x|_H^2 \, d\Sigma_n \bigg)^{1/2}, \qquad A \in \mathscr{B}(E \times E),$$

thus for $R \geq 0$ and $m > 0$ we obtain

$$\begin{aligned}
\int_{\{g\geq R\}} g \, d\Sigma_n &= \int_{\{g\geq R, |x_1|_H \leq m\}} g \, d\Sigma_n + \int_{\{g\geq R, |x_1|_H > m\}} g \, d\Sigma_n \\
&\leq \int_{\{|x_1|_H \leq m, |x_2|_H \geq \frac{R}{m}\}} g \, d\Sigma_n + \int_{\{g\geq R, |x_1|_H > m\}} g \, d\Sigma_n \\
&\leq m \int_{\{|x|_H \geq \frac{R}{m}\}} |x|_H \, d\nu_n + C \bigg(\int_{\{|x|_H > m\}} |x|_H^2 \, d\mu_n \bigg)^{1/2}.
\end{aligned}$$

Taking first the limes superior for $n \to \infty$, and then the limit for $R \to \infty$, we obtain using Lemma 6.13 and (6.7),

$$\lim_{R\to\infty} \varlimsup_{n\to\infty} \int_{\{g\geq R\}} g \, d\Sigma_n \leq C \varlimsup_{n\to\infty} \bigg(\int_{\{|x|_H > m\}} |x|_H^2 \, d\mu_n \bigg)^{1/2}.$$

Passing to the limit $m \to \infty$ and using (6.6) we obtain (6.8).

Step 2: For fixed $m \geq 1$ we define $f_m \in C(E \times E)$ by $f_m(x_1, x_2) := [\mathsf{P}_m x_1, \mathsf{P}_m x_2]_H$, and claim that

$$\lim_{n\to\infty} \int_{E\times E} f_m \, d\Sigma_n = \int_{E\times E} f_m \, d\Sigma.$$

For each integer $R > 0$, take a continuous function $\psi_R : [0, \infty) \to [0, 1]$ with $\psi_R([0, R]) = \{1\}$ and $\psi_R([R + 1, \infty)) = \{0\}$. We define $g_m \in C(E)$ by $g_m(x_1, x_2) := |\mathsf{P}_m x_1|_H |\mathsf{P}_m x_2|_H$ and write

$$f_m = f_m \cdot (\psi_R \circ g_m) + f_m \cdot (1 - \psi_R \circ g_m). \tag{6.9}$$

and analyse both terms separately.

Note that g is integrable with respect to Σ, since Lemma 6.2 and the assumptions on μ_n and ν_n imply that

$$\begin{aligned}
\int_{E\times E} g \, d\Sigma &\leq \bigg(\int_{E\times E} |x_1|_H^2 \, d\Sigma \bigg)^{1/2} \bigg(\int_{E\times E} |x_2|_H^2 \, d\Sigma \bigg)^{1/2} \\
&= \bigg(\int_E |x|_H^2 \, d\mu \bigg)^{1/2} \bigg(\int_E |x|_H^2 \, d\nu \bigg)^{1/2} \\
&\leq \varliminf_{n\to\infty} \bigg(\int_E |x|_H^2 \, d\mu_n \bigg)^{1/2} \varliminf_{n\to\infty} \bigg(\int_E |x|_H^2 \, d\nu_n \bigg)^{1/2} < \infty.
\end{aligned} \tag{6.10}$$

Since $|f_m| \le g$ we may use dominated convergence to obtain

$$\lim_{R\to\infty} \int_{E\times E} f_m \cdot (\psi_R \circ g_m)\, d\Sigma = \int_{E\times E} f_m \, d\Sigma. \tag{6.11}$$

To estimate the second term, we use again that $|f_m| \le g$ and find

$$\varlimsup_{n\to\infty} \int_{E\times E} |f_m| \cdot (1 - \psi_R \circ g_m)\, d\Sigma_n \le \varlimsup_{n\to\infty} \int_{\{|g|\ge R\}} g \, d\Sigma_n.$$

Passing to the limit $R \to \infty$ and using (6.8), we obtain

$$\lim_{R\to\infty} \varlimsup_{n\to\infty} \int_{E\times E} |f_m| \cdot (1 - \psi_R \circ g_m)\, d\Sigma_n \le \lim_{R\to\infty} \varlimsup_{n\to\infty} \int_{\{|g|\ge R\}} g \, d\Sigma_n = 0. \tag{6.12}$$

Since $f_m \cdot (\psi_R \circ g_m) \in C_{\mathrm{b}}(E \times E)$ and $\Sigma_n \rightharpoonup \Sigma$, we find

$$\begin{aligned}
&\varlimsup_{n\to\infty} \int_{E\times E} f_m \, d\Sigma_n \\
&\quad = \varlimsup_{n\to\infty} \left(\int_{E\times E} f_m \cdot (\psi_R \circ g_m)\, d\Sigma_n + \int_{E\times E} f_m \cdot (1 - \psi_R \circ g_m)\, d\Sigma_n \right) \\
&\quad = \int_{E\times E} f_m \cdot (\psi_R \circ g_m)\, d\Sigma + \varlimsup_{n\to\infty} \int_{E\times E} f_m \cdot (1 - \psi_R \circ g_m)\, d\Sigma_n.
\end{aligned} \tag{6.13}$$

Using (6.11) and (6.12) to pass to the limit $R \to \infty$ we arrive at

$$\varlimsup_{n\to\infty} \int_{E\times E} f_m \, d\Sigma_n = \int_{E\times E} f_m \, d\Sigma.$$

Similarly, replacing $\varlimsup$ by $\varliminf$ in (6.13), we obtain

$$\varliminf_{n\to\infty} \int_{E\times E} f_m \, d\Sigma_n = \int_{E\times E} f_m \, d\Sigma,$$

which completes the proof of Step 2.

Step 3: We complete the proof.

Using the Cauchy-Schwarz inequality, orthogonality, and (6.7), we have

$$\begin{aligned}
&\left| \int_{E\times E} [(I - \mathsf{P}_m)x_1, x_2]_H \, d\Sigma_n \right| \\
&\qquad \le \left(\int_{E\times E} |(I - \mathsf{P}_m)x_1|_H^2 \, d\Sigma_n \right)^{1/2} \left(\int_{E\times E} |x_2|_H^2 \, d\Sigma_n \right)^{1/2} \\
&\qquad \le C \left(\int_{E\times E} |x_1|_H^2 \, d\Sigma_n - \int_{E\times E} |\mathsf{P}_m x_1|_H^2 \, d\Sigma_n \right)^{1/2}.
\end{aligned}$$

Using (6.6) and the continuity of $x \mapsto |\mathsf{P}_m x|_H^2$ combined with Lemma 6.2, we arrive at

$$\overline{\lim_{n\to\infty}} \Big| \int_{E\times E} [(I-\mathsf{P}_m)x_1, x_2]_H \, d\Sigma_n \Big|$$
$$\le C\Big(\int_{E\times E} |x_1|_H^2 \, d\Sigma - \int_{E\times E} |\mathsf{P}_m x_1|_H^2 \, d\Sigma \Big)^{1/2}.$$

Passing to the limit $m\to\infty$, and using the dominated convergence theorem,

$$\lim_{m\to\infty} \overline{\lim_{n\to\infty}} \Big| \int_{E\times E} [(I-\mathsf{P}_m)x_1, x_2]_H \, d\Sigma_n \Big| = 0.$$

Passing to the limit $n\to\infty$ and then $m\to\infty$ in the identity

$$\int_{E\times E} [x_1, x_2]_H \, d\Sigma_n = \int_{E\times E} [\mathsf{P}_m x_1, \mathsf{P}_m x_2]_H \, d\Sigma_n + \int_{E\times E} [(I-\mathsf{P}_m)x_1, x_2]_H \, d\Sigma_n,$$

we obtain using Step 2 and dominated convergence (which can be applied since $|f|\le g$ and g is integrable with respect to Σ by (6.10)),

$$\begin{aligned}
\overline{\lim_{n\to\infty}} \int_{E\times E} [x_1, x_2]_H \, d\Sigma_n &= \lim_{m\to\infty} \overline{\lim_{n\to\infty}} \int_{E\times E} [\mathsf{P}_m x_1, \mathsf{P}_m x_2]_H \, d\Sigma_n \\
&= \lim_{m\to\infty} \int_{E\times E} [\mathsf{P}_m x_1, \mathsf{P}_m x_2]_H \, d\Sigma \qquad (6.14)\\
&= \int_{E\times E} [x_1, x_2]_H \, d\Sigma.
\end{aligned}$$

Replacing $\overline{\lim}$ by $\underline{\lim}$ in (6.14) we obtain

$$\underline{\lim_{n\to\infty}} \int_{E\times E} [x_1, x_2]_H \, d\Sigma_n = \int_{E\times E} [x_1, x_2]_H \, d\Sigma, \qquad (6.15)$$

which completes the proof. $\square$

7

Paths

In this chapter we study paths of probability measures on E which are absolutely continuous with respect to the Wasserstein metric W_H. It will be shown that a continuity equation can be associated with such paths. Furthermore we introduce tangent spaces for measures $\mu \in \mathscr{P}(E)$ and velocity fields associated with smooth paths. These objects will play an important role in the study of gradient flows in $(\mathscr{P}(E), W_H)$.

7.1 Absolute continuity in metric spaces

We start by recalling some general facts concerning absolute continuity in metric spaces. More information on this topic can be found in [4].

Let (X, d) be a complete metric space, let $J \subseteq \mathbb{R}$ be an interval, and let $1 \leq p \leq \infty$. We say that a function $u : J \to X$ belongs to $AC^p(J; X)$ if there exists $m \in L^p(J)$ such that, for any $s, t \in J$ with $s < t$,

$$d(u(s), u(t)) \leq \int_s^t m(r)\, dr. \tag{7.1}$$

If $p = 1$ we simply write $AC(J; X)$ and say that u is *absolutely continuous.*

We say that $u : J \to X$ is contained in $AC^p_{\mathrm{loc}}(J; X)$ if the restriction of u to each compact subinterval $J' \subseteq J$ is contained in $AC^p(J; X)$.

Theorem 7.1. *If $u \in AC(J; X)$, then the* metric derivative

$$|u'|(t) := \lim_{h \to 0} \frac{d(u(t+h), u(t))}{|h|}$$

exists a.e. on J, we have $|u'| \in L^1(J)$, and

$$d(u(s), u(t)) \leq \int_s^t |u'|(r)\, dr$$

for any $s, t \in J$ with $s < t$. Moreover, for any m satisfying (7.1) *we have* $|u'|(t) \le m(t)$ *a.e. on J.*

Proof. See [4, Theorem 1.1.2]. □

The results summarised above will be applied to the Wasserstein space $(\mathscr{P}(E), W_H)$. The fact that W_H is a pseudo-distance attaining the value $+\infty$ will not cause problems, since we may apply the theory in each component $\mathscr{P}_{H,\mu}(E)$ for $\mu \in \mathscr{P}(E)$.

7.2 Absolutely continuous paths of probability measures

We return to the setting described in Section 6.2:

- E is a real separable Banach space,
- H is a real separable Hilbert space,
- $i : H \hookrightarrow E$ is a continuous embedding.

Unless indicated otherwise, we will always endow $\mathscr{P}(E)$ with the pseudo-metric W_H.

Let $\mathcal{C}$ be the vector space consisting of all real-valued functions on E which can be written in the form

$$f(x) = \varphi(\langle \cdot, x_1^* \rangle, \dots, \langle \cdot, x_n^* \rangle), \qquad x \in E, \tag{7.2}$$

for some $n \ge 1$ and $\varphi \in C_{\mathrm{b}}^\infty(\mathbb{R}^n)$. Recall that the functionals $(x_k^*)_{k\ge1}$ appearing in this expression have been defined in (6.2). For $f \in \mathcal{C}$ of the form (7.2) we define its gradient $\nabla_H f : E \to H$ by

$$\nabla_H f(x) := \sum_{k=1}^n \partial_k \phi(\langle x, x_1^* \rangle, \dots, \langle x, x_n^* \rangle) i^* x_k^*, \qquad x \in E.$$

For $\mu \in \mathscr{P}(E)$ we consider the *tangent space*

$$T_\mu^H := \overline{\{\nabla_H f : f \in \mathcal{C}\}} \subseteq L^2(\mu; H),$$

where the closure is taken in $L^2(\mu; H)$. An explanation for this terminology will be given by Theorem 7.4 below.

Let $J \subseteq \mathbb{R}$ be an interval. For a weakly continuous path $(\mu_t)_{t\ge0} \subseteq \mathscr{P}(E)$ we let M_μ denote the Borel measure on the product $J \times E$ whose disintegration is given by

$$M_\mu := \int_J \mu_t \, dt.$$

The following result shows that a *continuity equation* can be associated with smooth paths of probability measures on E. The proof proceeds along the lines of [4, Theorem 8.3.1] and [59, Theorem 2.3].

Theorem 7.2. *Let $J \subseteq \mathbb{R}$ be an interval, and let $(\mu_t)_{t\in J} \in AC^2(J;\mathscr{P}(E))$. Then there exists a unique $Z \in L^2(M_\mu; H)$ such that $Z_t \in T^H_{\mu_t}$ a.e. and*

$$\partial_t \mu_t + \nabla_H \cdot (Z_t \mu_t) = 0$$

in the following sense: for all $\alpha \in C_{\mathrm{c}}^\infty(J)$ and $f \in \mathcal{C}$,

$$\int_J \int_E \Big(\alpha'(t) f(x) + \alpha(t)[\nabla_H f(x), Z_t(x)]_H\Big)\, d\mu_t(x)\, dt = 0. \tag{7.3}$$

Moreover, for a.e. $t \in J$ we have

$$\int_J \int_E |Z_t(x)|_H^2 \, d\mu_t(x)\, dt \le \int_J |\mu'|^2(t)\, dt,$$

where $|\mu'|$ denotes the metric derivative.

In this case we say that $Z = (Z_t)_{t\in J}$ is the *velocity field* along the path $\mu = (\mu_t)_{t\in J}$.

Proof. Let $\overline{V}$ be the closure in $L^2(M_\mu; H)$ of the space

$$V := \Big\{(t,x) \mapsto \sum_{i=1}^n \alpha_i(t) \nabla_H f_i(x) : n \ge 1, \alpha_i \in C_{\mathrm{c}}^\infty(J), f_i \in \mathcal{C}\Big\}.$$

The idea of the proof is to show that the linear functional

$$\ell : \sum_{i=1}^n \alpha_i \otimes \nabla_H f_i \mapsto \sum_{i=1}^n \int_J \int_E \alpha_i'(t) f_i(x)\, d\mu_t(x)\, dt$$

is well defined and bounded on V with respect to the norm of $L^2(M_\mu; H)$. Once this fact is established, we conclude from the Riesz representation theorem that there exists a unique $Z \in \overline{V}$ such that for any $\alpha \in C_{\mathrm{c}}^\infty(J)$ and $f \in \mathcal{C}$,

$$\begin{aligned}\int_J \int_E \alpha'(t) f(x)\, d\mu_t(x)\, dt &= \ell(\alpha \otimes \nabla_H f)\\ &= \int_J \int_E \alpha(t)[\nabla_H f(x), Z_t(x)]_H \, d\mu_t(x)\, dt,\end{aligned}$$

which is the desired result. We will derive the boundedness of ℓ in three steps.

Step 1: For $s, t \in J$, we take $\Sigma_s^t \in \Gamma_o(\mu_s, \mu_t)$, and observe that for $f \in \mathcal{C}$,

$$\begin{aligned}\int_E f(x)\, d\mu_t(x) - \int_E f(x)\, d\mu_{t+\eta}(x) &= \int_{E\times E} f(x) - f(y)\, d\Sigma_t^{t+\eta}(x,y)\\ &= \int_{E\times E} [K(x,y), x-y]_H \, d\Sigma_t^{t+\eta}(x,y),\end{aligned} \tag{7.4}$$

where

$$K(x,y) := \int_0^1 \nabla_H f((1-t)x + ty)\, dt, \qquad x, y \in E. \tag{7.5}$$

For any $\alpha \in C_c^\infty(J)$ and $\eta > 0$ we obtain, after extending α by 0 outside J,

$$\begin{aligned}
&\int_J \int_E \big(\alpha(t) - \alpha(t-\eta)\big) f(x)\, d\mu_t(x)\, dt \\
&\quad = \int_J \alpha(t) \Big(\int_E f(x)\, d\mu_t(x) - \int_E f(x)\, d\mu_{t+\eta}(x) \Big)\, dt \\
&\quad = \int_J \alpha(t) \Big(\int_{E\times E} [K(x,y), x-y]_H \, d\Sigma_t^{t+\eta}(x,y) \Big)\, dt.
\end{aligned}$$

Step 2: We will prove that for $t \in J$, for $c_1, \dots, c_m \in \mathbb{R}$, and for $f_1, \dots, f_m \in \mathcal{C}$, we have

$$\overline{\lim_{\eta \to 0}} \int_{E\times E} \Big| \sum_{j=1}^m c_j K_j(x,y) \Big|_H^2 \, d\Sigma_t^{t+\eta}(x,y) \le \Big\| \sum_{j=1}^m c_j \nabla_H f_j \Big\|_{L^2(\mu_t;H)}^2,$$

where $K_j(x,y) := \int_0^1 \nabla_H f_j((1-t)x + ty)\, dt$ for $x, y \in E$ and $j = 1, \dots, m$.

Fix $t \in J$ and take a sequence $\eta_n \to 0$ for which

$$\int_{E\times E} \Big| \sum_{j=1}^m c_j K_j(x,y) \Big|_H^2 \, d\Sigma_t^{t+\eta_n}(x,y)$$

converges (possibly to $+\infty$). Since $(\mu_{\eta_n})_{n\ge1}$ is W_H-convergent, it is tight by Lemma 6.9. This implies the tightness of $(\Sigma_t^{t+\eta_n})_{n\ge1}$, hence (up to a subsequence) this sequence has a weak limit $\widehat{\Sigma} \in \Gamma(\mu_t, \mu_t)$. Using Lemma 6.2 we obtain

$$\begin{aligned}
\int_{E\times E} |x-y|_H^2 \, d\widehat{\Sigma}(x,y) &\le \underline{\lim}_{n\to\infty} \int_{E\times E} |x-y|_H^2 \, d\Sigma_t^{t+\eta_n}(x,y) \\
&= \underline{\lim}_{n\to\infty} W_H^2(\mu_t, \mu_{t+\eta_n}) \\
&= 0.
\end{aligned}$$

It follows that $\widehat{\Sigma}(D) = 1$, where $D = \{(x,x) \in E \times E : x \in E\}$. Combined with the fact that $\widehat{\Sigma} \in \Gamma(\mu_t, \mu_t)$ this implies that $\widehat{\Sigma} = (I \times I)_\# \mu_t$. Since $(\Sigma_t^{t+\eta_n})$ converges weakly and $|\sum_{j=1}^m c_j K_j|^2 \in C_b(E \times E)$, we find

$$\begin{aligned}
\lim_{n\to\infty} \int_{E\times E} \Big| \sum_{j=1}^m c_j K_j(x,y) \Big|_H^2 \, d\Sigma_t^{t+\eta_n}(x,y) &= \int_{E\times E} \Big| \sum_{j=1}^m c_j K_j(x,y) \Big|_H^2 \, d\widehat{\Sigma}(x,y) \\
&= \int_E \Big| \sum_{j=1}^m c_j K_j(x,x) \Big|_H^2 \, d\mu_t(x) \\
&= \int_E \Big| \sum_{j=1}^m c_j \nabla_H f_j(x) \Big|_H^2 \, d\mu_t(x).
\end{aligned}$$

Step 3: We will show that ℓ is well defined, and that for any $\alpha_i \in C_c^\infty(J)$ and $f_i \in \mathcal{C}$ we have

$$\ell\Big(\sum_{i=1}^n \alpha_i \otimes \nabla_H f_i\Big) \leq \||\mu'|\|_{L^2(J)} \Big\|\sum_{i=1}^n \alpha_i \otimes \nabla_H f_i\Big\|_{\overline{V}}.$$

Using the dominated convergence theorem, Step 1, and the Cauchy-Schwarz inequality we obtain

$$\begin{aligned}
&\Big|\int_J \int_E \sum_{i=1}^n \alpha_i'(t) f_i(x)\, d\mu_t(x)\, dt\Big| \\
&\quad = \lim_{\eta\downarrow 0} \Big|\frac{1}{\eta}\int_J \int_E \sum_{i=1}^n \big(\alpha_i(t) - \alpha_i(t-\eta)\big) f_i(x)\, d\mu_t(x)\, dt\Big| \\
&\quad = \lim_{\eta\downarrow 0} \Big|\frac{1}{\eta}\int_J \int_{E\times E} \sum_{i=1}^n \alpha_i(t)[K_i(x,y), x-y]_H\, d\Sigma_t^{t+\eta}(x,y)\, dt\Big| \\
&\quad \leq \varlimsup_{\eta\downarrow 0} \Big(\int_J \int_{E\times E} \Big|\sum_{i=1}^n \alpha_i(t) K_i(x,y)\Big|_H^2\, d\Sigma_t^{t+\eta}(x,y)\, dt\Big)^{1/2} \qquad (7.6)\\
&\qquad\qquad \times \Big(\int_J \int_{E\times E} \frac{|x-y|_H^2}{\eta^2}\, d\Sigma_t^{t+\eta}(x,y)\, dt\Big)^{1/2} \\
&\quad \leq \varlimsup_{\eta\downarrow 0} \Big(\int_J \int_{E\times E} \Big|\sum_{i=1}^n \alpha_i(t) K_i(x,y)\Big|_H^2\, d\Sigma_t^{t+\eta}(x,y)\, dt\Big)^{1/2} \\
&\qquad\qquad \times \varlimsup_{\eta\downarrow 0} \Big(\int_J \int_{E\times E} \frac{|x-y|_H^2}{\eta^2}\, d\Sigma_t^{t+\eta}(x,y)\, dt\Big)^{1/2}.
\end{aligned}$$

Since $\int_E |\sum_{i=1}^n \alpha_i(t) K_i(x,y)|_H^2\, d\Sigma_t^{t+\eta}(x,y)$ is uniformly bounded for $t \in J$ and $\eta > 0$, we may apply Fatou's Lemma and Step 2 to obtain

$$\begin{aligned}
&\varlimsup_{\eta\downarrow 0}\Big(\int_J \int_{E\times E} \Big|\sum_{i=1}^n \alpha_i(t) K_i(x,y)\Big|_H^2\, d\Sigma_t^{t+\eta}(x,y)\, dt\Big)^{1/2} \\
&\quad \leq \Big(\int_J \varlimsup_{\eta\downarrow 0} \int_{E\times E} \Big|\sum_{i=1}^n \alpha_i(t) K_i(x,y)\Big|_H^2\, d\Sigma_t^{t+\eta}(x,y)\, dt\Big)^{1/2} \\
&\quad \leq \Big(\int_J \Big\|\sum_{i=1}^n \alpha_i(t) \nabla_H f_i\Big\|_{L^2(\mu_t;H)}^2\, dt\Big)^{1/2} \qquad (7.7)\\
&\quad = \Big\|\sum_{i=1}^n \alpha_i \otimes \nabla_H f_i\Big\|_{\overline{V}}.
\end{aligned}$$

For m as in (7.1), we have

$$\sup_{\eta>0} \frac{1}{\eta} W_H(\mu_t, \mu_{t+\eta}) \leq \sup_{\eta>0} \frac{1}{\eta}\int_t^{t+\eta} m(s)\, ds \leq m^*(t), \qquad \text{a.e. on } J,$$

where $m^*(t) := \sup_{\eta>0} \frac{1}{\eta}\int_t^{t+\eta} |m(s)|\,ds$ denotes the one-sided Hardy-Littlewood maximal function, which is contained in $L^2(J)$ by the maximal theorem [71, Theorem 2.1.6]. Therefore we can use the dominated convergence theorem and Theorem 7.1 to obtain

$$\begin{aligned}\lim_{\eta\downarrow 0}\int_J\int_E \frac{|x-y|_H^2}{\eta^2}\,d\Sigma_t^{t+\eta}(x,y)\,dt &= \lim_{\eta\downarrow 0}\int_J \frac{W_H^2(\mu_t,\mu_{t+\eta})}{\eta^2}\,dt\\ &= \int_J \lim_{\eta\downarrow 0}\frac{W_H^2(\mu_t,\mu_{t+\eta})}{\eta^2}\,dt \qquad (7.8)\\ &= \int_J |\mu_t'|^2\,dt.\end{aligned}$$

Combining (7.6), (7.7), and (7.8) we find that

$$\Big|\int_J\int_E \sum_{i=1}^n \alpha_i'(t) f_i(x)\,d\mu_t(x)\,dt\Big| \le \Big\|\sum_{i=1}^n \alpha_i\otimes\nabla_H f_i\Big\|_{\overline{V}}\Big(\int_J |\mu_t'|^2\,dt\Big)^{1/2},$$

which implies that ℓ is well defined and

$$\Big|\ell\Big(\sum_{i=1}^n \alpha_i\otimes\nabla_H f_i\Big)\Big| \le \Big\|\sum_{i=1}^n \alpha_i\otimes\nabla_H f_i\Big\|_{\overline{V}}\|\,|\mu'|\,\|_{L^2(J)}.$$

This completes the proof. $\square$

It has been shown in [150, Theorem 2.7] that a converse to Theorem 7.2 can be deduced from [4, Theorem 8.3.1]. The same argument works in our setting. Since the proof involves a finite dimensional approximation argument, it is natural to introduce

$$\mathscr{P}_f(E) := \{\mu\in\mathscr{P}(E) : (\mathsf{P}_n)_\#\mu \rightharpoonup \mu \text{ as } n\to\infty\}.$$

We remark that in the situation where $\mathscr{H}$ is the reproducing kernel Hilbert space of a Gaussian measure $\gamma\in\mathscr{P}(E)$ and $\mathscr{H}\subseteq H$ as subsets of E, we have $\mu\in\mathscr{P}_f(E)$ for each $\mu\in\mathscr{P}(E)$ which is absolutely continuous with respect to γ (see Proposition 10.8 below).

Theorem 7.3. *Suppose that $\mu := (\mu_t)_{t\in J}\subseteq\mathscr{P}_f(E)$ is a weakly continuous path satisfying the continuity equation*

$$\partial_t\mu_t + \nabla_H\cdot(Z_t\mu_t) = 0$$

in the sense of (7.3) for some $Z\in L^2(\mu;H)$. Then $\mu\in AC^2(J;\mathscr{P}(E))$ and $|\mu'|(t)\le\|Z_t\|_{L^2(\mu_t;H)}$ for a.a. $t\in J$.

Proof. See [150, Theorem 2.7]. $\square$

7.3 Linearisation of paths

In this section it will be shown that the velocity field obtained in Theorem 7.2 can be used to define a kind of linear approximation to a W_H-absolutely continuous path of probability measures on E.

Loosely speaking, the next theorem asserts that the velocity field along a smooth path can be obtained by taking a certain difference quotient of optimal plans, and pass to the limit. The result will be useful in Chapter 9.

A slightly weaker version of this result has been proved in [150, Proposition 2.11], where the authors work in Wiener spaces and show weak convergence. The Hilbertian version can be found in [4, Proposition 8.4.6].

Theorem 7.4. *Assume (H). Let $J \subseteq \mathbb{R}$ be an interval, and consider a path $(\mu_t)_{t\in J} \in AC^2(J; \mathscr{P}_f(E))$ with associated velocity field $Z \in L^2(M_\mu; H)$. For $s, t \in J$, take $\Sigma_s^t \in \Gamma_o(\mu_s, \mu_t)$. For a.a. $t \in J$ we have*

$$\big(\pi^1 \times \frac{1}{h}(\pi^2 - \pi^1)\big)_\# \Sigma_t^{t+h} \to (I_E \times Z_t)_\# \mu_t \qquad \textit{in } W_{H\times H}\textit{-distance, as } h \to 0,$$

and

$$\lim_{h\to 0} \frac{1}{h} W_H(\mu_{t+h}, (I_E + hZ_t)_\# \mu_t) = 0. \tag{7.9}$$

Proof. We proceed in several steps.
Step 1: *There exists a countable set $\mathcal{D} \subseteq \mathcal{C}$ which is dense in $\mathcal{C}$ with respect to the norm*

$$\|f\|_{1,\infty} = \sup_{x\in E} \big(|f(x)| + |\nabla_H f(x)|_H\big),$$

and a nullset $N \subseteq J$ such that

$$\lim_{h\to 0} \frac{W_H(\mu_t, \mu_{t+h})}{|h|} = |\mu'|(t) \le \|Z_t\|_{L^2(\mu_t;H)} < \infty, \tag{7.10}$$

and

$$\lim_{h\to 0} \frac{1}{h} \bigg(\int_E f\, d\mu_{t+h} - \int_E f\, d\mu_t \bigg) = \int_E [\nabla_H f, Z_t]_H \, d\mu_t, \tag{7.11}$$

for any $f \in \mathcal{D}$ and any $t \in J \setminus N$.

In the proof we let $h \downarrow 0$. The argument for $h \uparrow 0$ is similar. Let $f \in \mathcal{C}$, $\alpha \in C_c^\infty(J)$, and extend α by 0 outside J. By (7.3) and the dominated convergence theorem we have

$$\begin{aligned}\int_J\int_E &\alpha(t)[\nabla_H f(x), Z_t(x)]_H\, d\mu_t(x)\, dt\\ &= \int_J\int_E \alpha'(t) f(x)\, d\mu_t(x)\, dt\\ &= \lim_{h\downarrow 0}\int_J\int_E \frac{\alpha(t)-\alpha(t-h)}{h} f(x)\, d\mu_t(x)\, dt\\ &= \lim_{h\downarrow 0}\int_J \alpha(t)\frac{1}{h}\Big(\int_E f(x)\, d\mu_t(x) - \int_E f(x)\, d\mu_{t+h}(x)\Big)\, dt.\end{aligned} \tag{7.12}$$

We would like to apply the dominated convergence theorem to the right hand side. To find an integrable majorant function, we take K as in (7.5), set $C_f := \sup_{x\in E}|\nabla_H f(x)|_H$, take m as in (7.1) and let

$$m^*(t) := \sup_{h>0}\frac{1}{h}\int_t^{t+h}|m(s)|\, ds$$

denote the (one-sided) Hardy-Littlewood maximal function. Using (7.4) we obtain

$$\begin{aligned}\Big|\frac{1}{h}\int_E & f(x)\, d\mu_t(x) - \int_E f(x)\, d\mu_{t+h}(x)\Big|\\ &= \Big|\frac{1}{h}\int_{E\times E}[K(x,y), x-y]_H\, d\Sigma_t^{t+h}\Big|\\ &\le \frac{1}{h}\Big(\int_{E\times E}|K(x,y)|_H^2\, d\Sigma_t^{t+h}(x,y)\Big)^{1/2}\Big(\int_{E\times E}|x-y|_H^2\, d\Sigma_t^{t+h}\Big)^{1/2}\\ &\le C_f\frac{W_H(\mu_t,\mu_{t+h})}{h}\\ &\le C_f\frac{1}{h}\int_t^{t+h} m(s)\, ds\\ &\le C_f m^*(t).\end{aligned}$$

Since $m^*, \alpha \in L^2(J)$, we may apply dominated convergence to the right hand side of (7.12) to obtain

$$\begin{aligned}\int_J\int_E &\alpha(t)[\nabla_H f(x), Z_t(x)]_H\, d\mu_t(x)\, dt\\ &= \int_J \alpha(t)\Big[\lim_{h\downarrow 0}\frac{1}{h}\Big(\int_E f(x)\, d\mu_t(x) - \int_E f(x)\, d\mu_{t+h}(x)\Big)\Big]\, dt.\end{aligned}$$

Since $\alpha \in C_c^\infty(J)$ is arbitrary, we find for each $f \in \mathcal{C}$ a nullset $N_f \subseteq J$ such that (7.11) holds for any $t \in J\setminus N_f$.

The validity of (7.10) for all $t \in J$ ouside of a nullset N' follows from Theorem 7.3.

It remains to construct $\mathcal{D}$ and N. Let $\mathcal{F}_n \subseteq C_{\mathrm{b}}^\infty(\mathbb{R}^n)$ be countable and dense in the separable Banach space $C_{\mathrm{b}}^1(\mathbb{R}^n)$, and define $\mathcal{D}_n$ to be the set of all functions $f : E \to \mathbb{R}$ of the form

$$f(x) = \phi(\langle x, x_1^* \rangle, \ldots, \langle x, x_n^* \rangle), \qquad \phi \in \mathcal{F}_n.$$

Clearly, $\mathcal{D} := \cup_{n\geq 1} \mathcal{D}_n$ is dense in $\mathcal{C}$ with respect to the $\|\cdot\|_{1,\infty}$-norm. This completes the proof of the first step with $N := N' \cup \bigcup_{f\in\mathcal{D}} N_f$.

From now on we keep $t \in J \setminus N$ fixed, and write

$$\eta_h := \big(\pi^1, \frac{1}{h}(\pi^2 - \pi^1)\big)_{\#} \Sigma_t^{t+h}, \qquad h > 0.$$

Step 2: *The collection $(\eta_h)_{h\in(0,\delta)}$ is tight for some $\delta > 0$.*

First we observe that

$$\begin{aligned} \int_{E\times E} |y|_H^2 \, d\eta_h(x,y) &= \frac{1}{h^2} \int_{E\times E} |x-y|_H^2 \, d\Sigma_t^{t+h}(x,y) \\ &= \frac{1}{h^2} W_H^2(\mu_t, \mu_{t+h}). \end{aligned} \tag{7.13}$$

Let $\varepsilon > 0$ and take a compact set $K \subseteq E$ such that $\mu_t(K) \geq 1 - \varepsilon$. Writing $B_R(0) := \{x \in E : |x|_H \leq R\}$ and using (7.13) and (7.10), we obtain

$$\begin{aligned} \eta_h\big(\big(K \times B_R(0)\big)^c\big) &\leq \eta_h(K^c \times E) + \eta_h(E \times B_R(0)^c) \\ &\leq \varepsilon + \frac{1}{R^2} \int_{E\times E} |y|_H^2 \, d\eta_h(x,y) \\ &= \varepsilon + \frac{1}{R^2 h^2} \int_{E\times E} |x-y|_H^2 \, d\Sigma_t^{t+h}(x,y) \\ &= \varepsilon + \frac{W_H^2(\mu_t, \mu_{t+h})}{R^2 h^2} \\ &\leq 2\varepsilon, \end{aligned}$$

whenever R is large enough. Now the claim follows, since we assumed (H).

Step 3: *According to the previous step there exists $\eta_0 \in \mathscr{P}(E \times E)$ and a sequence $h_n \downarrow 0$ such that η_{h_n} converges weakly to η_0. Using disintegration we may write $\eta_0 = \int_E \eta_{0x} \, d\mu_t(x)$ with $(\eta_{0x})_{x\in E} \subseteq \mathscr{P}(E)$. We put $Y_t(x) := \int_E y \, d\eta_{0x}(y)$ and claim that*

$$\int_E [\nabla_H f, Y_t - Z_t]_H \, d\mu_t = 0, \qquad f \in \mathcal{D}.$$

Using (7.11) we find

$$\begin{aligned}
\frac{1}{h}\left(\int_E f\,d\mu_{t+h}-\int_E f\,d\mu_t\right) &= \frac{1}{h}\int_{E\times E} f(y)-f(x)\,d\Sigma_t^{t+h}(x,y)\\
&= \frac{1}{h}\int_{E\times E} f(x+hy)-f(x)\,d\eta_h(x,y)\\
&= \int_{E\times E}[\nabla_H f(x),y]_H+\omega_{x,y}(h)\,d\eta_h(x,y),
\end{aligned}\tag{7.14}$$

where $|\omega_{x,y}(h)|\le \frac{1}{2}hC_f|y|_H^2$ and $C_f:=\sup_{x\in E}\|D_H^2 f(x)\|_{\mathcal{L}(H)}$. Using (7.10) we obtain

$$\begin{aligned}
\overline{\lim_{h\to 0}}\int_{E\times E}|\omega_{x,y}(h)|\,d\eta_h(x,y) &\le \overline{\lim_{h\to 0}}\,\frac{hC_f}{2}\int_{E\times E}|y|_H^2\,d\eta_h(x,y)\\
&\le \overline{\lim_{h\to 0}}\,\frac{C_f}{2h}\int_{E\times E}|x-y|_H^2\,d\Sigma_t^{t+h}(x,y)\\
&= \overline{\lim_{h\to 0}}\,\frac{C_f}{2h}W_H^2(\mu_t,\mu_{t+h})\\
&= 0.
\end{aligned}\tag{7.15}$$

We will show by means of Proposition 6.14 that

$$\lim_{h\to 0}\int_E[\nabla_H f(x),y]_H\,d\eta_h(x,y)=\int_E[\nabla_H f(x),y]_H\,d\eta_0(x,y).\tag{7.16}$$

Indeed, since $\nabla_H f\in L^2(\mu_t;H)$ and $\pi^1_{\#}\eta_h=\mu_t$ for every $h\in(0,\delta)$, we have

$$\begin{aligned}
&\lim_{M\to\infty}\overline{\lim_{n\to\infty}}\int_{\{|\nabla_H f(x)|_H\ge M\}}|\nabla_H f(x)|_H^2\,d\eta_{h_n}(x,y)\\
&\quad=\lim_{M\to\infty}\int_{\{|\nabla_H f(x)|_H\ge M\}}|\nabla_H f(x)|_H^2\,d\mu_t(x)=0
\end{aligned}$$

and

$$\int_{E\times E}|\nabla_H f(x)|_H^2\,d\eta_{h_n}=\int_E|\nabla_H f(x)|_H^2\,d\mu_t=\int_{E\times E}|\nabla_H f(x)|_H^2\,d\eta_0.$$

Moreover, using (7.13) we obtain, as in Step 1,

$$\sup_{n\ge 1}\int_E|y|_H^2\,d\eta_{h_n}(x,y)=\sup_{n\ge 1}\frac{1}{h_n^2}W_H^2(\mu_t,\mu_{t+h_n})\le m^*(t)^2<\infty.$$

Therefore (7.16) follows from Proposition 6.14.

Passing to the limit in (7.14) and taking (7.11) and (7.15) into account, we obtain

$$\begin{aligned}\int_E [\nabla_H f(x), Z_t(x)]_H \, d\mu_t(x) &= \lim_{h\to 0} \int_{E\times E} [\nabla_H f(x), y]_H \, d\eta_h(x,y) \\ &= \int_{E\times E} [\nabla_H f(x), y]_H \, d\eta_0(x,y) \\ &= \int_{E\times E} [\nabla_H f(x), Y_t(x)]_H \, d\mu_t(x).\end{aligned}$$

Step 4: *We claim that*

$$\eta_0 = (I_E \times Z_t)_\# \mu_t.$$

Using Jensen's inequality, Lemma 6.2, (7.10) and (7.13), we find

$$\begin{aligned}\|Y_t\|^2_{L^2(\mu_t;H)} &= \int_E \left| \int_E y \, d\eta_{0x}(y) \right|_H^2 d\mu_t(x) \\ &\leq \int_{E\times E} |y|_H^2 \, d\eta_{0x}(y) \, d\mu_t(x) \\ &\leq \varliminf_{h\to 0} \int_{E\times E} |y|_H^2 \, d\eta_h(x,y) \\ &= \varliminf_{h\to 0} \frac{1}{h^2} W_H^2(\mu_t, \mu_{t+h}) \\ &\leq \|Z_t\|^2_{L^2(\mu_t;H)}.\end{aligned} \tag{7.17}$$

Since $\nabla_H(\mathcal{D})$ is dense in $T^H_{\mu_t}$ and $Z_t \in T^H_{\mu_t}$, Step 3 implies that $\int_E [Z_t, Y_t - Z_t]_H \, d\mu_t = 0$. Combined with the previous estimate we conclude that $Y_t = Z_t$ μ_t-a.e.

In particular, the first inequality in (7.17) is actually an equality, which implies that for μ_t-a.e. $x \in E$,

$$\left| \int_E y \, d\eta_{0x}(y) \right|_H^2 = \left(\int_E |y|_H \, d\eta_{0x}(y) \right)^2 = \int_E |y|_H^2 \, d\eta_{0x}(y).$$

These identities imply that there exist a unit vector $y_x \in H$, and $c_x, \alpha_x(y) \geq 0$ such that for η_{0x}-a.a. $y \in E$,

$$y = \alpha_x(y) y_x \text{ and } |y|_H = c_x,$$

which is absurd unless $\eta_{0x} = \delta_{c_x y_x}$. Since $Z_t(x) = Y_t(x) = \int_E y \, d\eta_{0x}(y)$ we conclude that $c_x y_x = Z_t(x)$ for μ_t-a.a $x \in E$, hence $\eta_0 = (I_E \times Z_t)_\# \mu_t$. This completes the proof of Step 4.

Step 5: *We claim that*

$$\int_E |Z_t(x)|_H^2 \, d\mu_t(x) \leq \varliminf_{h\to 0} \int_{E\times E} [Z_t(x), y]_H \, d\eta_h(x,y).$$

Since $Z_t \in L^2(\mu_t; H)$ and $\pi^1_{\#}\eta_h = \mu_t = \pi^1_{\#}\eta_0$ for any $h \in (0,\delta)$, we have

$$\lim_{M\to\infty} \overline{\lim_{n\to\infty}} \int_{\{|Z_t(x)|_H \ge M\}} |Z_t(x)|_H^2 \, d\eta_{h_n}(x,y) = \lim_{M\to\infty} \int_{\{|Z_t(x)|_H \ge M\}} |Z_t(x)|_H^2 \, d\mu_t(x) = 0$$

and for every $n \ge 1$,

$$\int_{E\times E} |Z_t(x)|_H^2 \, d\eta_{h_n}(x,y) = \int_E |Z_t(x)|_H^2 \, d\mu_t(x).$$

Moreover, by (7.13),

$$\sup_{h\in(0,\delta)} \int_{E\times E} |y|_H^2 \, d\eta_h(x,y) = \sup_{h\in(0,\delta)} \frac{1}{h^2} W_H^2(\mu_t, \mu_{t+h}) \le m^*(t)^2 < \infty.$$

Therefore Proposition 6.14 implies that

$$\lim_{n\to\infty} \int_{E\times E} [Z_t(x), y]_H \, d\eta_{h_n}(x,y) = \int_{E\times E} [Z_t(x), y]_H \, d\eta_0(x,y).$$

Since Step 4 implies that

$$\int_{E\times E} [Z_t(x), y]_H \, d\eta_0(x,y) = \int_E |Z_t(x)|_H^2 \, d\mu_t(x),$$

the proof of Step 5 is complete.

Step 6: *We have*

$$\lim_{h\to 0} \frac{1}{h} W_H(\mu_{t+h}, (I_E + hZ_t)_{\#}\mu_t) = 0 \quad \textit{and} \quad \lim_{h\to 0} W_{H\times H}(\eta_h, (I_E \times Z_t)_{\#}\mu_t) = 0.$$

To bound the first distance we estimate

$$\begin{aligned} \frac{1}{h^2} W_H^2(\mu_{t+h},&(I_E + hZ_t)_{\#}\mu_t) \\ &\le \frac{1}{h^2} \int_{E\times E} |x-y|_H^2 \, d(\pi^2 \times (I_E + hZ_t)\circ \pi^1)_{\#} \Sigma_t^{t+h}(x,y) \\ &= \int_{E\times E} \Big| \frac{1}{h}(y-x) - Z_t(x) \Big|_H^2 \, d\Sigma_t^{t+h}(x,y) \\ &= \int_{E\times E} |y - Z_t(x)|_H^2 \, d\eta_h(x,y). \end{aligned}$$

The second distance can be bounded by the same expression:

$$
\begin{aligned}
W_{H\times H}(\eta_h,(I_E\times Z_t)_{\#}\mu_t) &\le \int_{E^2\times E^2} |(x_1,x_2)-(y_1,y_2)|^2_{H\times H} \\
&\qquad d\big(I_{E\times E}\times\big((I_E\times Z_t)\circ\pi^1\big)\big)_{\#}\eta_h(x,y) \\
&= \int_{E\times E} |(x_1,x_2)-(x_1,Z_t(x_1))|^2_{H\times H}\, d\eta_h(x_1,x_2) \\
&= \int_{E\times E} |x_2 - Z_t(x_1)|^2_H\, d\eta_h(x_1,x_2).
\end{aligned}
$$

Therefore both statements are proved once we have shown that

$$
\lim_{h\to 0}\int_{E\times E} |y-Z_t(x)|^2_H\, d\eta_h(x,y) = 0.
$$

Combining (7.10) and (7.13) we arrive at

$$
\lim_{h\to 0}\int_{E\times E} |y|^2_H\, d\eta_h(x,y) \le \int_E |Z_t(x)|^2_H\, d\mu_t(x).
$$

Using this and Step 5 we obtain

$$
\begin{aligned}
&\overline{\lim_{h\to 0}}\int_{E\times E} |y-Z_t(x)|^2_H\, d\eta_h \\
&\quad = \overline{\lim_{h\to 0}}\int_{E\times E} |y|^2_H\, d\eta_h - 2\int_{E\times E}[Z_t(x),y]_H\, d\eta_h + \int_E |Z_t(x)|^2_H\, d\mu_t \\
&\quad \le 0,
\end{aligned}
$$

which completes the proof. □

8

Functionals

In this chapter we study functionals defined on the Wasserstein space. We study various notions of convexity and introduce subdifferentials for functionals, in the spirit of Hilbert space theory [18]. Following the approach of [4, 150], we investigate the properties of subdifferentials, which will be useful in the study of gradient flows in Chapter 9.

First we summarise some results for functionals on metric spaces from [4].

8.1 Functionals on metric spaces

Let (X, d) be a complete metric space. Let $\phi : X \to \mathbb{R} \cup \{\infty\}$ be proper, i.e.,

$$\mathsf{D}(\phi) := \{x \in X : \phi(x) < \infty\} \neq \varnothing.$$

and lower semicontinuous. For $h > 0$ and $x \in \overline{\mathsf{D}(\phi)}$ we consider the function

$$\Phi(h, x; \cdot) : X \to \mathbb{R} \cup \{\infty\}, \qquad y \mapsto \phi(y) + \frac{1}{2h} d^2(y, x).$$

We consider the following additional assumptions:

(A1) (Coercivity) There exist $\tilde{x} \in X$, $r > 0$, and $m \in \mathbb{R}$ such that $\phi(x) \geq m$ for every $x \in X$ with $d(x, \tilde{x}) \leq r$.

(A2) (Generalised λ-convexity) There exists $\lambda \in \mathbb{R}$ such that for every $y, x_0, x_1 \in \mathsf{D}(\phi)$ there exists a map $u : [0, 1] \to X$ satisfying $u(0) = x_0$, $u(1) = x_1$ and

$$\begin{aligned} \Phi(h, y; u(t)) \leq (1 - t)\Phi(h, y; x_0) + t\Phi(h, y; x_1) \\ - \frac{1}{2}(\frac{1}{h} + \lambda)t(1 - t)d^2(x_0, x_1) \end{aligned}$$

for every $t \in [0, 1]$ and every $h \in I_\lambda := \{t > 0 : 1 + t\lambda > 0\}$.

Under these assumptions, it has been shown in [4, Theorem 4.1.2] that for $h \in I_\lambda$ and $x \in \overline{\mathsf{D}(\phi)}$, the function $\Phi(h, x; \cdot)$ has a unique minimizer denoted by $J_h x$.

The Moreau-Yosida approximation is defined by

$$\phi_h(x) := \inf_{y \in X} \Phi(h, x; y) = \phi(J_h x) + \frac{1}{2h} d^2(J_h x, x), \qquad x \in \overline{\mathsf{D}(\phi)}.$$

Definition 8.1. *The local slope $|\partial\phi| : X \to [0, \infty]$ is defined by $|\partial\phi|(x) = 0$ if x is isolated in $\mathsf{D}(\phi)$, and otherwise*

$$|\partial\phi|(x) := \varlimsup_{y \to x, y \in \mathsf{D}(\phi)} \frac{\big(\phi(x) - \phi(y)\big)^+}{d(x, y)}.$$

Obviously, we have $\mathsf{D}(|\partial\phi|) \subseteq \mathsf{D}(\phi)$. In the next two propositions we collect some fundamental properties of J_h, ϕ_h, and $|\partial\phi|$.

Proposition 8.2. *Let $\phi : X \to \mathbb{R} \cup \{\infty\}$ be a proper lsc functional satisfying (A1) and (A2).*

(i) For $h \in I_\lambda$ and $x \in \overline{\mathsf{D}(\phi)}$ we have $J_h x \in \mathsf{D}(|\partial\phi|)$ and

$$|\partial\phi|(J_h x) \leq \frac{1}{h} d(J_h x, x). \tag{8.1}$$

(ii) $x \in \overline{\mathsf{D}(\phi)}$ iff

$$d(J_h x, x) \downarrow 0 \qquad \text{as } h \downarrow 0. \tag{8.2}$$

(iii) For $x \in \overline{\mathsf{D}(\phi)}$ we have

$$\phi(J_h x) \uparrow \phi(x) \qquad \text{as } h \downarrow 0. \tag{8.3}$$

(iv) For $x \in \overline{\mathsf{D}(\phi)}$ we have

$$\phi_h(x) \uparrow \phi(x) \qquad \text{as } h \downarrow 0. \tag{8.4}$$

Proof. See [4, Lemmas 3.1.2 & 3.1.3] or [35, Proposition 4.1]. □

Proposition 8.3. *Let $\phi : X \to \mathbb{R} \cup \{\infty\}$ be a proper lsc functional satisfying (A1) and (A2). For $h \in I_\lambda$ and $x \in \mathsf{D}(|\partial\phi|)$ we have*

$$\phi(x) \leq \phi_h(x) + \frac{h}{2} |\partial\phi|^2(x) \tag{8.5}$$

and

$$\begin{aligned} |\partial\phi|(x) &= \lim_{h \to 0} |\partial\phi|(J_h x) = \lim_{h \to 0} \frac{d(x, J_h x)}{h} \\ &= \lim_{h \to 0} \Big(2 \frac{\phi(x) - \phi_h(x)}{h} \Big)^{1/2} = \lim_{h \to 0} \Big(\frac{\phi(x) - \phi(J_h x)}{h} \Big)^{1/2}. \end{aligned} \tag{8.6}$$

Proof. See [4, Theorem 3.1.6] or [35, Proposition 4.3]. □

8.2 Convexity along generalised geodesics

Optimal transport plans provide a natural way to interpolate between probability measures. It has been discovered by McCann [118] in a seminal paper that various interesting functionals enjoy convexity properties along the interpolated paths given by optimal transport plans. This observation has many consequences; in particular to functional inequalities, Ricci curvature lower bounds, and gradient flows.

In [4] a more general type of interpolation between probability measures has been considered, which allows for the application of the abstract theory of gradient flows in metric spaces described in Section 8.1 to functionals on the Wasserstein space.

The crucial notion is the following:

Definition 8.4. *Let $\mu_0, \mu_1, \nu \in \mathscr{P}(E)$. A* generalised geodesic *joining μ_0 and μ_1 (with base ν) is a path $(\mu_t)_{t\in[0,1]} \subseteq \mathscr{P}(E)$ of the form*

$$\mu_t := (\pi_t^{2\to 3})_\# \Xi, \qquad t \in [0,1],$$

for some $\Xi \in \Gamma(\nu, \mu_0, \mu_1)$ satisfying

$$\pi_\#^{1,2} \Xi \in \Gamma_o(\nu, \mu_0), \qquad \pi_\#^{1,3} \Xi \in \Gamma_o(\nu, \mu_1).$$

Definition 8.5. *Let $\lambda \in \mathbb{R}$. A proper functional $\phi : \mathscr{P}(E) \to \mathbb{R} \cup \{\infty\}$ is said to be*

(i) λ-convex *if for any $\mu_0, \mu_1 \in \mathsf{D}(\phi)$ with $W_H(\mu_0, \mu_1) < \infty$, there exists an optimal transport plan $\Sigma \in \Gamma_o(\mu_0, \mu_1)$ such that $\mu_t := (\pi_t^{1\to 2})_\# \Sigma$ satisfies*

$$\phi(\mu_t) \le (1-t)\phi(\mu_0) + t\phi(\mu_1) - \frac{\lambda}{2} t(1-t) W_H^2(\mu_0, \mu_1), \qquad t \in [0,1].$$

(ii) λ-convex along generalised geodesics *if for any $\mu_0, \mu_1, \nu \in \mathsf{D}(\phi)$ with $W_H(\nu, \mu_i) < \infty$ for $i = 0, 1$, there exists a generalised geodesic $(\mu_t)_{t\in[0,1]}$ joining μ_0 and μ_1 with base ν, such that*

$$\phi(\mu_t) \le (1-t)\phi(\mu_0) + t\phi(\mu_1) - \frac{\lambda}{2} t(1-t) W_H^2(\mu_0, \mu_1), \qquad t \in [0,1].$$

Both convexity notions are related by the following result:

Proposition 8.6. *Let $\lambda \in \mathbb{R}$ and let $\phi : \mathscr{P}(E) \to \mathbb{R} \cup \{\infty\}$ be λ-convex along generalised geodesics. Then ϕ is λ-convex.*

Proof. Take $\mu_0, \mu_1 \in \mathscr{P}(E)$ with $W_H(\mu_0, \mu_1) < \infty$. Definition 8.5 applied to $\nu = \mu_0$ yields the existence of $\Xi \in \Gamma(\mu_0, \mu_0, \mu_1)$ with $\pi_\#^{1,2} \Xi \in \Gamma_o(\mu_0, \mu_0)$ and $\pi_\#^{1,3} \Xi \in \Gamma_o(\mu_0, \mu_1)$. Since $\Gamma_o(\mu_0, \mu_0) = \{(I_E \times I_E)_\# \mu_0\}$, it follows that $\Xi = \pi_\#^{1,1,2} \Sigma$ for some $\Sigma \in \Gamma_o(\mu_0, \mu_1)$. Since $(\pi_t^{2\to 3})_\# \Xi = (\pi_t^{1\to 2})_\# \Sigma$, the result follows. $\square$

The following result expresses a convexity property of the squared Wasserstein distance along generalised geodesics. In particular, it implies that functionals which are λ-convex along generalised geodesics satisfy (A2).

Proposition 8.7. *Let $\mu, \mu_0, \mu_1 \in \mathscr{P}(E)$ satisfy $W_H(\mu, \mu_i) < \infty$, for $i = 0, 1$ and let $\Sigma \in \mathscr{P}(E^3)$ be such that $\pi^{1,2}_{\#}\Sigma \in \Gamma_o(\mu, \mu_0)$ and $\pi^{1,3}_{\#}\Sigma \in \Gamma_o(\mu, \mu_1)$. Set $\mu_t := (\pi_t^{2\to 3})_{\#}\Sigma$. Then*

$$W_H^2(\mu, \mu_t) \le (1-t)W_H^2(\mu, \mu_0) + tW_H^2(\mu, \mu_1) - t(1-t)W_H^2(\mu_0, \mu_1).$$

Proof. We have

$$\begin{aligned} W_H^2(\mu, \mu_t) &\le \int_{E\times E} |x-y|_H^2 \, d(\pi^1 \times \pi_t^{2\to 3})_{\#}\Sigma \\ &= \int_{E^3} |(1-t)(x-y) + t(x-z)|_H^2 \, d\Sigma \\ &= (1-t)\int_{E^3} |x-y|_H^2 \, d\Sigma + t\int_{E^3} |x-z|_H^2 \, d\Sigma \\ &\qquad - t(1-t)\int_{E^3} |y-z|_H^2 \, d\Sigma \\ &\le (1-t)W_H^2(\mu, \mu_0) + tW_H^2(\mu, \mu_1) - t(1-t)W_H^2(\mu_0, \mu_1). \end{aligned}$$

□

The following result is proved in Hilbert spaces in [4, Lemma 7.2.1] and the proof remains valid in our setting. For the convenience of the reader we provide the details. The result will be useful in the proof of Theorem 8.11.

Lemma 8.8. *(i) Let $\mu_0, \mu_1 \in \mathscr{P}(E)$ with $W_H(\mu_0, \mu_1) < \infty$ and take $\Sigma \in \Gamma_o(\mu_0, \mu_1)$. Then $\mu_t := (\pi_t^{1\to 2})_{\#}\Sigma$ satisfies*

$$W_H(\mu_s, \mu_t) = |t-s|, \qquad s, t \in [0,1].$$

(ii) Let $(\mu_t)_{t\in[0,1]} \subseteq \mathscr{P}(E)$ be such that

$$W_H(\mu_s, \mu_t) = |t-s| W_H(\mu_0, \mu_1) < \infty$$

for every $s, t \in [0,1]$. For each $t \in (0,1)$ there exists an optimal plan $\Sigma \in \Gamma_o(\mu_0, \mu_1)$ and Borel maps $s_t^0, s_t^1 : E \to E$ such that

$$\begin{aligned} \Gamma_o(\mu_0, \mu_t) &= \{(\pi_t^{1,1\to 2})_{\#}\Sigma\} = \{(s_t^0 \times I_E)_{\#}\mu_t\}, \qquad \textit{and} \\ \Gamma_o(\mu_t, \mu_1) &= \{(\pi_t^{1\to 2,2})_{\#}\Sigma\} = \{(I_E \times s_t^1)_{\#}\mu_t\}. \end{aligned}$$

Proof. (i): Note that

$$\begin{aligned} W_H^2(\mu_s, \mu_t) &\le \int_{E\times E} |x-y|_H^2 \, d(\pi_s^{1\to 2} \times \pi_t^{1\to 2})_{\#}\Sigma \\ &= |t-s|^2 \int_{E\times E} |x-y|_H^2 \, d\Sigma = |t-s|^2 W_H^2(\mu_0, \mu_1). \end{aligned}$$

Since this inequality holds for every $s, t \in [0,1]$, the triangle inequality implies that the inequality above is in fact an equality, which gives the desired result.

(ii): Fix $0 < t < 1$. It is convenient to regard μ_0, μ_t, μ_1 as measures on distinct copies E_1, E_2, E_3 of E. Take $\Sigma^0 \in \Gamma_o(\mu_0, \mu_t)$ and $\Sigma^1 \in \Gamma_o(\mu_t, \mu_1)$, Using disintegration we find $\nu^0_{x_2} \in \mathscr{P}(E_1)$ and $\nu^1_{x_2} \in \mathscr{P}(E_3)$ satisfying

$$\Sigma^0 := \int_{E_2} \nu^0_{x_2} \, d\mu_t(x_2), \qquad \Sigma^1 := \int_{E_2} \nu^1_{x_2} \, d\mu_t(x_2).$$

Consider the probability measure $\Xi \in \mathscr{P}(E_1 \times E_2 \times E_3)$ defined by

$$\Xi := \int_{E_2} \nu^0_{x_2} \otimes \nu^1_{x_2} \, d\mu_t(x_2).$$

Since $\pi^{1,2}_{\#}\Xi \in \Gamma_o(\mu_0, \mu_t)$ and $\pi^{2,3}_{\#}\Xi \in \Gamma_o(\mu_t, \mu_1)$ we obtain

$$\begin{aligned} W_H(\mu_0, \mu_1) &\le \|\pi^1 - \pi^3\|_{L^2(\Xi;H)} \\ &\le \|\pi^1 - \pi^2\|_{L^2(\Xi;H)} + \|\pi^2 - \pi^3\|_{L^2(\Xi;H)} \\ &= W_H(\mu_0, \mu_t) + W_H(\mu_t, \mu_1) \\ &= W_H(\mu_0, \mu_1), \end{aligned}$$

hence all inequalities are actually equalities. In particular, we find that $\Sigma := \pi^{1,3}_{\#}\Xi \in \Gamma_o(\mu_0, \mu_1)$, and there exists $\alpha > 0$ such that $\pi^1 - \pi^2 = \alpha(\pi^1 - \pi^3)$, Ξ-a.e. Therefore

$$\begin{aligned} \|\pi^1 - \pi^2\|_{L^2(\Xi;H)} &= W_H(\mu_0, \mu_t) = tW_H(\mu_0, \mu_1) \\ &= t\|\pi^1 - \pi^3\|_{L^2(\Xi;H)} = \frac{t}{\alpha}\|\pi^1 - \pi^2\|_{L^2(\Xi;H)}, \end{aligned}$$

hence $\alpha = t$ and $\pi^2 = (1-t)\pi^1 + t\pi^3$, Ξ-a.e., which implies that

$$\Sigma^0 = \pi^{1,2}_{\#}\Xi = (\pi^{1,1\to 2}_t)_{\#}\Sigma \qquad \text{and} \qquad \Sigma^1 = \pi^{2,3}_{\#}\Xi = (\pi^{1\to 2,2}_t)_{\#}\Sigma.$$

This proves one part of the lemma.

To obtain the other part, we write

$$z^0(x_2) := \int_{E_1} x_1 d\nu^0_{x_2}(x_1), \qquad z^1(x_2) := \int_{E_3} x_3 d\nu^1_{x_2}(x_3).$$

Since $\pi^1 = \frac{\pi^2 - t\pi^3}{1-t}$ and $\pi^3 = \frac{\pi^2 - (1-t)\pi^1}{t}$, Ξ-a.e., we have

$$z^0(x_2) = \frac{x_2 - tx_3}{1-t}, \qquad \Sigma^1\text{-a.e.}, \qquad z^1(x_2) = \frac{x_2 - (1-t)x_1}{t}, \qquad \Sigma^0\text{-a.e.}$$

Therefore, we have Ξ-a.e.,

$$x_3 = s^1_t(x_2) := \frac{x_2 - (1-t)z^0(x_2)}{t}, \qquad x_1 = s^0_t(x_2) := \frac{x_2 - tz^1(x_2)}{1-t},$$

which implies that $\Sigma^1 := (I_E \times s_t^1)_\# \mu_t$ and $\Sigma^0 = (s_t^0 \times I_E)_\# \mu_t$.

Note that s_t^1 depends (through z^0) on Σ^0, but not on Σ^1. This implies that $\Gamma_o(\mu_t, \mu_1)$ contains only one element. Since the same argument works for $\Gamma_o(\mu_0, \mu_t)$, the proof is complete. □

8.3 Subdifferentials of convex functionals

In this section we define subdifferentials of functionals and study some of their properties under suitable convexity assumptions. First we introduce some notation.

For $\mu, \nu \in \mathscr{P}(E)$ and $\Sigma \in \mathscr{P}(E \times E)$ with $\pi^1_\# \Sigma = \mu$ we define

$$\Gamma_o(\Sigma, \nu) := \{\Xi \in \mathscr{P}(E^3) : \pi^{1,2}_\# \Xi = \Sigma, \pi^{1,3}_\# \Xi \in \Gamma_o(\mu, \nu)\}.$$

For $1 \le p < \infty$, $i = 1, \ldots, n$, and $\Xi \in \mathscr{P}(E^n)$, we set

$$|\Xi|_{i,p} := \left(\int_{E^n} |x_i|_H^p \, d\Xi \right)^{1/p}.$$

For $\Xi \in \Gamma(\mu^1, \mu^2, \mu^3)$ and $i, j \in \{1, 2, 3\}$ we write

$$W_\Xi(\mu^i, \mu^j) := \left(\int_{E^3} |x_i - x_j|_H^2 \, d\Xi \right)^{1/2}.$$

Following [4] and [59] we introduce the subdifferential associated with a functional on the Wasserstein space. The definition strongly resembles the corresponding definition for functionals on Hilbert spaces [18].

Definition 8.9. *Let $\phi : \mathscr{P}(E) \to \mathbb{R} \cup \{\infty\}$ be proper and lsc. Let $\mu \in \mathsf{D}(\phi)$ and $\Sigma \in \mathscr{P}(E \times E)$ be such that $\pi^1_\# \Sigma = \mu$ and $|\Sigma|_{2,2} < \infty$.*

(1) *We say that Σ is contained in the* subdifferential *of ϕ at μ, and write $\Sigma \in \partial\phi(\mu)$, if*

$$\phi(\nu) - \phi(\mu) \ge \inf_{\Xi \in \Gamma_o(\Sigma, \nu)} \int_{E^3} [x_2, x_3 - x_1]_H \, d\Xi + o(W_H(\mu, \nu))$$

as $W_H(\mu, \nu) \to 0$.

(2) *We say that Σ is contained in the* strong subdifferential *of ϕ at μ, and write $\Sigma \in \partial^s \phi(\mu)$, if for any $\Xi \in \Gamma(\Sigma, \nu)$,*

$$\phi(\nu) - \phi(\mu) \ge \int_{E^3} [x_2, x_3 - x_1]_H \, d\Xi + o(W_\Xi(\mu, \nu))$$

as $W_\Xi(\mu, \nu) \to 0$.

Remark 8.10. Let us be more explicit and write for $\varXi \in \varGamma(\varSigma, \nu)$,

$$I(\varXi) := \int_{E^3} [x_2, x_3 - x_1]_H \, d\varXi.$$

With this notation, the assertion $\varSigma \in \partial\phi(\mu)$ means that

$$\begin{aligned} &\forall\, \nu \in \mathscr{P}_{H,\mu}(E)\ \exists\, \varXi \in \varGamma_o(\varSigma,\nu)\ \exists\, \varPsi(\varXi) \in \mathbb{R} : \\ &\qquad \phi(\nu) - \phi(\mu) \geq I(\varXi) + \varPsi(\varXi) \qquad \text{and} \quad \lim_{W_\varXi(\mu,\nu)\to 0} \frac{\varPsi(\varXi)}{W_\varXi(\mu,\nu)} = 0. \end{aligned}$$

Similarly, $\varSigma \in \partial^s\phi(\mu)$ means that

$$\begin{aligned} &\forall\, \nu \in \mathscr{P}_{H,\mu}(E)\ \forall\, \varXi \in \varGamma(\varSigma,\nu)\ \exists\, \varPsi(\varXi) \in \mathbb{R} : \\ &\qquad \phi(\nu) - \phi(\mu) \geq I(\varXi) + \varPsi(\varXi) \qquad \text{and} \quad \lim_{W_\varXi(\mu,\nu)\to 0} \frac{\varPsi(\varXi)}{W_\varXi(\mu,\nu)} = 0. \end{aligned}$$

From this description it is clear that the following inclusion holds:

$$\partial^s\phi(\mu) \subseteq \partial\phi(\mu).$$

We will use the notation

$$\begin{aligned} \mathsf{D}(\partial\phi) &:= \{\mu \in \mathsf{D}(\phi) : \partial\phi(\mu) \neq \varnothing\}, \\ \mathsf{D}(\partial^s\phi) &:= \{\mu \in \mathsf{D}(\phi) : \partial^s\phi(\mu) \neq \varnothing\}. \end{aligned}$$

The next theorem provides a useful description of the subdifferential for λ-convex functionals. We refer to [4, Theorem 10.3.6] and [150, Theorem 3.8] for the corresponding results in the Hilbert and Wiener space setting.

Theorem 8.11. *Let $\phi : \mathscr{P}(E) \to \mathbb{R} \cup \{\infty\}$ be proper, lsc, and λ-convex for some $\lambda \in \mathbb{R}$. Let $\mu \in \mathsf{D}(\phi)$ and let $\varSigma \in \mathscr{P}(E \times E)$ satisfy $\pi^1_\# \varSigma = \mu$ and $|\varSigma|_{2,2} < \infty$. Then $\varSigma \in \partial\phi(\mu)$ if and only if for every $\nu \in \mathscr{P}_{H,\mu}(E)$ there exists $\varXi \in \varGamma_o(\varSigma,\nu)$ such that*

$$\int_{E^3} [x_2, x_3 - x_1]_H \, d\varXi + \frac{\lambda}{2} W_H^2(\mu,\nu) + \phi(\mu) \leq \phi(\nu). \tag{8.7}$$

Proof. Clearly, (8.7) implies that $\varSigma \in \partial\phi(\mu)$.

Conversely, assume that $\varSigma \in \partial\phi(\mu)$. Let $\nu \in \mathsf{D}(\phi)$ and let $(\mu_t)_{t\in[0,1]} \subseteq \mathscr{P}(E)$ such that $\mu_0 = \mu$, $\mu_1 = \nu$, and

$$\begin{aligned} W_H(\mu_s, \mu_t) &= |t-s| W_H(\mu,\nu), \\ \phi(\mu_t) &\leq (1-t)\phi(\mu) + t\phi(\nu) - \frac{\lambda}{2} t(1-t) W_H^2(\mu,\nu). \end{aligned}$$

Since $\varSigma \in \partial\phi(\mu)$ there exists $\widehat{\varXi}_t \in \varGamma_o(\varSigma, \mu_t)$ satisfying

$$\begin{aligned}\phi(\mu_t)-\phi(\mu) &\geq \int_{E^3}[x_2,x_3-x_1]_H\,d\widehat{\Xi}_t + o(W_H(\mu,\mu_t)) \\ &= t\int_{E^3}[x_2,x_3-x_1]_H\,d\Xi_t + o(t),\end{aligned} \tag{8.8}$$

where $\Xi_t := (\pi^1 \times \pi^2 \times \frac{\pi^3-(1-t)\pi^1}{t})_\# \widehat{\Xi}_t$, or equivalently $\widehat{\Xi}_t := (\pi_t^{1,2,1\to 3})_\# \Xi_t$.

We claim that $\Xi_t \in \Gamma_o(\Sigma,\nu)$. Indeed, by Lemma 8.8 we have $\pi^{1,3}_\# \widehat{\Xi}_t = (\pi_t^{1,1\to 2})_\# \Upsilon$ for some $\Upsilon \in \Gamma_o(\mu,\nu)$. Therefore

$$\pi^3_\# \Xi_t = \Big(\frac{\pi^3-(1-t)\pi^1}{t}\Big)_\# \widehat{\Xi}_t = \pi^2_\# \Upsilon = \nu.$$

Since $\pi^{1,2}_\# \Xi_t = \pi^{1,2}_\# \widehat{\Xi}_t = \Sigma$, it follows that $\Xi_t \in \Gamma(\Sigma,\nu)$. The optimality follows from

$$\int_{E^3}|x_1-x_3|_H^2\,d\Xi_t = \frac{1}{t^2}\int_{E^3}|x_1-x_3|_H^2\,d\widehat{\Xi}_t = \frac{1}{t^2}W_H^2(\mu,\mu_t) = W_H^2(\mu,\nu).$$

This proves the claim.

Combining the λ-convexity of ϕ with (8.8),

$$\begin{aligned}\phi(\nu)-\phi(\mu) &\geq \frac{\phi(\mu_t)-\phi(\mu)+\frac{\lambda}{2}t(1-t)W_H^2(\mu,\nu)}{t} \\ &\geq \int_{E^3}[x_2,x_3-x_1]_H\,d\Xi_t + o(1) + \frac{\lambda}{2}(1-t)W_H^2(\mu,\nu).\end{aligned}$$

Therefore, to complete the proof, it suffices to find $\Xi \in \Gamma_o(\Sigma,\nu)$ and a vanishing sequence $t_n \downarrow 0$ for which

$$\lim_{n\to\infty}\int_{E^3}[x_2,x_3-x_1]_H\,d\Xi_{t_n} = \int_{E^3}[x_2,x_3-x_1]_H\,d\Xi.$$

Since $\pi^{1,2}_\# \Xi_t = \Sigma$ and $\pi^3_\# \Xi_t = \nu$, the collection $(\Xi_t)_{t\in(0,1)}$ is tight. Let $t_n \downarrow 0$ such that Ξ_{t_n} converges weakly to $\Xi \in \mathscr{P}(E^3)$. Since $\Xi_t \in \Gamma_o(\Sigma,\nu)$ we clearly have $\Xi \in \Gamma(\Sigma,\nu)$. Moreover, by Lemma 6.2,

$$\int_{E^3}|x_1-x_3|_H^2\,d\Xi \leq \varliminf_{n\to\infty}\int_{E^3}|x_1-x_3|_H^2\,d\Xi_{t_n} = W_H^2(\mu,\nu),$$

which implies that $\Xi \in \Gamma_o(\Sigma,\nu)$.

Define $\Delta_t := (\pi^2 \times (\pi^3-\pi^1))_\# \Xi_t$ and observe that $\Delta_{t_n} \rightharpoonup \Delta := (\pi^2 \times (\pi^3-\pi^1))_\# \Xi$. Since $\pi^1_\# \Delta_t = \pi^2_\# \Sigma$, we have

$$\int_{E\times E}|x_1|_H^2\,d\Delta_t = \int_{E\times E}|x_2|_H^2\,d\Sigma = |\Sigma|_{2,2}^2 < \infty.$$

Moreover,

$$\int_{E\times E} |x_2|_H^2 \, d\Delta_t = \int_{E^3} |x_3 - x_1|_H^2 \, d\Xi_t = W_H^2(\mu,\nu).$$

Since $\Xi \in \Gamma_o(\Sigma,\nu)$ we also have

$$\int_{E\times E} |x_2|_H^2 \, d\Delta = \int_{E^3} |x_3 - x_1|_H^2 \, d\Xi = W_H^2(\mu,\nu).$$

Therefore, we can apply Proposition 6.14 to obtain

$$\begin{aligned}\lim_{n\to\infty} \int_{E^3} [x_2, x_3 - x_1]_H \, d\Xi_{t_n} &= \lim_{n\to\infty} \int_{E\times E} [x_1, x_2]_H \, d\Delta_{t_n} \\ &= \int_{E\times E} [x_1, x_2]_H \, d\Delta = \int_{E^3} [x_2, x_3 - x_1]_H \, d\Xi,\end{aligned}$$

which completes the proof. □

We proceed with a variation of [4, Lemma 10.1.2], which is based on an argument first used by Otto. This result has also been useful in the investigation of invariance of closed convex sets under Wasserstein gradient flows [104].

Proposition 8.12. *Let $\phi : \mathscr{P}(E) \to \mathbb{R}\cup\{\infty\}$ be proper, lsc, and λ-convex for some $\lambda \in \mathbb{R}$. Let $\mu \in \overline{\mathsf{D}(\phi)}$ and $\nu \in \mathsf{D}(\phi)$, and let $\widetilde{\Sigma}_h \in \Gamma_o(J_h\mu, \mu)$ for some $h \in I_\lambda$. Take $\widetilde{\Xi}_h \in \Gamma(\widetilde{\Sigma}_h, \nu)$, and consider the rescaled plans*

$$\Sigma_h := (\pi^1 \times \frac{\pi^2 - \pi^1}{h})_\# \widetilde{\Sigma}_h, \qquad \Xi_h := (\pi^1 \times \frac{\pi^2 - \pi^1}{h} \times \pi^3)_\# \widetilde{\Xi}_h.$$

Then

$$\phi(\nu) - \phi(J_h\mu) \ge \int_{E^3} [x_2, x_3 - x_1] \, d\Xi_h - \frac{1}{2h} W_{\Xi_h}^2 (J_h\mu, \nu).$$

In particular,

$$\Sigma_h \in \partial^s \phi(J_h\mu).$$

Proof. The minimizing property of $J_h\mu$ combined with the identity $\frac{1}{2}|a|_H^2 - \frac{1}{2}|b|_H^2 = [a, a-b]_H - \frac{1}{2}|a-b|_H^2$ implies that

$$\begin{aligned}\phi(\nu) - \phi(J_h\mu) &\ge \frac{1}{2h} W_H^2(J_h\mu, \mu) - \frac{1}{2h} W_H^2(\nu, \mu) \\ &\ge \frac{1}{2h} \int_{E^3} |x_2 - x_1|_H^2 \, d\widetilde{\Xi}_h - \frac{1}{2h} \int_{E^3} |x_3 - x_2|_H^2 \, d\widetilde{\Xi}_h \\ &= \int_{E^3} [\frac{x_2 - x_1}{h}, x_3 - x_1]_H \, d\widetilde{\Xi}_h - \frac{1}{2h} \int_{E^3} |x_3 - x_1|_H^2 \, d\widetilde{\Xi}_h \\ &= \int_{E^3} [x_2, x_3 - x_1]_H \, d\Xi_h - \frac{1}{2h} \int_{E^3} |x_3 - x_1|_H^2 \, d\Xi_h \\ &= \int_{E^3} [x_2, x_3 - x_1]_H \, d\Xi_h - \frac{1}{2h} W_{\Xi_h}^2 (J_h\mu, \nu).\end{aligned}$$

The last statement is an immediate consequence. □

8.4 Regularity and interpolation of subdifferentials

The following result expresses a regularity property of λ-convex functionals (see also [4, Lemma 10.3.8] and [150, Theorem 3.11]).

Proposition 8.13. *Let $\phi : \mathscr{P}(E) \to \mathbb{R} \cup \{\infty\}$ be proper, lsc, and λ-convex for some $\lambda \in \mathbb{R}$. Let $\mu \in \mathsf{D}(\phi)$ and $\Sigma \in \mathscr{P}(E \times E)$. If $\mu_n \in \mathsf{D}(\partial\phi)$ and $\Sigma_n \in \partial\phi(\mu_n)$ satisfy*

$$\lim_{n\to\infty} W_H(\mu_n, \mu) = 0, \qquad \Sigma_n \rightharpoonup \Sigma, \qquad \sup_{n\geq 1} |\Sigma_n|_{2,2} < \infty,$$

then $\mu \in \mathsf{D}(\partial\phi)$ and $\Sigma \in \partial\phi(\mu)$.

Proof. Let $\nu \in \mathsf{D}(\phi)$ be such that $W_H(\mu,\nu) < \infty$. We may assume that $W_H(\mu_n,\nu) < \infty$ for all $n \geq 1$. By Theorem 8.11 there exists $\Xi_n \in \Gamma_o(\Sigma_n, \nu)$ such that

$$\phi(\nu) - \phi(\mu_n) \geq \int_{E^3} [x_2, x_3 - x_1]_H \, d\Xi_n + \frac{\lambda}{2} W_H^2(\mu_n, \nu). \tag{8.9}$$

Since ϕ is lsc, there exists a subsequence $(\mu_{n_k})_{k\geq 1}$ for which

$$\lim_{k\to\infty} \phi(\mu_{n_k}) = \varliminf_{n\to\infty} \phi(\mu_n) \geq \phi(\mu).$$

Since $\mu_{n_k} \rightharpoonup \mu$ by Proposition 6.10 and $\Sigma_{n_k} \rightharpoonup \Sigma$, the collection $(\Xi_n)_{n\geq 1}$ is tight. Let $(\Xi_{n_k})_{k\geq 1}$ be a subsequence converging weakly to $\Xi \in \Gamma(\Sigma, \nu)$. Note that $\Xi \in \Gamma_o(\Sigma, \nu)$, since Lemma 6.2 implies that

$$\begin{aligned}
\int_{E^3} |x_1 - x_3|_H^2 \, d\Xi &\leq \varliminf_{k\to\infty} \int_{E^3} |x_1 - x_3|_H^2 \, d\Xi_{n_k} \\
&= \varliminf_{k\to\infty} W_H^2(\mu_{n_k}, \nu) = W_H^2(\mu, \nu).
\end{aligned}$$

Using (8.9) we obtain

$$\begin{aligned}
\phi(\nu) - \phi(\mu) &\geq \lim_{k\to\infty} \phi(\nu) - \phi(\mu_{n_k}) \\
&\geq \varliminf_{k\to\infty} \int_{E^3} [x_2, x_3 - x_1]_H \, d\Xi_{n_k} + \frac{\lambda}{2} W_H^2(\mu_{n_k}, \nu) \\
&= \varliminf_{k\to\infty} \int_{E\times E} [x_1, x_2]_H \, d\Delta_{n_k} + \frac{\lambda}{2} W_H^2(\mu, \nu),
\end{aligned} \tag{8.10}$$

where $\Delta_n := (\pi^2 \times (\pi^3 - \pi^1))_\# \Xi_n$. Note that $\Delta_{n_k} \rightharpoonup \Delta := (\pi^2 \times (\pi^3 - \pi^1))_\# \Xi$. Since $\pi^1_\# \Delta_{n_k} = \pi^2_\# \Xi_{n_k} = \pi^2_\# \Sigma_{n_k}$, we have

$$\sup_{k\geq 1} \int_{E\times E} |x_1|_H^2 \, d\Delta_{n_k} = \sup_{k\geq 1} |\Sigma_{n_k}|_{2,2}^2 < \infty.$$

Moreover, since $W_H(\mu_{n_k},\mu) \to 0$ and $\Xi_{n_k} \rightharpoonup \Xi$, Proposition 6.11 implies that

$$\lim_{M\to\infty} \overline{\lim_{k\to\infty}} \int_{\{|x_2|_H \geq M\}} |x_2|_H^2 \, d\Delta_{n_k}$$
$$= \lim_{M\to\infty} \overline{\lim_{k\to\infty}} \int_{\{|x_3-x_1|_H \geq M\}} |x_3 - x_1|_H^2 \, d\Xi_{n_k} = 0.$$

Combining this with the identity

$$\lim_{k\to\infty} \int_{E\times E} |x_2|_H^2 \, d\Delta_{n_k} = \lim_{k\to\infty} W_H^2(\mu_{n_k},\nu) \to W_H^2(\mu,\nu) = \int_{E\times E} |x_2|_H^2 \, d\Delta,$$

we may apply Proposition 6.14 to obtain

$$\lim_{k\to\infty} \int_{E\times E} [x_1,x_2]_H \, d\Delta_{n_k} = \int_{E\times E} [x_1,x_2]_H \, d\Delta = \int_{E^3} [x_2, x_3 - x_1]_H \, d\Xi.$$

It follows from (8.10) that

$$\phi(\nu) - \phi(\mu) \geq \int_{E^3} [x_2, x_3 - x_1]_H \, d\Xi + \frac{\lambda}{2} W_H^2(\mu,\nu),$$

which together with Theorem 8.11 implies the desired result. □

In the next lemma, taken from [4, Proposition 7.3.1], we write E_i to denote identical copies of E.

Lemma 8.14. *Let $t \in (0,1)$, and let $\Xi^l, \Xi^r \in \mathscr{P}(E^3)$ satisfy the compatibility condition $(\pi_t^{1,2\to3})_\# \Xi^l = \pi_\#^{1,2} \Xi^r$. Then there exists $\Upsilon \in \mathscr{P}(E^4)$ such that*

$$\pi_\#^{1,2,3} \Upsilon = \Xi^l, \qquad (\pi_t^{1,2\to3,4})_\# \Upsilon = \Xi^r.$$

Proof. Set $\widehat{\Xi}^l := (\pi_t^{1,2,2\to3})_\# \Xi^l$. A measure $\Upsilon \in \mathscr{P}(E^4)$ has the desired properties if and only if $\widehat{\Upsilon} := (\pi_t^{1,2,2\to3,4})_\# \Upsilon$ satisfies

$$\pi_\#^{1,2,3} \widehat{\Upsilon} = \widehat{\Xi}^l, \qquad \pi_\#^{1,3,4} \widehat{\Upsilon} = \Xi^r.$$

This can easily be arranged. Indeed, set $\Sigma := \pi_\#^{1,3} \widehat{\Xi}^l = \pi_\#^{1,2} \Xi^r$ and use disintegration to write

$$\widehat{\Xi}^l = \int_{E_1\times E_3} \mu_{x_1,x_3} \, d\Sigma(x_1,x_3), \qquad \Xi^r = \int_{E_1\times E_2} \nu_{x_1,x_2} \, d\Sigma(x_1,x_2),$$

for suitable $\mu_{x_1,x_3} \in \mathscr{P}(E_2)$ and $\nu_{x_1,x_2} \in \mathscr{P}(E_3)$. Then

$$\widehat{\Upsilon} := \int_{E_1\times E_3} \mu_{x_1,x_3} \otimes \nu_{x_1,x_3} \, d\Sigma(x_1,x_3),$$

has the required properties. □

It turns out that strong subdifferentials enjoy good interpolation properties. The next result is proved in Hilbert spaces in [4, Lemma 10.3.12].

Proposition 8.15 (Interpolation of strong subdifferentials). *Let $\phi : \mathscr{P}(E) \to \mathbb{R} \cup \{\infty\}$ be proper and lsc. Let $\mu \in \mathsf{D}(\partial^s\phi)$ and take $\Sigma^2, \Sigma^3 \in \partial^s\phi(\mu)$. Let $\Xi \in \mathscr{P}(E^3)$ satisfy $\pi^{1,2}_{\#}\Xi = \Sigma^2$ and $\pi^{1,3}_{\#}\Xi = \Sigma^3$. For $t \in [0,1]$ we have*

$$\Sigma_t := (\pi_t^{1,2\to 3})_{\#}\Xi \in \partial^s\phi(\mu).$$

Proof. Let $\nu \in \mathsf{D}(\phi)$ and $\Xi_t \in \Gamma(\Sigma_t, \nu)$. By Lemma 8.14 there exists $\Upsilon \in \mathscr{P}(E^4)$ such that

$$\pi^{1,2,3}_{\#}\Upsilon = \Xi, \qquad (\pi_t^{1,2\to 3,4})_{\#}\Upsilon = \Xi_t.$$

Observe that

$$\pi^{1,2,4}_{\#}\Upsilon \in \Gamma(\Sigma^2, \nu), \qquad \pi^{1,3,4}_{\#}\Upsilon \in \Gamma(\Sigma^3, \nu).$$

Since $\Sigma^2, \Sigma^3 \in \partial\phi^s(\mu)$,

$$\begin{aligned}
\phi(\nu) - \phi(\mu) &\geq \int_{E^4} [x_2, x_4 - x_1]_H \, d\Upsilon + o(W_\Upsilon(\mu,\nu)), \\
\phi(\nu) - \phi(\mu) &\geq \int_{E^4} [x_3, x_4 - x_1]_H \, d\Upsilon + o(W_\Upsilon(\mu,\nu)).
\end{aligned}$$

Taking weighted averages and using that

$$W_\Upsilon^2(\mu,\nu) = \int_{E^4} |x_1 - x_4|_H^2 \, d\Upsilon = \int_{E^3} |x_1 - x_3|_H^2 \, d\Xi_t = W_{\Xi_t}^2(\mu,\nu),$$

we obtain

$$\begin{aligned}
\phi(\nu) - \phi(\mu) &\geq \int_{E^4} [(1-t)x_2 + t x_3, x_4 - x_1]_H \, d\Upsilon + o(W_{\Xi_t}(\mu,\nu)) \\
&\geq \int_{E^3} [x_2, x_3 - x_1]_H \, d\Xi_t + o(W_{\Xi_t}(\mu,\nu)),
\end{aligned}$$

which means that $\Xi_t \in \partial^s\phi(\mu)$. □

8.5 Minimal selection

In this section we shall show that for proper lsc functionals which are λ-convex along generalised geodesics, the domain of the slope and the domain of the subdifferential coincide. Moreover, we shall prove that the subdifferential contains a unique element of minimal length if the subdifferential is non-empty. This element will play a distinguished role in the theory of gradient flows in Chapter 9.

In the proof of Theorem 8.17 we will use the following lemma which is taken from [4].

Lemma 8.16. *Let X be a separable metric space, and let $r_n : X \to X$ be a sequence of Borel maps which converge uniformly on compact sets to a continuous map r. If $(\mu_n)_{n\geq 1} \subseteq \mathscr{P}(E)$ is tight and weakly converging to $\mu \in \mathscr{P}(E)$, then $(r_n)_\# \mu_n$ converges weakly to $r_\# \mu$.*

Proof. See [4, Lemma 5.2.1]. $\square$

Theorem 8.17. *Assume (H). Let $\phi : \mathscr{P}(E) \to \mathbb{R} \cup \{\infty\}$ be a proper lsc functional satisfying (A1), which is λ-convex along generalised geodesics for some $\lambda \in \mathbb{R}$. Then*

$$\mathsf{D}(|\partial\phi|) = \mathsf{D}(\partial\phi),$$

and for any $\mu \in \mathsf{D}(|\partial\phi|)$ we have

$$|\partial\phi|(\mu) = \min\{|\Sigma|_{2,2} : \Sigma \in \partial\phi(\mu)\}.$$

The existence of a minimizer is part of the assertion.

Proof. Suppose first that $\mu \in \mathsf{D}(\partial\phi)$ and $\Sigma \in \partial\phi(\mu)$. For every $\nu \in \mathsf{D}(\phi)$ we can find $\Xi \in \Gamma_o(\Sigma, \nu)$ such that

$$\begin{aligned}
\phi(\mu) - \phi(\nu) &\leq \int_{E^3} [x_2, x_1 - x_3]_H \, d\Xi + o(W_H(\mu,\nu)) \\
&\leq \left(\int_{E^3} |x_2|_H^2 \, d\Xi \right)^{1/2} \left(\int_{E^3} |x_1 - x_3|_H^2 \, d\Xi \right)^{1/2} + o(W_H(\mu,\nu)) \\
&= |\Sigma|_{2,2} W_H(\mu,\nu) + o(W_H(\mu,\nu)),
\end{aligned}$$

from which we infer that $\mu \in \mathsf{D}(|\partial\phi|)$ and

$$|\partial\phi|(\mu) = \varlimsup_{W_H(\mu,\nu)\to 0} \frac{(\phi(\mu) - \phi(\nu))^+}{W_H(\mu,\nu)} \leq |\Sigma|_{2,2}. \tag{8.11}$$

This proves that $\mathsf{D}(|\partial\phi|) \supseteq \mathsf{D}(\partial\phi)$ and

$$|\partial\phi|(\mu) \leq \inf\big\{|\Sigma|_{2,2} : \Sigma \in \partial\phi(\mu)\big\}. \tag{8.12}$$

Conversely, (8.6) implies that for $\mu \in \mathsf{D}(|\partial\phi|)$,

$$|\partial\phi|^2(\mu) = \lim_{h\downarrow 0} \frac{W_H^2(J_h\mu, \mu)}{h^2} = 2 \lim_{h\downarrow 0} \frac{\phi(\mu) - \phi_h(\mu)}{h}. \tag{8.13}$$

On the other hand, for $\tilde{\Sigma}_h \in \Gamma_o(J_h\mu, \mu)$ and $\Sigma_h := (\pi^1 \times \frac{\pi^2 - \pi^1}{h})_\# \tilde{\Sigma}_h$, we have

$$|\Sigma_h|_{2,2}^2 = \int_{E\times E} |x_2|_H^2 \, d\Sigma_h = \frac{1}{h^2} \int_{E\times E} |x_1 - x_2|_H^2 \, d\tilde{\Sigma}_h = \frac{W_H^2(J_h\mu, \mu)}{h^2}, \tag{8.14}$$

for each $h \in I_\lambda$. Moreover, Proposition 8.12 implies that $\Sigma_h \in \partial^s\phi(J_h\mu)$. Our goal is to produce an element in $\partial\phi(\mu)$ by appealing to Proposition 8.13. We observe that

- $W_H(J_h\mu,\mu) \to 0$ as $h \downarrow 0$ by (8.2).
- $(\Sigma_{h_n})_{n\geq 1}$ is tight for some sequence $h_n \downarrow 0$. Indeed, to prove this, it suffices to show the tightness of both marginals. Since $\pi^1_\# \Sigma_h = J_h\mu$ and $W_H(J_h\mu,\mu) \to 0$ as $h \downarrow 0$ by (8.13), the tightness of $(\pi^1_\# \Sigma_{h_n})_{n\geq 1}$ follows from Proposition 6.10. Using (8.13) and (8.14) we find

$$\sup_{n\geq 1} |\Sigma_{h_n}|^2_{2,2} = \sup_{n\geq 1} \int_E |x|^2_H \, d\pi^2_\# \Sigma_{h_n} = \sup_{n\geq 1} \frac{W_H^2(J_{h_n}\mu,\mu)}{h_n^2} < \infty,$$

so that Proposition 6.12 implies that $(\pi^2_\# \Sigma_{h_n})_{n\geq 1}$ is tight. By passing to a subsequence we may assume that Σ_{h_n} converges weakly to some $\bar\Sigma \in \mathscr{P}(E \times E)$.

Proposition 8.13 implies that $\mu \in \mathsf{D}(\partial\phi)$ and $\bar\Sigma \in \partial\phi(\mu)$. This proves that $\mathsf{D}(|\partial\phi|) \subseteq \mathsf{D}(\partial\phi)$.

To prove the final assertion, we note that by Lemma 6.2 and (8.13),

$$|\bar\Sigma|^2_{2,2} \leq \varliminf_{n\to\infty} |\Sigma_{h_n}|^2_{2,2} = \varliminf_{n\to\infty} \frac{W_H^2(J_{h_n}\mu,\mu)}{h_n^2} = |\partial\phi|^2(\mu).$$

Combining this with (8.12) yields the desired result. □

Using the method devised in [4, Theorem 10.3.11] we show that the minimizer obtained in Theorem 8.17 is unique.

Theorem 8.18. *Assume (H). Let $\phi : \mathscr{P}(E) \to \mathbb{R} \cup \{\infty\}$ be a proper lsc functional satisfying (A1), which is λ-convex along generalised geodesics for some $\lambda \in \mathbb{R}$. Let $\mu \in \mathsf{D}(\partial\phi)$ and suppose that $\Sigma^2, \Sigma^3 \in \partial\phi(\mu)$ satisfy*

$$|\Sigma^2|_{2,2} = |\Sigma^3|_{2,2} = |\partial\phi|(\mu).$$

Then $\Sigma^2 = \Sigma^3$.

In this situation, we will denote by $\partial^\circ\phi(\mu)$ the unique element in $\partial\phi(\mu)$ satisfying

$$|\partial^\circ\phi(\mu)|_{2,2} = |\partial\phi|(\mu),$$

which exists in view of Theorem 8.17

Proof. We proceed in several steps.

Step 1: We shall show that, for $i = 2,3$, there exists a sequence $h_n \downarrow 0$ and *strong* subdifferentials $\Sigma^i_{h_n} \in \partial^s\phi(J_{h_n}\mu)$ such that

$$\Sigma^i_{h_n} \rightharpoonup \Sigma^i, \qquad |\Sigma^i_{h_n}|_{2,2} \to |\Sigma^i|_{2,2} = |\partial\phi|(\mu).$$

Suppose that $\Sigma \in \partial\phi(\mu)$ and $|\Sigma|_{2,2} = |\partial\phi|$. For each $h \in I_\lambda$ there exists $\widehat{\Xi}_h \in \Gamma_o(\Sigma, J_h\mu)$ such that

$$\phi(J_h\mu) - \phi(\mu) \geq \int_{E^3} [x_2, x_3 - x_1]_H \, d\widehat{\Xi}_h + o(W_H(J_h\mu, \mu)). \tag{8.15}$$

We define $\Xi_h := (\pi^1 \times \pi^2 \times \frac{\pi^1 - \pi^3}{h})_\# \widehat{\Xi}_h$ and claim that $(\Xi_h)_{h\in(0,\delta)}$ is tight for some $\delta > 0$. Indeed,

$$\int_E |x|_H^2 \, d(\pi^3_\# \Xi_h) = \int_{E^3} |x_3|_H^2 \, d\Xi_h = \frac{1}{h^2} \int_{E^3} |x_1 - x_3|_H^2 \, d\widehat{\Xi}_h = \frac{1}{h^2} W_H^2(J_h\mu, \mu).$$

Since

$$\lim_{h\to 0} \frac{1}{h^2} W_H^2(\mu, J_h\mu) = |\partial\phi|^2(\mu)$$

by (8.6), the tightness of $(\pi^3_\# \Xi_h)_{h\in(0,\delta)}$ for some $\delta > 0$ follows from Proposition 6.12. Since $\pi^{1,2}_\# \Xi = \Sigma$, we conclude that $(\Xi_h)_{h\in(0,\delta)}$ is tight.

Consider a sequence $h'_n \downarrow 0$ and let $h_n \downarrow 0$ be a subsequence such that Ξ_{h_n} converges weakly to some limit point Ξ. Lemma 6.2 implies that

$$\int_{E^3} |x_3|_H^2 \, d\Xi \leq \varliminf_{n\to\infty} \int_{E^3} |x_3|_H^2 \, d\Xi_{h_n} = |\partial\phi|^2(\mu). \tag{8.16}$$

On the other hand, since $\pi^{1,2}_\# \Xi = \Sigma$,

$$\int_{E^3} |x_2|_H^2 \, d\Xi = |\Sigma|_{2,2}^2 = |\partial\phi|^2(\mu). \tag{8.17}$$

Using (8.6) and (8.15) we arrive at

$$\begin{aligned} |\partial\phi|^2(\mu) &= \lim_{n\to\infty} \frac{\phi(\mu) - \phi(J_{h_n}\mu)}{h_n} \\ &\leq \varliminf_{n\to\infty} \frac{1}{h_n} \int_{E^3} [x_2, x_1 - x_3]_H \, d\widehat{\Xi}_{h_n} + \frac{o(W_H(J_{h_n}\mu, \mu))}{h_n} \\ &= \varliminf_{n\to\infty} \int_{E^3} [x_2, x_3]_H \, d\Xi_{h_n} \\ &= \int_{E^3} [x_2, x_3]_H \, d\Xi. \end{aligned} \tag{8.18}$$

The last step in this computation is justified by Proposition 6.14, which can be applied since $\pi^{1,2}_\# \Xi_h = \Sigma$, hence

$$\lim_{R\to\infty} \varlimsup_{h\downarrow 0} \int_{\{|x_2|_H \geq R\}} |x_2|_H^2 \, d\Xi_h = \lim_{R\to\infty} \int_{\{|x_2|_H \geq R\}} |x_2|_H^2 \, d\Sigma = 0,$$

and, for $\delta > 0$ small enough,

$$\sup_{h\in(0,\delta)} \int_{E^3} |x_3|_H^2 \, d\Xi_h = \sup_{h\in(0,\delta)} \frac{1}{h^2} W_H^2(J_h\mu, \mu) < \infty.$$

Combining (8.16), (8.17) and (8.18), we arrive at

$$\frac{1}{2}\int_{E^3} |x_2|_H^2 + |x_3|_H^2 \, d\varXi \le |\partial\phi|^2(\mu) \le \int_{E^3} [x_2, x_3]_H \, d\varXi,$$

from which we infer that $\varXi(\{x_2 = x_3\}) = 1$. Combined with the fact that $\pi^{1,2}_{\#}\varXi = \varSigma$, we conclude that $\varXi = (\pi^1 \times \pi^2 \times \pi^2)_{\#}\varSigma$. This implies that

$$\pi^{1,3}_{\#}\varXi_{h_n} \rightharpoonup \pi^{1,3}_{\#}\varXi = \pi^{1,2}_{\#}\varXi = \varSigma.$$

On the other hand, an application of Lemma 8.16 with $r_n := (\pi^1 - h_n\pi^3) \times \pi^3$ gives

$$\varSigma_{h_n} := \big((\pi^1 - h_n\pi^3) \times \pi^3\big)_{\#}\varXi_{h_n} \rightharpoonup \pi^{1,3}_{\#}\varXi = \varSigma.$$

Since $\pi^{3,1}_{\#}\widehat{\varXi}_{h_n} \in \varGamma_o(J_{h_n}\mu, \mu)$, Proposition 8.12 implies that

$$\varSigma_{h_n} = \Big(\pi^3 \times \frac{\pi^1 - \pi^3}{h_n}\Big)_{\#}\widehat{\varXi}_{h_n} = \Big(\pi^1 \times \frac{\pi^2 - \pi^1}{h_n}\Big)_{\#}\big(\pi^{3,1}_{\#}\widehat{\varXi}_{h_n}\big) \in \partial^s\phi(J_{h_n}\mu).$$

It follows from (8.6) that

$$\varlimsup_{n\to\infty} \int_{E^3} |x_3|_H^2 \, d\varXi_{h_n} = \varlimsup_{n\to\infty} \frac{1}{h_n^2} W_H^2(J_{h_n}\mu, \mu) = |\partial\phi|^2(\mu),$$

which together with (8.17) yields that

$$|\varSigma_{h_n}|_{2,2}^2 = \int_{E^3} |x_3|_H^2 \, d\varXi_{h_n} \to |\varSigma|_{2,2}^2,$$

hence $\varSigma$ has the desired properties. The proof of Step 1 is completed by first applying this procedure to $\varSigma := \varSigma^2$ and then to $\varSigma := \varSigma^3$.

Step 2: For $i = 2, 3$, let $(\varSigma^i_{h_n})_{n\ge1}$ be as in Step 1, and take $\varUpsilon_{h_n} \in \mathscr{P}(E^3)$ with $\pi^{1,i}_{\#}\varUpsilon_{h_n} = \varSigma^i_{h_n}$. The tightness of $(\varSigma^i_{h_n})_{n\ge1}$ implies that $(\varUpsilon_{h_n})_{n\ge1}$ is tight as well. Passing to a subsequence we may assume that $\varUpsilon_{h_n} \rightharpoonup \varUpsilon$. We will show that

$$\widehat{\varSigma} := (\pi^{1,2\to3}_{1/2})_{\#}\varUpsilon \in \partial\phi(\mu).$$

To show this, we define $\widehat{\varSigma}_{h_n} := (\pi^{1,2\to3}_{1/2})_{\#}\varUpsilon_{h_n}$ and check the conditions from Proposition 8.13:

- Proposition 8.2(ii) asserts that $J_{h_n}\mu \to \mu$ in W_H-distance.
- Proposition 8.15 implies that $\widehat{\varSigma}_{h_n} \in \partial^s\phi(J_{h_n}\mu)$.
- Since $\varUpsilon_{h_n} \rightharpoonup \varUpsilon$, we have $\widehat{\varSigma}_{h_n} \rightharpoonup \widehat{\varSigma}$.

– Finally, for each $n \geq 1$ we have, using the triangle inequality in $L^2(\Upsilon_{h_n}; H)$ and the fact that $\pi^{1,i}_{\#}\Upsilon_{h_n} = \Sigma^i_{h_n}$ for $i = 2, 3$,

$$\begin{aligned}
|\widehat{\Sigma}_{h_n}|_{2,2} &= \left(\int_{E\times E} |x_2|^2_H \, d\widehat{\Sigma}_{h_n} \right)^{1/2} \\
&= \left(\int_{E^3} |\frac{1}{2}x_2 + \frac{1}{2}x_3|^2_H \, d\Upsilon_{h_n} \right)^{1/2} \\
&\leq \frac{1}{2}\left(\int_{E^3} |x_2|^2_H \, d\Upsilon_{h_n} \right)^{1/2} + \frac{1}{2}\left(\int_{E^3} |x_3|^2_H \, d\Upsilon_{h_n} \right)^{1/2} \\
&= \frac{1}{2}\left(\int_{E\times E} |x_2|^2_H \, d\Sigma^2_{h_n} \right)^{1/2} + \frac{1}{2}\left(\int_{E\times E} |x_2|^2_H \, d\Sigma^3_{h_n} \right)^{1/2} \\
&= \frac{1}{2}|\Sigma^2_{h_n}|_{2,2} + \frac{1}{2}|\Sigma^3_{h_n}|_{2,2}.
\end{aligned}$$

Taking Step 1 into account, we infer that $\sup_{n\geq 1} |\widehat{\Sigma}_{h_n}|_{2,2} < \infty$.

Now we can apply Proposition 8.13 to conclude that $\mu \in \mathsf{D}(\partial\phi)$ and $\widehat{\Sigma} \in \partial\phi(\mu)$.

Step 3: We complete the proof. Using the fact that $\widehat{\Sigma} = (\pi^{1,2\to 3}_{1/2})_{\#}\Upsilon$ and the parallelogram identity in $L^2(\Upsilon; H)$, we obtain

$$\begin{aligned}
|\widehat{\Sigma}|^2_{2,2} &= \|\frac{1}{2}\pi^2 + \frac{1}{2}\pi^3\|^2_{L^2(\Upsilon;H)} \\
&= \frac{1}{2}\|\pi^2\|^2_{L^2(\Upsilon;H)} + \frac{1}{2}\|\pi^3\|^2_{L^2(\Upsilon;H)} - \frac{1}{4}\|\pi^2 - \pi^3\|^2_{L^2(\Upsilon;H)} \\
&= |\partial\phi|^2(\mu) - \frac{1}{4}\|\pi^2 - \pi^3\|^2_{L^2(\Upsilon;H)}.
\end{aligned}$$

Theorem 8.17 enforces that $\pi^2 = \pi^3$ Υ-a.e., hence $\Sigma^2 = \pi^{1,2}_{\#}\Upsilon = \pi^{1,3}_{\#}\Upsilon = \Sigma^3$. □

9

Gradient Flows

In this chapter we will consider gradient flows in the space $\mathscr{P}(E)$ endowed with the Wasserstein metric W_H. Gradient flows can be defined in purely metric terms, by means of an evolution variational inequality. Alternatively, using the velocity fields and subdifferentials considered in the preceding chapters, a more differential geometric formulation can be given in the Wasserstein space. In this chapter we will show that both approaches are equivalent under appropriate convexity conditions on the functional. Hilbert space and Wiener space versions of such results can be found in [4] and [150] respectively.

First we collect some fundamental facts from the theory of gradient flows in metric spaces developed in [4].

9.1 Metric properties

Let (X, d) be a complete metric space.

The following definition of a gradient flow is based on an evolution variational inequality.

Definition 9.1. *Let $\phi : X \to \mathbb{R} \cup \{\infty\}$ be a proper, lsc functional. A map $u \in C([0,\infty); X) \cap AC_{\mathrm{loc}}((0,\infty); X)$ is said to be a gradient flow for ϕ if there exists $\lambda \in \mathbb{R}$ such that for any $y \in \mathsf{D}(\phi)$,*

$$\frac{1}{2}\partial_t d^2(u(t), y) + \frac{\lambda}{2} d^2(u(t), y) \leq \phi(y) - \phi(u(t)) \tag{9.1}$$

a.e. on $(0, \infty)$.

The following result is one of the main results in the general theory of gradient flows in metric spaces from [4]. This theory generalises the Hilbert space results, which can be found in [18].

Let $J \subseteq \mathbb{R}$ be an open interval. For a function $u : J \to X$ and $t \in J$ we let

$$|u'_+|(t) := \lim_{h\downarrow 0} \frac{d(u(t+h),u(t))}{h}$$

denote the *right metric derivative*, provided this limit exists.

Theorem 9.2. *Let $\phi : X \to \mathbb{R}\cup\{\infty\}$ be a proper lsc functional satisfying the assumptions (A1) and (A2) from Section 8.1. For each $u_0 \in \overline{\mathsf{D}(\phi)}$ there exists a unique gradient flow*

$$u(\cdot) := S(\cdot)u_0 \in C([0,\infty);X) \cap AC_{\mathrm{loc}}((0,\infty);X)$$

(in the sense of Definition 9.1 with λ as in (A2)) satisfying $u(0) = u_0$. Moreover, the following properties hold:

(i) (Exponential formula) For $t > 0$ we have

$$u(t) = \lim_{n\to\infty} J^n_{t/n} u_0.$$

(ii) (Regularising effect) For $t > 0$ we have $u(t) \in \mathsf{D}(|\partial\phi|) \subseteq \mathsf{D}(\phi)$ and

$$\phi(u(t)) \le \phi(\sigma) + \frac{1}{2t} d^2(u_0,\sigma), \qquad \sigma \in \mathsf{D}(\phi),$$
$$|\partial\phi|^2(u(t)) \le |\partial\phi|^2(\sigma) + \frac{1}{t^2} d^2(u_0,\sigma), \qquad \sigma \in \mathsf{D}(|\partial\phi|).$$

(iii) (λ-contractive semigroup) For $s,t \ge 0$ and $\widehat{u}_0 \in \overline{\mathsf{D}(\phi)}$ we have

$$S(s+t)u_0 = S(s)S(t)u_0, \qquad d(S(t)u_0, S(t)\widehat{u}_0) \le e^{-\lambda t} d(u_0,\widehat{u}_0).$$

(iv) If $u_0 \in \mathsf{D}(\phi)$ and $t > 0$ we have

$$-\partial_t^+ \phi(u(t)) = |\partial\phi|^2(u(t)) = |u'_+|^2(t) = |\partial\phi|(u(t))\,|u'_+|(t).$$

In the first term of (iv), ∂_t^+ denotes the right derivative.

Proof. See [4, Theorems 2.4.15 & 4.0.4]. □

9.2 Differential properties

- Throughout this section we assume that (H) holds.

The very definition of a gradient flow (9.1) suggests that it might be useful to calculate the derivative of the squared Wasserstein distance along absolutely continuous paths:

Proposition 9.3. *Let $J \subseteq \mathbb{R}$ be an interval, let $(\mu_t)_{t\in J} \in AC^2(J;\mathscr{P}_f(E))$ with velocity field $Z \in L^2(M_\mu;H)$ as defined in Theorem 7.2, and let $\nu \in \mathscr{P}_{H,\mu_s}(E)$ for some $s \in J$. For a.e. $t \in J$ and any $\Sigma \in \Gamma_o(\mu_t,\nu)$ we have*

$$\frac{1}{2}\partial_t W_H^2(\mu_t,\nu) = \int_{E\times E} [Z_t(x), x-y]_H \, d\Sigma(x,y).$$

Proof. For any $t \in J$ for which (7.9) holds, we have

$$\partial_t W_H^2(\mu_t, \nu) = \lim_{h \to 0} \frac{W_H^2((I + hZ_t)_\# \mu_t, \nu) - W_H^2(\mu_t, \nu)}{h},$$

and for any $h \in \mathbb{R}$ we find

$$\begin{aligned} W_H^2((I + hZ_t)_\# \mu_t, \nu) &- W_H^2(\mu_t, \nu) \\ &\leq \int_{E \times E} |x + hZ_t(x) - y|^2 - |x - y|^2 \, d\Sigma(x, y) \\ &= \int_{E \times E} 2h[Z_t(x), x - y]_H + h^2 |Z_t(x)|_H^2 \, d\Sigma(x, y). \end{aligned}$$

Letting $h \downarrow 0$, we arrive at

$$\frac{1}{2} \partial_t W_H^2(\mu_t, \nu) \leq \int_{E \times E} [Z_t(x), x - y]_H \, d\Sigma(x, y).$$

The reverse inequality follows by passing to the limit $h \uparrow 0$. □

The next theorem gives a differential characterisation of gradient flows. In its proof we need the following simple lemma.

Lemma 9.4. *Let X, Y, Z be Polish spaces, let $\Xi \in \mathscr{P}(X \times Y \times Z)$, let $\mu \in \mathscr{P}(X)$, and suppose that there exists a Borel mapping $r : X \to Y$ satisfying $\pi_\#^{1,2} \Xi = (I_X \times r)_\# \mu$. For every Borel measurable $f : X \times Y \times Z \to [0, \infty]$ we have*

$$\int_{X \times Y \times Z} f(x, y, z) \, d\Xi = \int_{X \times Y \times Z} f(x, r(x), z) \, d\Xi.$$

Proof. By the Disintegration Theorem 6.3 there exists a family of Borel probability measures $(\gamma_{x,y})_{(x,y) \in X \times Y} \subseteq \mathscr{P}(Z)$ such that

$$\Xi = \int_{X \times Y} \gamma_{x,y} \, d(I_X \times r)_\# \mu(x, y).$$

In particular,

$$\begin{aligned} \int\limits_{X \times Y \times Z} f(x, y, z) \, d\Xi &= \int\limits_{X \times Y} \int\limits_Z f(x, y, z) \, d\gamma_{x,y}(z) \, d(I_X \times r)_\# \mu(x, y) \\ &= \int\limits_X \int\limits_Z f(x, r(x), z) \, d\gamma_{x,r(x)}(z) \, d\mu(x), \end{aligned}$$

while on the other hand we have

$$\int\limits_{X\times Y\times Z} f(x,r(x),z)\,d\Xi = \int\limits_{X\times Y}\int\limits_{Z} f(x,r(x),z)\,d\gamma_{x,y}(z)\,d(I_X\times r)_{\#}\mu(x,y)$$
$$= \int\limits_{X}\int\limits_{Z} f(x,r(x),z)\,d\gamma_{x,r(x)}(z)\,d\mu(x).$$

This proves the result. □

The next result shows that the metric definition of a gradient flow is equivalent to a differential inclusion. The second assertion should be interpreted as a Wasserstein version of the differential inclusion

$$u'(t) + \partial\phi(u(t)) \ni 0,$$

which has been studied in Hilbert spaces in [18].

Theorem 9.5. *Let $\phi : \mathscr{P}(E) \to \mathbb{R}\cup\{\infty\}$ be proper, lsc, and λ-convex for some $\lambda \in \mathbb{R}$. Let $(\mu_t)_{t\geq 0} \in C([0,\infty);\mathscr{P}_f(E)) \cap AC^2_{\mathrm{loc}}([0,\infty);\mathscr{P}_f(E))$, and let $Z \in L^2_{\mathrm{loc}}(M_\mu;H)$ be its velocity field as in Theorem 7.2. The following assertions are equivalent:*

(1) $(\mu_t)_{t\geq 0}$ is a gradient flow for ϕ in the sense of Definition 9.1;
(2) For a.e. $t \geq 0$ the following differential inclusion holds:

$$(I_E \times (-Z_t))_{\#}\mu_t \in \partial\phi(\mu_t).$$

Here, the space $L^2_{\mathrm{loc}}(M_\mu;H)$ consists of all (equivalence classes of) Borel functions $Z : [0,\infty)\times E \to H$ satisfying $\int_J\int_E |Z_t(x)|^2_H\,d\mu_t(x)\,dt < \infty$ for all compact subintervals $J \subseteq [0,\infty)$.

Proof. To show that (1) implies (2) we take $\nu \in \mathscr{P}_{H,\mu_0}(E)$. By Definition 9.1 and Proposition 9.3 we obtain for a.e. $t \geq 0$ and any $\Sigma_t \in \Gamma_o(\mu_t,\nu)$,

$$\int_{E\times E} [Z_t(x_1), x_1 - x_2]_H\,d\Sigma_t + \frac{\lambda}{2}W_H^2(\mu_t,\nu) + \phi(\mu_t) \leq \phi(\nu).$$

Put $\Xi_t := (\pi^1\times(-Z_t\circ\pi^1)\times\pi^2)_{\#}\Sigma_t$ and note that $\Xi_t \in \Gamma_o((I_E\times(-Z_t))_{\#}\mu_t,\nu)$. Since the integral appearing above equals

$$\int_{E^3} [x_2, x_3 - x_1]_H\,d\Xi_t,$$

we obtain (2) by virtue of Theorem 8.11.

Conversely, it follows from (2) and Theorem 8.11 that for any $\nu \in \mathscr{P}_{H,\mu_0}(E)$ there exists $\Xi_t \in \Gamma_o((I_E\times(-Z_t))_{\#}\mu_t,\nu)$ such that

$$\int_{E^3} [x_2, x_3 - x_1]_H\,d\Xi_t + \frac{\lambda}{2}W_H^2(\mu_t,\nu) + \phi(\mu_t) \leq \phi(\nu).$$

By Lemma 9.4 the integral equals

$$\int_{E^3} [Z_t(x_1), x_1 - x_3]_H \, d\varXi_t.$$

Since $\pi^{1,3}_{\#}\varXi_t \in \varGamma_o(\mu_t, \nu)$, the result follows from Proposition 9.3. □

The following result is a variation of (11.2.7) – (11.2.9) in [4, Theorem 11.2.1].

Theorem 9.6. *Let $\phi : \mathscr{P}(E) \to \mathbb{R}\cup\{\infty\}$ be a proper, lsc functional satisfying (A1), which is λ-convex along generalised geodesics for some $\lambda \in \mathbb{R}$. Let $\mu_0 \in \mathsf{D}(\phi)$, and let $(\mu_t)_{t\geq 0}$ be the corresponding gradient flow. For $t, h > 0$ there exists $\widehat{\varSigma}_{t,h} \in \varGamma_o(\mu_t, \mu_{t+h})$ such that for any $t > 0$,*

$$\partial^\circ \phi(\mu_t) = \lim_{h\downarrow 0} \Big(\pi^1 \times \frac{\pi^1 - \pi^2}{h}\Big)_{\#} \widehat{\varSigma}_{t,h}, \tag{9.2}$$

where the convergence is understood with respect to $W_{H\times H}$. Moreover,

$$-\partial_t^+ \phi(\mu_t) = |\mu'_+|^2(t) = |\partial\phi|^2(\mu_t) = |\partial^\circ\phi(\mu_t)|^2_{2,2}. \tag{9.3}$$

Proof. Let $t > 0$. The first two equalities in (9.3) follow from the general metric theory of Theorem 9.2(iii), and the last identity has been proved in Theorem 8.17.

Let us prove (9.2). From Theorem 8.17 and Theorem 9.2(ii) we know that $\partial\phi(\mu_t) \neq \varnothing$. Set $\varSigma_t := \partial^\circ\phi(\mu_t)$. Since $\varSigma_t \in \partial\phi(\mu_t)$, for any $h > 0$ there exists $\widehat{\varXi}_{t,h} \in \varGamma_o(\varSigma_t, \mu_{t+h})$ such that

$$\begin{aligned} \frac{\phi(\mu_{t+h}) - \phi(\mu_t)}{h} &\geq \frac{1}{h}\int_{E^3} [x_2, x_3 - x_1]_H \, d\widehat{\varXi}_{t,h} + \frac{o(W_H(\mu_t, \mu_{t+h}))}{h}. \\ &= -\int_{E^3} [x_2, x_3]_H \, d\varXi_{t,h} + \frac{o(W_H(\mu_t, \mu_{t+h}))}{h}, \end{aligned} \tag{9.4}$$

where $\varXi_{t,h} := (\pi^1 \times \pi^2 \times \frac{\pi^1 - \pi^3}{h})_{\#}\widehat{\varXi}_{t,h}$. Since $\mu_t \in \mathsf{D}(|\partial\phi|)$ by Theorem 9.2, we have

$$\varlimsup_{h\downarrow 0} \frac{W_H(\mu_t, \mu_{t+h})}{h} \leq |\mu'|(t) < \infty.$$

Therefore we obtain, as $h \downarrow 0$ in (9.4),

$$\partial_t^+ \phi(\mu_t) \geq \varlimsup_{h\downarrow 0} \Big(-\int_{E^3} [x_2, x_3]_H \, d\varXi_{t,h} \Big).$$

On the other hand, since $\pi^2_{\#}\varXi_{t,h} = \pi^2_{\#}\varSigma_t$ for any $h > 0$, Theorem 9.2(iv) implies that

$$-\partial_t^+ \phi(\mu_t) = |\partial\phi|^2(\mu_t) = |\Sigma_t|_{2,2}^2 = \int_{E^3} |x_2|_H^2 \, d\Xi_{t,h}.$$

Applying Theorem 9.2(iv) once more,

$$\begin{aligned} -\partial_t^+ \phi(\mu_t) = |\mu_+'|^2(t) &= \lim_{h\downarrow 0} \frac{W_H^2(\mu_t, \mu_{t+h})}{h^2} \\ &= \lim_{h\downarrow 0} \frac{1}{h^2} \int_{E^3} |x_3 - x_1|_H^2 \, d\widehat{\Xi}_{t,h} = \lim_{h\downarrow 0} \int_{E^3} |x_3|_H^2 \, d\Xi_{t,h}. \end{aligned}$$

Combining the latter three statements, we obtain

$$\begin{aligned} \overline{\lim_{h\downarrow 0}} \int_{E^3} |x_2 - x_3|_H^2 \, d\Xi_{t,h} = \overline{\lim_{h\downarrow 0}} & \int_{E^3} |x_2|_H^2 \, d\Xi_{t,h} \\ & - 2\int_{E^3} [x_2, x_3]_H \, d\Xi_{t,h} + \int_{E^3} |x_3|_H^2 \, d\Xi_{t,h} \le 0. \end{aligned} \tag{9.5}$$

Set $\widehat{\Sigma}_{t,h} := \pi_{\#}^{1,3} \widehat{\Xi}_{t,h}$ and note that $\widehat{\Sigma}_{t,h} \in \Gamma_o(\mu_t, \mu_{t+h})$. Then

$$\begin{aligned} & W_{H\times H}^2 \Big(\Sigma_t, \Big(\pi^1 \times \frac{\pi^1 - \pi^2}{h} \Big)_{\#} \widehat{\Sigma}_{t,h} \Big) \\ & \quad \le \int_{E^4} |(x_1, x_2) - (y_1, y_2)|_{H\times H}^2 \, d\Big(\pi^{1,2,1} \times \frac{\pi^1 - \pi^3}{h} \Big)_{\#} \widehat{\Xi}_{t,h} \\ & \quad = \int_{E^3} \Big| (x_1, x_2) - \Big(x_1, \frac{x_1 - x_3}{h} \Big) \Big|_{H\times H}^2 d\widehat{\Xi}_{t,h} \\ & \quad = \int_{E^3} \Big| x_2 - \frac{x_1 - x_3}{h} \Big|_H^2 d\widehat{\Xi}_{t,h} \\ & \quad = \int_{E^3} |x_2 - x_3|_H^2 \, d\Xi_{t,h}, \end{aligned}$$

and therefore (9.2) follows from (9.5). □

The next result shows, loosely speaking, that the velocity field along the path of a gradient flow selects the element of minimal length in the subdifferential.

Corollary 9.7. *Let $\phi : \mathscr{P}(E) \to \mathbb{R} \cup \{\infty\}$ be a proper, lsc functional satisfying (A1), which is λ-convex along generalised geodesics for some $\lambda \in \mathbb{R}$, and assume that $\mathsf{D}(\phi) \subseteq \mathscr{P}_f(E)$. Let $\mu \in \overline{\mathsf{D}(\phi)}$, let $(\mu_t)_{t\ge 0}$ be the corresponding gradient flow with $\mu_0 = \mu$, and let $Z \in L^2_{\mathrm{loc}}(M_\mu; H)$ be its velocity field. Then, for a.e. $t > 0$,*

$$(I_E \times (-Z_t))_{\#} \mu_t = \partial^\circ \phi(\mu_t).$$

Proof. Since $\mu_t \in \mathsf{D}(\phi)$ for any $t > 0$ by Theorem 9.2, this follows immediately by combining Theorem 7.4 and Theorem 9.6. □

10

Entropy and Fokker-Planck Equations

In this chapter we study a class of entropy functionals on the Wasserstein space $(\mathscr{P}(E), W_H)$. Under suitable assumptions on the reproducing kernel Hilbert space, it will be shown that Gaussian entropy functionals are displacement convex in the sense of McCann [118] and Ambrosio, Gigli, and Savaré [4].

This result will be applied to relative entropy functionals associated with invariant measures of linear stochastic differential equations in Banach spaces. We will prove that the associated Wasserstein gradient flows satisfy a Fokker-Planck equation corresponding to the SDE, thereby establishing a connection between Parts I and II of this thesis. The underlying Hilbert space in the definition of the Wasserstein metric is the reproducing kernel Hilbert space of the noise term in the SDE.

10.1 Entropy functionals

For $\nu \in \mathscr{P}(E)$ we consider the relative entropy functional (also known as Kullback-Leibler divergence)

$$\mathcal{H}_\nu : \mathscr{P}(E) \to [0, \infty], \qquad \mathcal{H}_\nu(\mu) := \begin{cases} \int_E \rho \log \rho \, d\nu, & \mu \ll \nu, \mu = \rho\nu \\ \infty, & \text{otherwise.} \end{cases}$$

The asserted nonnegativity of $\mathcal{H}_\nu(\mu)$ follows from the observation that $1 - t + t \log t \geq 0$ for $t \geq 0$, together with the identity

$$\int_E \rho \log \rho \, d\nu = \int_E 1 - \rho + \rho \log \rho \, d\nu.$$

We will use some continuity and contractivity properties of relative entropy functionals, which have been proved in a Hilbert space setting in [4, Lemma 9.4.3] and [4, Lemma 9.4.5]. The proofs remains valid in our setting (see also [166, Theorem 29.20] for related results).

Lemma 10.1. *Let $(\mu^n)_{n\geq 1}$ and $(\nu^n)_{n\geq 1}$ be sequences in $\mathscr{P}(E)$ converging weakly to $\mu, \nu \in \mathscr{P}(E)$ respectively. Then*

$$\mathcal{H}_\nu(\mu) \leq \varliminf_{n\to\infty} \mathcal{H}_{\nu^n}(\mu^n).$$

Lemma 10.2. *Let $\pi : E \to E$ be a Borel map. For all $\mu, \nu \in \mathscr{P}(E)$ we have*

$$\mathcal{H}_{\pi_\# \nu}(\pi_\# \mu) \leq \mathcal{H}_\nu(\mu). \tag{10.1}$$

10.2 Displacement convexity of Gaussian entropy

In this section we assume that

- γ is a Gaussian measure on E. Let $\mathscr{H}$ be its reproducing kernel Hilbert space, let $\iota : \mathscr{H} \hookrightarrow E$ be the canonical embedding, and set $\mathscr{Q} := \iota\iota^*$.

In some parts of this section we will impose the following additional assumption:

(B) There exists $\beta > 0$ such that $\langle \mathscr{Q}x^*, x^* \rangle \leq \beta^2 \langle Qx^*, x^* \rangle$ for any $x^* \in E^*$.

It is not difficult to prove that Assumption (B) holds if and only if the following equivalent conditions are satisfied:

– The mapping

$$U : i^* x^* \mapsto \iota^* x^*, \qquad x^* \in E^*, \tag{10.2}$$

is well-defined and extends uniquely to a bounded linear operator $U \in \mathcal{L}(H, \mathscr{H})$ of norm $\leq \beta$;

– As subsets of E, we have the inclusion $\iota\mathscr{H} \subseteq iH$, together with the norm estimate

$$|h|_H \leq \beta |h|_{\mathscr{H}}, \qquad h \in \mathscr{H}. \tag{10.3}$$

In this situation, the operator $j := U^* \in \mathscr{L}(\mathscr{H}, H)$ is the inclusion mapping.

This assumption guarantees that the following version of Talagrand's inequality [161] holds.

Proposition 10.3. *Assume (B). Then $\mathsf{D}(\mathcal{H}_\gamma) \subseteq \mathscr{P}_{H,\gamma}(E)$, and for all $\mu \in \mathsf{D}(\mathcal{H}_\gamma)$ we have*

$$\frac{1}{2\beta^2} W_H^2(\mu, \gamma) \leq \mathcal{H}_\gamma(\mu),$$

where $\beta > 0$ has been defined in (B).

Proof. It follows from (10.3) that $W_H(\mu_0,\mu_1) \le \beta W_{\mathscr{H}}(\mu_0,\mu_1)$ for all $\mu_0,\mu_1 \in \mathscr{P}(E)$. Combining this inequality with Talagrand's inequality [161] in abstract Wiener spaces (see [62, Theorem 3.1], [54, Section 5], or [66, Théorème 5.8.7]):

$$\frac{1}{2}W^2_{\mathscr{H}}(\mu,\gamma) \le \mathcal{H}_\gamma(\mu), \qquad \mu \in \mathscr{P}(E),$$

we obtain the result. □

The goal of this section is to prove that, under Assumption (B), Gaussian entropy functionals are β^{-2}-convex along generalised geodesics. The proof proceeds along the lines of [150], where the case $H = \mathscr{H}$ has been considered.

Theorem 10.4. *Assume (B) and let $\sigma,\mu_i \in \mathsf{D}(\mathcal{H}_\gamma)$ for $i = 0,1$. Let $\Xi \in \mathscr{P}(E^3)$ be such that $\pi^{1,2}_\#\Xi \in \Gamma_o(\sigma,\mu_0)$ and $\pi^{1,3}_\#\Xi \in \Gamma_o(\sigma,\mu_1)$. For all $t \in [0,1]$ we have*

$$\mathcal{H}_\gamma(\mu_t) \le (1-t)\mathcal{H}_\gamma(\mu_0) + t\mathcal{H}_\gamma(\mu_1) - \frac{1}{2\beta^2}t(1-t)W^2_H(\mu_0,\mu_1), \tag{10.4}$$

where $\mu_t := (\pi^{2\to3}_t)_\#\Xi$, and $\beta > 0$ has been defined in (B).

The proof of this result relies on a finite dimensional approximation procedure.

A finite dimensional result

First we will state the finite dimensional displacement convexity result which will be used in the approximation argument.

In this subsection we will work in $E = \mathbb{R}^n$ endowed with the Euclidean metric $|\cdot|$. We consider $\sigma \in \mathscr{P}(\mathbb{R}^n)$ of the form $\sigma = Z^{-1}e^{-V}\mathscr{L}^n$, where $Z := \int_{\mathbb{R}^n} e^{-V}\,d\mathscr{L}^n$ is a normalising constant, and $V \in C^2(\mathbb{R}^n)$ is K-convex for some $K \ge 0$, i.e.,

$$V\big((1-t)x+ty\big) \le (1-t)V(x) + tV(y) - \frac{1}{2}Kt(1-t)|x-y|^2_{\mathbb{R}^n}$$

for every $x,y \in \mathbb{R}^n$ and $t \in [0,1]$, or equivalently, $D^2V(x) \ge K$ in the ordering of positive matrices for every $x \in \mathbb{R}^n$.

The following displacement convexity result for the relative entropy functional $\mathcal{H}_\sigma$ is well known (see, e.g., [166, Theorem 17.15] for the case of convexity along an optimal plan, and [4, Theorem 9.4.11] for a version with generalised geodesics and $K = 0$). We use the notation $\Gamma^{\mathbb{R}^n}_o$ to denote optimal plans with respect to the Wasserstein distance $W_{\mathbb{R}^n}$ induced by the Euclidean metric on $\mathbb{R}^n$.

Lemma 10.5. *Take $\sigma,\mu_0,\mu_1 \in \mathsf{D}(\mathcal{H}_\sigma)$ and $\Xi \in \mathscr{P}\big(\mathbb{R}^{3n}\big)$ such that $\pi^{1,2}_\#\Xi \in \Gamma^{\mathbb{R}^n}_o(\sigma,\mu_0)$ and $\pi^{1,3}_\#\Xi \in \Gamma^{\mathbb{R}^n}_o(\sigma,\mu_1)$, and set $\mu_t := \big(\pi^{2\to3}_t\big)_\#\Xi$ for $t \in [0,1]$. Then*

$$\mathcal{H}_\sigma(\mu_t) \le (1-t)\mathcal{H}_\sigma(\mu_0) + t\mathcal{H}_\sigma(\mu_1) - \frac{1}{2}Kt(1-t)W^2_{\mathbb{R}^n}(\mu_0,\mu_1), \quad t \in [0,1].$$

Cylindrical approximation

We continue with finite dimensional approximation of probability measures. Recall that the collection $\mathcal{C}$ and the functionals $(x_n^*)_{n\geq 1}$ and $(y_n^*)_{n\geq 1}$ have been introduced in Section 6.3.

Lemma 10.6. *The set $\mathcal{C}$ is dense in $L^p(\gamma)$ for all $1 \leq p < \infty$.*

Proof. We claim that the linear subspace S spanned by $(\iota^* x_n^*)_{n\geq 1}$ is dense in $\mathscr{H}$. Indeed, it suffices to show that this subspace is weak*-dense, i.e., separating. Suppose that $[h, \iota^* x_n^*]_{\mathscr{H}} = 0$ for some $h \in \mathscr{H}$ and all $n \geq 1$. Then $\langle \iota h, x_n^* \rangle = 0$, which implies that $h = 0$, since the subspace spanned by $(y_n^*)_{n\geq 1}$ (which equals the subspace spanned by $(x_n^*)_{n\geq 1}$) was chosen to be weak*-dense in E^*.

Consequently, for each $z^* \in E^*$ we can find a sequence of functionals $(\iota^* z_n^*)_{n\geq 1} \subseteq S$ such that $\iota^* z_n^* \to \iota^* z^*$ in $\mathscr{H}$. By Proposition 1.12 and Theorem 1.18 this implies that $\langle \cdot, z_n^* \rangle \to \langle \cdot, z^* \rangle$ in $L^p(\gamma)$ for any $1 \leq p < \infty$. Using this fact and an easy truncation argument, we obtain that, for any Hermite polynomial H_m with $m \geq 0$, the function $H_m(\langle \cdot, z^* \rangle)$ can be approximated in $L^p(\gamma)$ with elements from $\mathcal{C}$. The result follows from this observation, since the functions of the form $H_m(\langle \cdot, z^* \rangle)$ span a dense subspace of $L^p(\gamma)$ by (1.2) and Theorem 1.18. $\square$

The following easy lemma will be used in the proof of Proposition 10.8 below.

Lemma 10.7. *Let $(\mu_n)_{n\geq 1} \subseteq \mathscr{P}(E)$ be tight and suppose that the Fourier transform $(\widehat{\mu}_n(x^*))_{n\geq 1}$ converges for each $x^* \in E^*$. Then $(\mu_n)_{n\geq 1}$ converges weakly.*

Proof. See [130, Lemma 2.18]. $\square$

Proposition 10.8. *Assume (B). For all $\mu \in \mathscr{P}(E)$ satisfying $\mu \ll \gamma$ we have $(\mathsf{P}_n)_{\#}\mu \rightharpoonup \mu$ as $n \to \infty$.*

Proof. We proceed in several steps.

Step 1: *We prove the result for $\mu = \gamma$.*

The Fourier transforms of $\gamma_n := (\mathsf{P}_n)_{\#}\gamma$ and γ are given for $x^* \in E^*$ by

$$\widehat{\gamma_n}(x^*) = \exp\Big(-\frac{1}{2} \langle \mathsf{P}_n \mathscr{Q} \mathsf{P}_n^* x^*, x^* \rangle \Big),$$
$$\widehat{\gamma}(x^*) = \exp\Big(-\frac{1}{2} \langle \mathscr{Q} x^*, x^* \rangle \Big).$$

In view of the identity $\mathsf{P}_n^* x^* = \sum_{k=1}^n \langle Q x_k^*, x^* \rangle x_k^*$ and (10.3), we obtain

$$\begin{aligned}
\langle \mathsf{P}_n \mathscr{Q} \mathsf{P}_n^* x^*, x^* \rangle &= \Big\langle \mathscr{Q} \sum_{j=1}^n \langle Q x_j^*, x^* \rangle x_j^*, \sum_{k=1}^n \langle Q x_k^*, x^* \rangle x_k^* \Big\rangle \\
&= \Big\| \sum_{k=1}^n [i^* x_k^*, i^* x^*] \iota^* x_k^* \Big\|_{\mathscr{H}}^2 = \| U \mathsf{P}_n i^* x^* \|_{\mathscr{H}}^2 \\
&\le \|U\|^2 \, \|i^* x^*\|_H^2 = \beta^2 \, \langle Q x^*, x^* \rangle,
\end{aligned}$$

where $\beta > 0$ has been defined in (B). Since Q is the covariance of a Gaussian measure on E, this estimate implies that the sequence $(\gamma_n)_{n \ge 1}$ is tight by covariance domination (Proposition 1.5). Using the boundedness of U once more, for each $x^* \in E^*$ we obtain, as $n \to \infty$,

$$\langle \mathsf{P}_n \mathscr{Q} \mathsf{P}_n^* x^*, x^* \rangle = \| U \mathsf{P}_n i^* x^* \|_{\mathscr{H}}^2 \to \| U i^* x^* \|_{\mathscr{H}}^2 = \| \iota^* x^* \|_{\mathscr{H}}^2 = \langle \mathscr{Q} x^*, x^* \rangle,$$

hence $\widehat{\gamma_n}(x^*) \to \widehat{\gamma}(x^*)$. Therefore Lemma 10.7 yields the desired conclusion.

Step 2: *The result holds for $\mu = \rho\gamma$, where $\rho \in \mathcal{C}$.*

Indeed, since $\rho \circ \mathsf{P}_n = \rho$ for n large enough, we obtain for any $\varphi \in C_{\mathrm{b}}(E)$,

$$\begin{aligned}
\lim_{n\to\infty} \int_E \varphi \, d(\mathsf{P}_n)_\#(\rho\gamma) &= \lim_{n\to\infty} \int_E (\varphi \circ \mathsf{P}_n) \cdot \rho \, d\gamma \\
&= \lim_{n\to\infty} \int_E (\varphi \circ \mathsf{P}_n) \cdot (\rho - \rho \circ \mathsf{P}_n) \, d\gamma + \int_E (\varphi \cdot \rho) \circ \mathsf{P}_n \, d\gamma \\
&= \lim_{n\to\infty} \int_E \varphi\rho \, d\gamma_n = \int_E \varphi\rho \, d\gamma,
\end{aligned}$$

where we used Step 1 and the fact that $\varphi\rho \in C_{\mathrm{b}}(E)$.

Step 3: *We prove the result for $\mu = \rho\gamma$, where $\rho \in L^1(\gamma)$ is an arbitrary probability density on E.*

For this purpose, let $\varepsilon > 0$ and take, using Lemma 10.6, $\tilde\rho \in \mathcal{C}$ such that $\|\rho - \tilde\rho\|_{L^1(\gamma)} < \varepsilon$. For $\varphi \in C_{\mathrm{b}}(E)$ and n large enough we obtain from Step 2,

$$\begin{aligned}
\Big| \int_E (\varphi \circ \mathsf{P}_n) \cdot \rho \, d\gamma - \int_E \varphi\rho \, d\gamma \Big| &\le \Big| \int_E (\varphi \circ \mathsf{P}_n) \cdot \big(\rho - \tilde\rho\big) \, d\gamma \Big| \\
&\quad + \Big| \int_E \big(\varphi \circ \mathsf{P}_n - \varphi\big) \tilde\rho \, d\gamma \Big| + \Big| \int_E \varphi \big(\tilde\rho - \rho\big) \, d\gamma \Big| \\
&\le \varepsilon \|\varphi\|_\infty + \varepsilon + \varepsilon \|\varphi\|_\infty,
\end{aligned}$$

which gives the result. □

Proof of the convexity of the Gaussian entropy

Proof (of Theorem 10.4). Let us first remark that $W_H(\sigma, \mu_i) < \infty$ for $i = 0, 1$, as a consequence of Talagrand's inequality (Proposition 10.3) and the triangle inequality.

Put $\sigma^n := (\mathsf{P}_n)_\#\sigma$ and $\mu_i^n := (\mathsf{P}_n)_\#\mu_i$. Proposition 10.8 implies that $\sigma^n \rightharpoonup \sigma$ and $\mu_i^n \rightharpoonup \mu_i$. We claim that $W_H(\mu_0^n, \mu_1^n) \le W_H(\mu_0, \mu_1)$. Indeed, for $\widehat{\Sigma} \in \Gamma_o(\mu_0, \mu_1)$ and $\widehat{\Sigma}^n := (\mathsf{P}_n \times \mathsf{P}_n)_\#\widehat{\Sigma}$ we have

$$\begin{aligned} W_H^2(\mu_0^n, \mu_1^n) &\le \int_{E\times E} |x-y|_H^2 \, d\widehat{\Sigma}^n = \int_{E\times E} |\mathsf{P}_n(x-y)|_H^2 \, d\widehat{\Sigma} \\ &\le \int_{E\times E} |x-y|_H^2 \, d\widehat{\Sigma} = W_H^2(\mu_0, \mu_1). \end{aligned} \tag{10.5}$$

By the same argument we obtain that $W_H(\sigma^n, \mu_i^n) \le W_H(\sigma, \mu_i)$ for $i = 0, 1$. Take $\Xi^n \in \mathscr{P}(E^3)$ satisfying

$$\pi_\#^{1,2}\Xi^n \in \Gamma_o(\sigma^n, \mu_0^n), \qquad \pi_\#^{1,3}\Xi^n \in \Gamma_o(\sigma^n, \mu_1^n).$$

Since the collections $(\sigma^n)_{n\ge1}$ and $(\mu_i^n)_{n\ge1}$ for $i = 0, 1$, are tight by Prokhorov's Theorem 6.1, the collection $(\Xi^n)_{n\ge1}$ is tight as well. By passing to a subsequence we may assume that Ξ^n converges weakly to $\Xi \in \Gamma(\sigma, \mu_0, \mu_1)$. Using Lemma 6.2 we obtain

$$\begin{aligned} W_H^2(\sigma, \mu_0) &\le \int_{E^3} |x-y|_H^2 \, d\Xi(x,y,z) \\ &\le \varliminf_{n\to\infty} \int_{E^3} |x-y|_H^2 \, d\Xi^n(x,y,z) = \varliminf_{n\to\infty} W_H^2(\sigma^n, \mu_0^n). \end{aligned} \tag{10.6}$$

Since we already obtained the inequality $W_H(\sigma^n, \mu_i^n) \le W_H(\sigma, \mu_i)$ for all $n \ge 1$, we infer that equality must hold in (10.6). In particular this implies that $\pi_\#^{1,2}\Xi \in \Gamma_o(\sigma, \mu_0)$, and by the same argument, $\pi_\#^{1,3}\Xi \in \Gamma_o(\sigma, \mu_1)$.

Note that $\gamma^n := \mathsf{P}_\#^n\gamma$ is a Gaussian measure on E, which is supported on the finite dimensional Hilbertian subspace

$$H^{(n)} := \mathrm{lin}(i^*x_k^*)_{1\le k\le n}.$$

We use the orthogonal basis $(i^*x_k^*)_{1\le k\le n}$ to identify $H^{(n)}$ with $\mathbb{R}^n$, and remark that under this identification the covariance operator $\mathsf{P}^n\mathscr{Q}\mathsf{P}^{n*}$ of γ^n is represented by the matrix $R^n = (\langle \mathscr{Q}x_k^*, x_l^*\rangle)_{k,l=1}^n$. For $a_1, \ldots, a_n \in \mathbb{R}$ we set $x^* := \sum_{k=1}^n a_k x_k^*$. Using (B) we obtain

$$\begin{aligned} \sum_{k,l=1}^n \langle \mathscr{Q}x_k^*, x_l^*\rangle a_k a_l &= \langle \mathscr{Q}x^*, x^*\rangle \\ &\le \beta^2\langle Qx^*, x^*\rangle = \beta^2 \sum_{k,l=1}^n \langle Qx_k^*, x_l^*\rangle a_k a_l = \beta^2 \sum_{k=1}^n a_k^2, \end{aligned}$$

which shows that $R^n \le \beta^2$ in the ordering of positive matrices. Note that the measure γ_n has a density with respect to Lebesgue measure of the form $Z^{-1}e^{-V}$, where $Z > 0$ and $V(\xi) := \frac{1}{2}[(R^n)^{-1}\xi, \xi]$. For $0 < t < 1$, put

$$\mu_t^n := (\pi_t^{2\to 3})_\# \varXi^n.$$

Since $D^2V(\xi) = (R^n)^{-1} \ge \beta^{-2}$, it follows from Lemma 10.5 that

$$\begin{aligned} \mathcal{H}_{\gamma^n}(\mu_t^n) &\le (1-t)\mathcal{H}_{\gamma^n}(\mu_0^n) + t\mathcal{H}_{\gamma^n}(\mu_1^n) - \frac{1}{2\beta^2}t(1-t)W^2_{H^{(n)}}(\mu_0^n, \mu_1^n) \\ &\le (1-t)\mathcal{H}_\gamma(\mu_0) + t\mathcal{H}_\gamma(\mu_1) - \frac{1}{2\beta^2}t(1-t)W_H^2(\mu_0^n, \mu_1^n). \end{aligned} \tag{10.7}$$

Since μ_t^n converges weakly to $\mu_t := (\pi_t^{2\to 3})_\# \varXi$ as $n \to \infty$, Lemma 10.1 implies that $\mathcal{H}_\gamma(\mu_t) \le \underline{\lim}_{n\to\infty} \mathcal{H}_{\gamma^n}(\mu_t^n)$. Therefore, in order to prove (10.4), it suffices to prove that

$$\lim_{n\to\infty} W_H^2(\mu_0^n, \mu_1^n) = W_H^2(\mu_0, \mu_1). \tag{10.8}$$

For this purpose, take $\varUpsilon^n \in \varGamma_o^H(\mu_0^n, \mu_1^n)$. The tightness of $(\mu_0^n)_{n\ge 1}$ and $(\mu_1^n)_{n\ge 1}$ implies that $(\varUpsilon^n)_{n\ge 1}$ is tight. Passing to a subsequence we may assume that $\varUpsilon^n$ converges weakly to some $\varUpsilon \in \varGamma(\mu_0, \mu_1)$. Applying Lemma 6.2 we arrive at

$$\begin{aligned} W_H^2(\mu_0, \mu_1) &\le \int_{E\times E} |x-y|_H^2 \, d\varUpsilon(x,y) \\ &\le \underline{\lim}_{n\to\infty} \int_{E\times E} |x-y|_H^2 \, d\varUpsilon^n(x,y) = \underline{\lim}_{n\to\infty} W_H^2(\mu_0^n, \mu_1^n), \end{aligned} \tag{10.9}$$

and combining this with (10.5), we obtain (10.8). This completes the proof of (10.4). □

10.3 Entropy gradient flows and Fokker-Planck equations

In this section we establish a connection between Parts I and II of this thesis by applying the theory developed in the current chapter to the study of Fokker-Planck equations associated with a class of infinite dimensional stochastic processes.

We will follow the line of argumentation from the paper by Fang, Shao, and Sturm [59], who adapted the approach of Jordan, Kinderlehrer, and Otto [85] to the Wiener space setting.

We consider the following setup:

- $-\mathcal{A}$ is the generator of a C_0-semigroup $(\mathcal{S}(t))_{t\ge 0}$ of bounded linear operators on a separable Banach space E.

- As before, H is a separable Hilbert space, $i : H \hookrightarrow E$ is a continuous embedding, and $Q := ii^* \in \mathcal{L}(E^*, E)$.

Furthermore, we will assume in the sequel that the functionals $(x_n^*)_{n\geq 1}$ appearing in the definition of $\mathcal{C}$ in Section 6.2 are contained in $\mathsf{D}(\mathcal{A}^*)$. This can be assumed without loss of generality, since $\mathsf{D}(\mathcal{A}^*)$ is weak*-dense in E^*.

We consider the following assumptions:

(C1) For each $t > 0$, the operator $Q_t \in \mathcal{L}(E^*, E)$ defined by

$$Q_t := \int_0^t \mathcal{S}(s)Q\mathcal{S}^*(s)x^*\,ds, \qquad x^* \in E^*,$$

is the covariance of a Gaussian measure μ_t on E. Moreover, $Q_\infty := \lim_{t\to\infty} Q_t$ exists in the weak operator topology, and Q_∞ is the covariance of a Gaussian measure μ_∞ on E, whose reproducing kernel Hilbert space (see Section 1.2) will be denoted by H_∞.

Assumption (C1) has already been imposed in Section 2.1 of Part I. It allows the construction of the associated Ornstein-Uhlenbeck semigroup P, which is defined for $t \geq 0$ and $f \in C_{\mathrm{b}}(E)$ by

$$(P(t)f)(x) := \int_E f(\mathcal{S}(t)x + y)\,d\mu_t(y), \qquad x \in E.$$

The measure μ_∞ is an invariant measure for P in the sense that for all $t \geq 0$,

$$\int_E P(t)f\,d\mu_\infty = \int_E f\,d\mu_\infty, \qquad f \in C_{\mathrm{b}}(E).$$

Furthermore, the semigroup P extends to a C_0-semigroup of positive contractions on $L^p(\mu_\infty)$ for $1 \leq p < \infty$. The generator is denoted by $-L$ and admits the expression

$$\begin{aligned} Lf(x) = -\frac{1}{2}\sum_{k,\ell=1}^{n} &\langle Qx_k^*, x_\ell^*\rangle \partial_k\partial_\ell\varphi(\langle x, x_1^*\rangle, \ldots, \langle x, x_n^*\rangle) \\ &+ \sum_{k=1}^{n}\langle x, \mathcal{A}^*x_k^*\rangle \partial_k\varphi(\langle x, x_1^*\rangle, \ldots, \langle x, x_n^*\rangle), \qquad x \in E, \end{aligned} \tag{10.10}$$

for functions $f \in \mathcal{C}$ of the form (7.2) (see also (4.2)). We refer to Chapter 2 for more information on Ornstein-Uhlenbeck semigroups.

(C2) For $t \in [0, \infty)$ we have $Q\mathcal{S}^*(t) = \mathcal{S}(t)Q$.

In Proposition 2.18 various characterisations of this assumption have been stated. In particular, it has been shown that this assumption is equivalent to the selfadjointness of the Ornstein-Uhlenbeck semigroup P.

(C3) There exists a constant $\beta > 0$ such that

$$\langle Q_\infty x^*, x^* \rangle \le \beta^2 \langle Qx^*, x^* \rangle, \qquad x^* \in E^*.$$

It has already been discussed in Section 10.2 that Assumption (C3) is equivalent to the existence of a bounded operator $U \in \mathcal{L}(H, H_\infty)$ of norm $\le \beta$ satisfying $U(i^*x^*) = i_\infty^* x^*$ for every $x^* \in E^*$. This condition holds if and only if $H_\infty \subseteq H$ and

$$|h|_H \le \beta |h|_{H_\infty}, \qquad h \in H, \tag{10.11}$$

in which case the operator $j := U^*$ is the inclusion mapping from H_∞ into H.

The next result, taken from [33, Theorem 4.2] and [70, Lemma 5.2], provides equivalent conditions in terms of decay rates for various semigroups. Recall from Chapter 2 that the semigroups $\mathcal{S}_\infty$ and $\mathcal{S}_H$ denote the restrictions of the drift semigroup $\mathcal{S}$ to the Hilbert spaces H_∞ and H respectively.

Proposition 10.9. *Assume (C1) and (C2). The following conditions are equivalent for $\beta > 0$:*

(1) *Assumption (C3) holds;*
(2) $\|\mathcal{S}_\infty(t)\|_{\mathcal{L}(H_\infty)} \le \exp(-\frac{t}{2\beta^2})$ *for $t \ge 0$;*
(3) $\|\mathcal{S}_H(t)\|_{\mathcal{L}(H)} \le \exp(-\frac{t}{2\beta^2})$ *for $t \ge 0$;*
(4) $\|P(t)f\|_{L^2(\mu_\infty)} \le \exp(-\frac{t}{2\beta^2})\|f\|_{L^2(\mu_\infty)}$ *for $t \ge 0$ and all $f \in L^2(\mu_\infty)$ satisfying $\int_E f \, d\mu_\infty = 0$.*

Further equivalent conditions can be given in terms of a logarithmic Sobolev inequality for the generator $-L$ and a hypercontractivity estimate for the semigroup P. We refer to [33] for the details.

- From now on we assume that (C1) – (C3) are satisfied.

As an application of the theory developed in the first part of this chapter we obtain the following result.

Theorem 10.10. *The relative entropy functional $\mathcal{H}_{\mu_\infty}$ is (β^{-2})-convex along generalised geodesics in $(\mathscr{P}(E), W_H)$. More precisely, let $\sigma, \nu_i \in \mathsf{D}(\mathcal{H}_{\mu_\infty})$ for $i = 0, 1$. For each $\Xi \in \mathscr{P}(E^3)$ satisfying $\pi^{1,2}_\# \Xi \in \Gamma_o(\sigma, \nu_0)$ and $\pi^{1,3}_\# \Xi \in \Gamma_o(\sigma, \nu_1)$, and for any $t \in [0, 1]$ we have*

$$\mathcal{H}_{\mu_\infty}(\nu_t) \le (1-t)\mathcal{H}_{\mu_\infty}(\nu_0) + t\mathcal{H}_{\mu_\infty}(\nu_1) - \frac{1}{2\beta^2} t(1-t) W_H^2(\nu_0, \nu_1), \tag{10.12}$$

where $\nu_t := (\pi_t^{2\to 3})_\# \Xi$, and $\beta > 0$ has been defined in (C3).

Proof. This follows immediately from Theorem 10.4 and (C3). □

Combining this result with the abstract theory developed in [4] we obtain the existence of a gradient flow on $\mathscr{P}(E)$ associated with the functional $\mathcal{H}_{\mu_\infty}$ and the (pseudo)-metric W_H.

Theorem 10.11. *Let $\sigma \in \overline{\mathsf{D}(\mathcal{H}_{\mu_\infty})}$. There exists a unique gradient flow $u \in C([0,\infty);\mathscr{P}(E)) \cap AC_{\mathrm{loc}}((0,\infty);\mathscr{P}(E))$ associated with the functional $\mathcal{H}_{\mu_\infty}$ and satisfying $u(0) = \sigma$. Moreover, for any $\sigma_0, \sigma_1 \in \overline{\mathsf{D}(\mathcal{H}_{\mu_\infty})}$ we have*

$$W_H(u_0(t), u_1(t)) \le e^{-t/\beta^2} W_H(\sigma_0, \sigma_1), \qquad t \ge 0, \tag{10.13}$$

where u_0 and u_1 are the gradient flows starting from σ_0 and σ_1 respectively.

More explicitly, in view of Definition 9.1, this result asserts that there exists a unique function $u \in C([0,\infty);\mathscr{P}(E)) \cap AC_{\mathrm{loc}}((0,\infty);\mathscr{P}(E))$ satisfying $u(0) = \sigma$ and, for any $\nu \in \mathsf{D}(\mathcal{H}_{\mu_\infty})$, the evolution variational inequality

$$\frac{1}{2}\partial_t W_H^2(u(t),\nu) + \frac{1}{2\beta^2} W_H^2(u(t),\nu) \le \mathcal{H}_{\mu_\infty}(\nu) - \mathcal{H}_{\mu_\infty}(u(t)) \tag{10.14}$$

a.e. on $(0,\infty)$.

Proof. This is a consequence of Proposition 8.7, Theorem 9.2, and Theorem 10.10. $\square$

Let us introduce some notation. We will denote by

$$\delta_H : \mathsf{D}(\delta_H) \subseteq L^2(\mu_\infty; H) \to L^2(\mu_\infty)$$

the adjoint of the gradient ∇_H. Similarly, we let

$$\delta_{H_\infty} : \mathsf{D}(\delta_{H_\infty}) \subseteq L^2(\mu_\infty; H_\infty) \to L^2(\mu_\infty)$$

be the adjoint of the gradient ∇_{H_∞}.

Remark 10.12. The simplest operator L contained in the framework of this section is the classical Ornstein-Uhlenbeck operator (also known as the number operator). To obtain this operator we let Q be the covariance of a Gaussian measure μ_Q on E, and set $\mathcal{S}(t) := e^{-t} I_E$ for $t \ge 0$. In this case we have $Q_\infty = \frac{1}{2}Q$, and the conditions (C1) – (C3) are trivially fulfilled with $\beta = \frac{1}{\sqrt{2}}$. The Ornstein-Uhlenbeck operator L can be written as

$$L = \delta_{H_\infty} \nabla_{H_\infty} = \frac{1}{2}\delta_H \nabla_H.$$

Theorem 10.10 and Theorem 10.11 imply that $\mathcal{H}_{\mu_\infty}$ is 2-convex along generalised geodesics, and $W_H(u_0(t), u_1(t)) \le e^{-2t} W_H(\sigma_0, \sigma_1)$, for $t \ge 0$, where $u_0, u_1 : [0,\infty) \to \mathscr{P}(E)$ are the gradient flows starting from σ_0 and σ_1 respectively.

Since $|h|_H = \frac{1}{\sqrt{2}}|h|_{H_\infty}$ for any $h \in H$, it follows that the same functional $\mathcal{H}_{\mu_\infty}$ is 1-convex with respect to the different Wasserstein metric W_{H_∞}, which corresponds to the result proved in [59]. Accordingly, it follows that $W_{H_\infty}(\tilde{u}_0(t), \tilde{u}_1(t)) \le e^{-t} W_{H_\infty}(\sigma_0, \sigma_1)$ for the associated gradient flow with respect to the metric W_{H_∞}.

The difference between the rates of exponential decay is due to the fact that $\tilde{u}_i(2t) = u_i(t)$ for $t \ge 0$, which can be seen from the exponential formulae (Theorem 9.2(i)).

In the remainder of this section we will show that the gradient flow associated with the functional $\mathcal{H}_{\mu_\infty}$ and the metric W_H satisfies a Fokker-Planck equation associated with the operator L. The proof proceeds along the lines of [59, 85] with some modifications.

We will use the following result of Cruzeiro [42] on the existence of flows associated with Malliavin differentiable vector fields on Wiener spaces. Generalisations have been recently proved in [3, 58]. We formulate the result in the Wiener space $(E, H_\infty, \mu_\infty)$. The spaces $W^{k,p}_{H_\infty}(\mu_\infty)$ and $W^{k,p}_{H_\infty}(\mu_\infty; H_\infty)$ appearing below are the Gaussian Sobolev spaces for the Malliavin derivative, which correspond to the spaces considered in Section 11.4 in the special case where $\mu = \mu_\infty$, $\underline{H} = H_\infty$, and $V = I_{H_\infty}$.

Proposition 10.13. *Let $Y \in \bigcap_{k\ge 1} W^{k,2}_{H_\infty}(\mu_\infty; H_\infty)$ be such that*

$$\exp\big(\varepsilon_0 |Y|_{H_\infty}\big),\ \exp\big(\kappa_0 \|\nabla_{H_\infty} Y\|_{\mathcal{L}(H_\infty)}\big),\ \exp\big(\lambda_0 |\delta_{H_\infty} Y|\big) \in L^1(\mu_\infty)$$

for every $\varepsilon_0, \kappa_0, \lambda_0 > 0$. Then there exists a collection $(U_t)_{t\in\mathbb{R}}$ of Borel maps $U_t : E \to E$ such that

$$\begin{cases} U_{s+t} = U_s \circ U_t, & s, t \in \mathbb{R}, \\ U_t(x) = x + \int_0^t Y(U_s(x))\, ds, & t \in \mathbb{R},\ \textit{for } \mu_\infty\textit{-a.e. } x \textit{ in } E. \end{cases} \tag{10.15}$$

Moreover, if $\delta_{H_\infty} Y \in W^{1,16}_{H_\infty}(\mu_\infty)$, we have for $t \in \mathbb{R}$,

$$(U_t)_\# \mu_\infty = \exp\bigg(\int_0^t \delta_{H_\infty} Y(U_{-s}(x))\, ds\bigg)\mu_\infty.$$

Proof. See [42, Theorem 1.4.1]. □

The next proposition is a variation of this result involving H-valued (instead of H_∞-valued) vector fields.

Proposition 10.14. *For $f \in \mathcal{C}$ there exists a collection $(U_t)_{t\in\mathbb{R}}$ of Borel maps $U_t : E \to E$ satisfying*

$$\begin{cases} U_{s+t} = U_s \circ U_t, & s, t \in \mathbb{R}, \\ U_t(x) = x + \int_0^t \nabla_H f(U_s(x))\, ds, & t \in \mathbb{R},\ \textit{for } \mu_\infty\textit{-a.e. } x \textit{ in } E. \end{cases} \tag{10.16}$$

Moreover, for $t \in \mathbb{R}$ we have

$$(U_t)_{\#}\mu_\infty = \exp\left(2\int_0^t Lf(U_{-s}(x))\,ds\right)\mu_\infty. \tag{10.17}$$

Proof. We claim that $i(\nabla_H f(x)) = 2i_\infty(\mathcal{A}_\infty^* \nabla_{H_\infty} f(x))$ for all $x \in E$. To prove the claim, we note that $i_\infty^*(\mathsf{D}(\mathcal{A}^*)) \subseteq \mathsf{D}(\mathcal{A}_\infty^*)$ by Lemma 2.9. In view of (C2), Proposition 2.18, and Corollary 2.12, we obtain for $x^*, y^* \in \mathsf{D}(\mathcal{A}^*)$,

$$\begin{aligned}\langle i_\infty(\mathcal{A}_\infty^* i_\infty^* x^*), y^*\rangle &= [\mathcal{A}_\infty^* i_\infty^* x^*, i_\infty^* y^*]\\ &= \frac{1}{2}[\mathcal{A}_\infty^* i_\infty^* x^*, i_\infty^* y^*] + \frac{1}{2}[i_\infty^* x^*, \mathcal{A}_\infty^* i_\infty^* y^*]\\ &= \frac{1}{2}[i^* x^*, i^* y^*] = \frac{1}{2}\langle ii^* x^*, y^*\rangle.\end{aligned}$$

Since $i_\infty^*(\mathsf{D}(A^*))$ is weak*-dense in H_∞, we infer that $2i_\infty(\mathcal{A}_\infty^* i_\infty^* x^*) = i(i^* x^*)$, which implies the claimed identity.

Therefore the result follows from Proposition 10.13 applied to $Y := 2\mathcal{A}_\infty^* \nabla_{H_\infty} f$, and the observation that $\delta_{H_\infty} Y = 2Lf$, which is a consequence of Theorem 2.16 and Theorem 4.3. □

In Lemma 10.15 and Proposition 10.16 we fix $f \in \mathcal{C}$ and let $(U_t)_{t\in\mathbb{R}}$ be the associated collection of Borel maps $U_t : E \to E$ obtained in Proposition 10.14. For $t \in \mathbb{R}$ we consider the functions $K_t, A_t : E \to E$ defined for μ_∞-a.e. $x \in E$ by

$$K_t(x) := \exp\left(2\int_0^t Lf(U_{-s}(x))\,ds\right), \qquad A_t(x) := \frac{1}{t}\int_0^t Lf(U_{t-s}(x))\,ds,$$

with the understanding that $A_0 = Lf$. The following lemma will be used in the proof of Proposition 10.16.

Lemma 10.15. *For $\mu \in \mathsf{D}(\mathcal{H}_{\mu_\infty})$ and $T > 0$, the collection $(A_t)_{0\le t\le T}$ is uniformly bounded in $L^2(\mu)$.*

Proof. We proceed in three steps.

Step 1. We claim that for any $t > 0$,

$$\sup_{|s|\le t} \|K_s\|_{L^2(\mu_\infty)}^2 \le \int_E \exp\Big(8t|Lf(x)|\Big)\,d\mu_\infty(x) < \infty. \tag{10.18}$$

To prove the claim, we use Jensen's inequality (in the second estimate) and (10.17) to obtain

$$
\begin{aligned}
\|K_t\|^2_{L^2(\mu_\infty)} &\le \int_E \exp\left(\frac{1}{t}\int_0^t 4t|Lf(U_{-s}(x))|\,ds\right) d\mu_\infty(x) \\
&\le \int_E \frac{1}{t}\int_0^t \exp\Big(4t|Lf(U_{-s}(x))|\Big)\,ds\,d\mu_\infty(x) \\
&= \frac{1}{t}\int_0^t \int_E \exp\Big(4t|Lf(x)|\Big) K_{-s}(x)\,d\mu_\infty(x)\,ds \\
&\le \frac{1}{t}\int_0^t \left(\int_E \exp\Big(8t|Lf(x)|\Big)\,d\mu_\infty(x)\right)^{1/2} \|K_{-s}\|_{L^2(\mu_\infty)}\,ds \\
&= \frac{1}{t}\int_0^t \|K_{-s}\|_{L^2(\mu_\infty)}\,ds \left(\int_E \exp\Big(8t|Lf(x)|\Big)\,d\mu_\infty(x)\right)^{1/2}.
\end{aligned}
$$

Replacing f by $-f$ and arguing similarly we find that

$$
\|K_{-t}\|^2_{L^2(\mu_\infty)} \le \frac{1}{t}\int_0^t \|K_s\|_{L^2(\mu_\infty)}\,ds \left(\int_E \exp\Big(8t|Lf(x)|\Big)\,d\mu_\infty(x)\right)^{1/2}.
$$

Combining these estimates we obtain

$$
\begin{aligned}
&\max\left\{\|K_{-t}\|^2_{L^2(\mu_\infty)}, \|K_t\|^2_{L^2(\mu_\infty)}\right\} \\
&\qquad\qquad \le \sup_{|s|\le t} \|K_s\|_{L^2(\mu_\infty)} \left(\int_E \exp\Big(8t|Lf(x)|\Big)\,d\mu_\infty(x)\right)^{1/2},
\end{aligned}
$$

and therefore

$$
\sup_{|s|\le t} \|K_s\|^2_{L^2(\mu_\infty)} \le \int_E \exp\Big(8t|Lf(x)|\Big)\,d\mu_\infty(x).
$$

The finiteness of the right-hand side follows by combining Fernique's Theorem 1.3 with the fact that $Lf(x) = b(x) + \langle x, F(x)\rangle$ for some bounded functions $b \in C_{\mathrm{b}}(E)$ and $F \in C_{\mathrm{b}}(E; E^*)$ according to (10.10). This completes the proof of (10.18).

Step 2. There exists $\varepsilon_0 > 0$ such that for any $t > 0$ we have

$$
\begin{aligned}
\int_E \exp\Big(\varepsilon_0|A_t(x)|^2\Big)\,d\mu_\infty(x) &\le \sup_{0\le s\le t} \|K_s\|_{L^2(\mu_\infty)} \\
&\quad \times \left(\int_E \exp\Big(2\varepsilon_0|Lf(x)|^2\Big)\,d\mu_\infty(x)\right)^{1/2} < \infty.
\end{aligned}
\tag{10.19}
$$

To prove this estimate, we write $Lf(x) = b(x) + \langle x, F(x)\rangle$ as in Step 1. Fernique's Theorem 1.3 implies that there exists $\varepsilon_0 > 0$ such that

$$
\int_E \exp\big(2\varepsilon_0|Lf(x)|^2\big)\,d\mu_\infty(x) < \infty.
$$

Using (10.17) and the inequalities of Jensen (twice) and Cauchy-Schwarz, we obtain

$$\begin{aligned}
&\int_E \exp\left(\varepsilon_0 |A_t(x)|^2\right) d\mu_\infty(x)\\
&\quad = \int_E \exp\left(\varepsilon_0 \left|\frac{1}{t}\int_0^t Lf(U_{t-s}(x))\,ds\right|^2\right) d\mu_\infty(x)\\
&\quad \leq \int_E \exp\left(\frac{\varepsilon_0}{t}\int_0^t |Lf(U_{t-s}(x))|^2\,ds\right) d\mu_\infty(x)\\
&\quad \leq \int_E \frac{1}{t}\int_0^t \exp\left(\varepsilon_0 |Lf(U_{t-s}(x))|^2\right) ds\, d\mu_\infty(x)\\
&\quad = \frac{1}{t}\int_0^t \int_E \exp\left(\varepsilon_0 |Lf(x)|^2\right) K_{t-s}(x)\, d\mu_\infty(x)\, ds\\
&\quad \leq \frac{1}{t}\int_0^t \|K_{t-s}\|_{L^2(\mu_\infty)} \left(\int_E \exp\left(2\varepsilon_0 |Lf(x)|^2\right) d\mu_\infty(x)\right)^{1/2} ds\\
&\quad \leq \sup_{0\leq s\leq t} \|K_s\|_{L^2(\mu_\infty)} \left(\int_E \exp\left(2\varepsilon_0 |Lf(x)|^2\right) d\mu_\infty(x)\right)^{1/2},
\end{aligned}$$

which implies (10.19).

Step 3. To finish the proof, we write $\mu = \rho\mu_\infty$ and use Young's inequality $uv \leq e^u + v\log v$ for $u, v \geq 0$ to obtain in view of (10.18) and (10.19),

$$\begin{aligned}
&\int_E |A_t(x)|^2\, d\mu(x)\\
&\quad = \int_E \varepsilon_0 |A_t(x)|^2 \frac{\rho(x)}{\varepsilon_0}\, d\mu_\infty(x)\\
&\quad \leq \int_E \exp\left(\varepsilon_0 |A_t(x)|^2\right) d\mu_\infty(x) + \frac{1}{\varepsilon_0}\mathcal{H}_{\mu_\infty}(\mu) - \frac{1}{\varepsilon_0}\log\varepsilon_0\\
&\quad \leq \left(\int_E \exp\left(8t|Lf(x)|\right) d\mu_\infty(x)\right)^{1/2} \left(\int_E \exp\left(2\varepsilon_0 |Lf(x)|^2\right) d\mu_\infty(x)\right)^{1/2}\\
&\qquad + \frac{1}{\varepsilon_0}\mathcal{H}_{\mu_\infty}(\mu) - \frac{1}{\varepsilon_0}\log\varepsilon_0,
\end{aligned}$$

which implies that $\sup_{0\leq t\leq T} \|A_t\|_{L^2(\mu)} < \infty$. $\square$

Proposition 10.16. *Take $\mu \in \mathsf{D}(\mathcal{H}_{\mu_\infty})$ and $h > 0$. For $\Sigma \in \Gamma_o(\mu, J_h\mu)$ and $f \in \mathcal{C}$ we have*

$$\frac{1}{h}\int_{E\times E} [\nabla_H f(y), x - y]_H\, d\Sigma(x,y) = 2\int_E Lf(x)\, d(J_h\mu)(x). \tag{10.20}$$

Proof. For each $t > 0$ we have

$$\begin{aligned} &\frac{1}{2t}\big(W_H^2(\mu,(U_t)_\#(J_h\mu)) - W_H^2(\mu,J_h\mu)\big) \\ &\quad\le \frac{1}{2t}\int_{E\times E} |x-U_t(y)|_H^2 - |x-y|_H^2\,d\Sigma \\ &\quad= \frac{1}{2t}\int_{E\times E} |y-U_t(y)|_H^2\,d\Sigma + \frac{1}{t}\int_{E\times E}[x-y,y-U_t(y)]_H\,d\Sigma. \end{aligned} \tag{10.21}$$

For μ_∞-a.e. y in E we have

$$|y-U_t(y)|_H = \left|\int_0^t \nabla_H f(U_s(y))\,ds\right|_H \le \|\nabla_H f\|_\infty t,$$

from which we infer that the first summand in the right hand side of (10.21) tends to 0 as $t\downarrow 0$. Moreover, the same estimate shows that $t\mapsto U_t(y)$ is right-continuous at $t=0$ for μ_∞-a.e. $y\in E$, hence

$$\frac{U_t(y)-y}{t} = \frac{1}{t}\int_0^t \nabla_H f(U_s(y))\,ds \to \nabla_H f(y), \qquad \mu_\infty\text{-a.e.}$$

Since $J_h\mu \ll \mu_\infty$, this implies that for Σ-a.e. $(x,y)\in E\times E$,

$$\frac{1}{t}[x-y,y-U_t(y)]_H \to -[x-y,\nabla_H f(y)]_H.$$

Since

$$\frac{1}{t}|[x-y,y-U_t(y)]_H| \le |x-y|_H\|\nabla_H f\|_\infty$$

and

$$\int_{E\times E}|x-y|_H\,d\Sigma \le \left(\int_{E\times E}|x-y|_H^2\,d\Sigma\right)^{1/2} = W_H(\mu,J_h\mu)<\infty,$$

we can use the dominated convergence theorem to pass to the limit in (10.21) to obtain

$$\overline{\lim_{t\downarrow 0}}\,\frac{1}{2t}\big(W_H^2(\mu,(U_t)_\# J_h\mu) - W_H^2(\mu,J_h\mu)\big) = -\int_{E\times E}[x-y,\nabla_H f(y)]_H\,d\Sigma.$$

By definition of $J_h\mu$ we have

$$\frac{h}{t}\Big(\mathcal{H}_{\mu_\infty}(J_h\mu) - \mathcal{H}_{\mu_\infty}\big((U_t)_\# J_h\mu\big)\Big) \le \frac{1}{2t}\Big(W_H^2(\mu,(U_t)_\# J_h\mu) - W_H^2(\mu,J_h\mu)\Big),$$

which leads to

$$\overline{\lim_{t\downarrow 0}}\,\frac{1}{t}\Big(\mathcal{H}_{\mu_\infty}(J_h\mu) - \mathcal{H}_{\mu_\infty}\big((U_t)_\# J_h\mu\big)\Big) \le -\frac{1}{h}\int_{E\times E}[x-y,\nabla_H f(y)]_H\,d\Sigma.$$

Replacing f by $-f$ and arguing similarly, we arrive at

$$\varliminf_{t\downarrow 0} \frac{1}{t}\Big(\mathcal{H}_{\mu_\infty}(J_h\mu) - \mathcal{H}_{\mu_\infty}\big((U_t)_\# J_h\mu\big)\Big) \ge -\frac{1}{h}\int_{E\times E} [x-y, \nabla_H f(y)]_H \, d\Sigma,$$

and combining these inequalities we infer that

$$\lim_{t\downarrow 0} \frac{1}{t}\Big(\mathcal{H}_{\mu_\infty}(J_h\mu) - \mathcal{H}_{\mu_\infty}\big((U_t)_\# J_h\mu\big)\Big) = -\frac{1}{h}\int_{E\times E} [x-y, \nabla_H f(y)]_H \, d\Sigma. \tag{10.22}$$

On the other hand, (10.17) implies that for $\nu = \rho\mu_\infty \in \mathsf{D}(\mathcal{H}_{\mu_\infty})$ and $\varphi \in C_{\mathrm{b}}(E)$,

$$\begin{aligned}
\int_E \varphi(x)\, d((U_t)_\#\nu)(x) &= \int_E \varphi(U_t(x))\rho(x)\, d\mu_\infty(x) \\
&= \int_E \varphi(x)\rho(U_{-t}(x))\, d((U_t)_\#\mu_\infty)(x) \\
&= \int_E \varphi(x)\rho(U_{-t}(x))K_t(x)\, d\mu_\infty(x),
\end{aligned}$$

and thus $(U_t)_\#\nu = \rho_t\mu_\infty$, where $\rho_t(x) := \rho(U_{-t}(x))K_t(x)$ for μ_∞-a.e. x in E. Taking into account that $\log(K_t \circ U_t) = 2tA_t \in L^1(\nu)$ by Lemma 10.15, it follows that

$$\begin{aligned}
\mathcal{H}_{\mu_\infty}((U_t)_\#\nu) &= \int_E \log\Big(\rho(U_{-t}(x))K_t(x)\Big)\, d((U_t)_\#\nu)(x) \\
&= \int_E \log\Big(\rho(x)K_t(U_t(x))\Big)\, d\nu(x) \\
&= \mathcal{H}_{\mu_\infty}(\nu) + \int_E \log K_t(U_t(x))\, d\nu(x) \\
&= \mathcal{H}_{\mu_\infty}(\nu) + 2t\int_E A_t(x)\, d\nu(x),
\end{aligned}$$

hence

$$\frac{\mathcal{H}_{\mu_\infty}((U_t)_\#\nu) - \mathcal{H}_{\mu_\infty}(\nu)}{t} = 2\int_E A_t(x)\, d\nu(x).$$

Lemma 10.15 implies that $(A_t)_{t\in[0,T]}$ is uniformly bounded in $L^2(\nu)$, and therefore uniformly integrable with respect to ν. Since $t \to U_t(x)$ is right-continuous at $t = 0$ for ν-a.e. $x \in E$, it follows that $A_t(x) \to Lf(x)$ for ν-a.e. $x \in E$ as $t \downarrow 0$, hence

$$\lim_{t\downarrow 0} \frac{\mathcal{H}_{\mu_\infty}((U_t)_\#\nu) - \mathcal{H}_{\mu_\infty}(\nu)}{t} = 2\int_E Lf(x)\, d\nu(x).$$

Applying this result to $\nu = J_h\mu$ and using (10.22) we obtain (10.20). $\square$

Now we are in a position to show that the gradient flow associated with $\mathcal{H}_{\mu_\infty}$ and W_H solves a Fokker-Planck equation associated with L. The factor 2 appearing in the statement of the result is not essential and can be removed by multiplying the entropy functional or the Hilbert space norm by a scaling factor (see also Remark 10.12).

Theorem 10.17. *For $\sigma \in \mathsf{D}(\mathcal{H}_{\mu_\infty})$, let $(\sigma_t)_{t\geq 0}$ be the gradient flow in $(\mathscr{P}(E), W_H)$ associated with the functional $\mathcal{H}_{\mu_\infty}$ and satisfying $\sigma_0 = \sigma$, which has been obtained in Theorem 10.11. The measures $(\sigma_t)_{t\geq 0}$ satisfy the Fokker-Planck equation*

$$\partial_t \sigma_t + 2L^* \sigma_t = 0$$

in the following sense: for all $f \in \mathcal{C}$ and $\alpha \in C_c^\infty[0,\infty)$ we have

$$\begin{aligned} -\int_0^\infty \alpha'(t) \int_E f(x)\, d\sigma_t(x)\, dt \qquad & \\ + \int_0^\infty \alpha(t) \int_E 2Lf(x)\, d\sigma_t(x)\, dt &= \alpha(0) \int_E f(x)\, d\sigma(x). \end{aligned} \tag{10.23}$$

Proof. Take $f \in \mathcal{C}$ and $\alpha \in C_c^\infty[0,\infty)$, and let $T > 0$ be such that $\operatorname{supp}(\alpha) \subseteq [0,T)$. For $h \in (0,1)$ set $N := \lfloor \frac{T}{h} \rfloor + 1$. For $k = 0, \ldots, N-1$, we take $\Sigma^k \in \Gamma_o(J_h^k \sigma, J_h^{k+1}\sigma)$ and set $\sigma_t^h := J_h^k \sigma$ whenever $t \in ((k-1)h, kh]$. With this notation we have

$$\begin{aligned} &\int_0^\infty \alpha'(t) f(x)\, d\sigma_t^h(x)\, dt \\ &\quad = \sum_{k=0}^{N-1} \big(\alpha((k+1)h) - \alpha(kh)\big) \int_E f(x)\, d(J_h^{k+1}\sigma)(x) \\ &\quad = \sum_{k=0}^{N-1} \alpha(kh) \bigg(\int_E f(x)\, d(J_h^k \sigma)(x) - \int_E f(x)\, d(J_h^{k+1}\sigma)(x) \bigg) \\ &\qquad - \alpha(0) \int_E f(x)\, d\sigma(x) \\ &\quad = \sum_{k=0}^{N-1} \alpha(kh) \bigg(\int_{E\times E} f(x) - f(y)\, d\Sigma^k(x,y) \bigg) - \alpha(0) \int_E f(x)\, d\sigma(x). \end{aligned}$$

Moreover, by Proposition 10.16,

$$\begin{aligned} &\int_0^\infty \alpha(t) \int_E 2Lf(x)\, d\sigma_t^h(x)\, dt \\ &\quad = \sum_{k=0}^{N-1} \int_{kh}^{(k+1)h} \alpha(t)\, dt \int_E 2Lf(x)\, d(J_h^{k+1}\sigma)(x) \\ &\quad = \frac{1}{h} \sum_{k=0}^{N-1} \int_{kh}^{(k+1)h} \alpha(t)\, dt \int_{E\times E} [\nabla_H f(y), x-y]_H\, d\Sigma^k(x,y). \end{aligned}$$

Combining these two identities we infer that

$$\begin{aligned}
\int_0^\infty \alpha'(t) \int_E f(x)\, d\sigma_t^h(x)\, dt - \int_0^\infty \alpha(t) \int_E 2Lf(x)\, d\sigma_t^h(x)\, dt \\
= \sum_{k=0}^{N-1} \alpha(kh) I_k + \sum_{k=0}^{N-1} \beta_k \int_{E\times E} [\nabla_H f(y), x-y]_H \, d\Sigma^k(x,y) \\
- \alpha(0) \int_E f(x)\, d\sigma(x) \\
= (I) + (II) - (III),
\end{aligned} \tag{10.24}$$

where

$$\begin{aligned}
I_k &:= \int_{E\times E} f(x) - f(y) - [\nabla_H f(y), x-y]_H \, d\Sigma^k(x,y), \\
\beta_k &:= \alpha(kh) - \frac{1}{h}\int_{kh}^{(k+1)h} \alpha(t)\, dt.
\end{aligned}$$

Before estimating both sums in (10.24) individually, we observe that by definition of $J_h^{k+1}\sigma$,

$$\frac{1}{2h} W_H^2(J_h^k\sigma, J_h^{k+1}\sigma) \le \mathcal{H}_{\mu_\infty}(J_h^k\sigma) - \mathcal{H}_{\mu_\infty}(J_h^{k+1}\sigma),$$

hence

$$\begin{aligned}
\frac{1}{2h}\sum_{k=0}^{N-1} W_H^2(J_h^k\sigma, J_h^{k+1}\sigma) &\le \sum_{k=0}^{N-1}\Big(\mathcal{H}_{\mu_\infty}(J_h^k\sigma) - \mathcal{H}_{\mu_\infty}(J_h^{k+1}\sigma)\Big) \\
&= \mathcal{H}_{\mu_\infty}(\sigma) - \mathcal{H}_{\mu_\infty}(J_h^N\sigma).
\end{aligned} \tag{10.25}$$

To estimate (I) we use (10.25) to obtain

$$\begin{aligned}
|(I)| &\le \frac{1}{2}\|\alpha\|_\infty \|\nabla_H^2 f\|_\infty \sum_{k=0}^{N-1} \int_{E\times E} |x-y|^2 \, d\Sigma^k \\
&= \frac{1}{2}\|\alpha\|_\infty \|\nabla_H^2 f\|_\infty \sum_{k=0}^{N-1} W_H^2(J_h^k\sigma, J_h^{k+1}\sigma) \\
&\le h\|\alpha\|_\infty \|\nabla_H^2 f\|_\infty \Big(\mathcal{H}_{\mu_\infty}(\sigma) - \mathcal{H}_{\mu_\infty}(J_h^N\sigma)\Big),
\end{aligned}$$

from which we infer that $(I) \to 0$ as $h \to 0$.

To estimate (II) we use the estimates $|\beta_k| \le h\|\alpha'\|_\infty$ and $N \le 2\frac{T}{h}$, the Cauchy-Schwarz inequality, and (10.25) once more, to arrive at

$$
\begin{aligned}
|(II)| &\le h\|\alpha'\|_\infty \|\nabla_H f\|_\infty \sum_{k=0}^{N-1} \int_{E\times E} |x-y|_H \, d\Sigma^k(x,y) \\
&\le h\|\alpha'\|_\infty \|\nabla_H f\|_\infty \sqrt{N} \Big(\sum_{k=0}^{N-1} \int_{E\times E} |x-y|_H^2 \, d\Sigma^k(x,y) \Big)^{1/2} \\
&\le \sqrt{2Th}\, \|\alpha'\|_\infty \|\nabla_H f\|_\infty \Big(\sum_{k=0}^{N-1} W_H^2 (J_h^k \sigma, J_h^{k+1}\sigma) \Big)^{1/2} \\
&\le \sqrt{2Th}\, \|\alpha'\|_\infty \|\nabla_H f\|_\infty \sqrt{2h \mathcal{H}_{\mu_\infty}(\sigma)},
\end{aligned}
$$

which converges to 0 as $h \downarrow 0$.

Combining the estimates for (I) and (II) with (10.24) we arrive at

$$
\begin{aligned}
&\int_0^\infty \alpha'(t) \int_E f(x)\, d\sigma_t^h(x)\, dt \\
&\quad - \int_0^\infty \alpha(t) \int_E 2Lf(x)\, d\sigma_t^h(x)\, dt + \alpha(0) \int_E f(x)\, d\sigma(x) \to 0,
\end{aligned} \tag{10.26}
$$

as $h \downarrow 0$.

It remains to analyse the limit behaviour of the first two terms in (10.26) individually. Since $\sigma_t^h \rightharpoonup \sigma_t$ as $h \downarrow 0$, we have for each $t > 0$,

$$
\int_E f(x)\, d\sigma_t^h(x) \to \int_E f(x)\, d\sigma_t(x), \tag{10.27}
$$

hence by dominated convergence,

$$
\int_0^\infty \alpha'(t) \int_E f(x)\, d\sigma_t^h(x)\, dt \to \int_0^\infty \alpha'(t) \int_E f(x)\, d\sigma_t(x)\, dt.
$$

We claim that

$$
\int_0^\infty \alpha(t) \int_E Lf(x)\, d\sigma_t^h(x)\, dt \to \int_0^\infty \alpha(t) \int_E Lf(x)\, d\sigma_t(x)\, dt \tag{10.28}
$$

as $h \downarrow 0$. The latter two identities together with (10.26) imply (10.23). Therefore, to complete the proof, it remains to show (10.28).

For this purpose, we use the explicit formula (10.10) to write $Lf(x) := b(x) + \langle x, F(x) \rangle$ for some $b \in C_{\mathrm{b}}(E)$ and $F \in C_{\mathrm{b}}(E; E^*)$. For $R > 0$ let $\zeta_R \in C_{\mathrm{b}}(\mathbb{R})$ be a cut-off function satisfying $\operatorname{supp} \zeta_R \subseteq [-2R, 2R]$, $\|\zeta_R\|_\infty \le 1$, and $(\zeta_R)|_{[-R,R]} = 1$. By Fernique's Theorem 1.3 there exists $\varepsilon > 0$ such that

$$
\int_E \exp\Big(\varepsilon |x|_E^2\Big)\, d\mu_\infty(x) < \infty.
$$

Using Young's inequality $uv \le e^u + v \log v$ for $u, v \ge 0$ we obtain for $t > 0$ and $h > 0$,

$$
\begin{aligned}
\int_E & |\langle x, F(x)\rangle| \big(1 - \zeta_R(|x|_E)\big)\, d\sigma_t^h(x) \\
&\le \frac{\|F\|_\infty}{R} \int_E |x|_E^2 \, d\sigma_t^h(x) \\
&\le \frac{\|F\|_\infty}{R} \bigg(\int_E \exp\Big(\varepsilon |x|_E^2\Big)\, d\mu_\infty(x) + \frac{1}{\varepsilon} \mathcal{H}_{\mu_\infty}(\sigma_t^h) - \frac{1}{\varepsilon}\log\varepsilon \bigg) \qquad (10.29) \\
&\le \frac{\|F\|_\infty}{R} \bigg(\int_E \exp\Big(\varepsilon |x|_E^2\Big)\, d\mu_\infty(x) + \frac{1}{\varepsilon} \mathcal{H}_{\mu_\infty}(\sigma) - \frac{1}{\varepsilon}\log\varepsilon \bigg) \\
&=: \frac{C}{R},
\end{aligned}
$$

with C not depending on h and R. Combining this estimate with the fact that $b \in C_{\mathrm{b}}(E)$, it follows that

$$
\int_E |Lf| \, d\sigma_t^h < \infty.
$$

Furthermore, as $\mathcal{H}_{\mu_\infty}(\sigma_t) < \infty$, the same argument shows that

$$
\int_E |Lf(x)| \, d\sigma_t(x) < \infty. \tag{10.30}
$$

Taking into account that $\sigma_t^h \rightharpoonup \sigma_t$ and the functions b and $\langle \cdot, F(\cdot)\rangle \zeta_R(|\cdot|_E)$ are contained in $C_{\mathrm{b}}(E)$, we obtain using (10.29),

$$
\begin{aligned}
\varlimsup_{h\downarrow 0} & \int_E Lf(x)\, d\sigma_t^h(x) \\
&= \varlimsup_{R\to\infty} \varlimsup_{h\downarrow 0} \bigg(\int_E b(x)\, d\sigma_t^h(x) + \int_E \langle x, F(x)\rangle \zeta_R(|x|_E)\, d\sigma_t^h(x) \\
&\qquad\qquad + \int_E \langle x, F(x)\rangle \big(1 - \zeta_R(|x|_E)\big)\, d\sigma_t^h(x) \bigg) \\
&\le \varlimsup_{R\to\infty} \bigg(\int_E b(x)\, d\sigma_t(x) + \int_E \langle x, F(x)\rangle \zeta_R(|x|_E)\, d\sigma_t(x) + \frac{C}{R} \bigg) \\
&\le \int_E b(x)\, d\sigma_t(x) + \int_E \langle x, F(x)\rangle\, d\sigma_t(x) \\
&= \int_E Lf(x)\, d\sigma_t(x),
\end{aligned}
$$

where the last inequality uses the dominated convergence theorem, which can be applied since $\int_E |\langle x, F(x)\rangle|\, d\mu_t < \infty$ in view of (10.30). The same argument shows that

$$
\varliminf_{h\downarrow 0} \int_E Lf(x)\, d\sigma_t^h(x) \ge \int_E Lf(x)\, d\sigma_t(x),
$$

and therefore we arrive at

$$\lim_{h\downarrow 0}\int_E Lf(x)\,d\sigma_t^h(x)=\int_E Lf(x)\,d\sigma_t(x).$$

Since $|\int_E Lf(x)\,d\sigma_t^h(x)|\le \|b\|_\infty+\frac{C}{R}+2R\|F\|_\infty$ for any $h>0$, $t>0$, and $R>0$, the dominated convergence theorem (which can be used due to the fact that $\alpha\in C_c^\infty[0,\infty))$ implies that

$$\lim_{h\downarrow 0}\int_0^\infty \alpha(t)\int_E Lf(x)\,d\sigma_t^h(x)\,dt=\int_0^\infty \alpha(t)\int_E Lf(x)\,d\sigma_t(x)\,dt,$$

which proves (10.28). This completes the proof. $\square$

Part III

Malliavin Calculus in Banach Spaces

11

Banach Space-valued Analysis on Wiener Spaces

This chapter deals with the Malliavin calculus for Banach space-valued random variables. Using radonifying operators instead of symmetric tensor products we extend the Wiener-Itô isometry to Banach spaces. In the white noise case we obtain two sided L^p-estimates for multiple stochastic integrals in arbitrary Banach spaces. It is shown that the Malliavin derivative is bounded on vector-valued Wiener-Itô chaoses. Our main tools are decoupling inequalities for vector-valued random variables. In the opposite direction we use Meyer's inequalities to give a new proof of a decoupling result for Gaussian chaoses in UMD Banach spaces.

11.1 Preliminaries

We start by collecting some preliminary results on decoupling, γ-radonifying operators, and UMD Banach spaces.

Decoupling

Decoupling inequalities go back to the work of McConnell and Taqqu [119, 120], Kwapień [96], Arcones and Giné [6], and de la Peña and Montgomery-Smith [50] among others. We refer to the monographs [49, 97] for extensive information on this topic.

First we introduce some notation which will be used throughout this chapter. For $j \geq 1$ and a finite sequence $\mathbf{i} = (i_1, \ldots, i_n)$ with values in $\{1, 2, \ldots\}$ we set

$$j(\mathbf{i}) = \#\{i_k : 1 \leq k \leq n, i_k = j\}, \quad |\mathbf{i}| = n, \quad |\mathbf{i}|_\infty := \max_{1 \leq k \leq n} i_k, \quad \mathbf{i}! = \prod_{k=1}^{n} k(\mathbf{i})!.$$

Let $(\gamma_n)_{n\geq 1}$ be a Gaussian sequence, i.e. a sequence of independent standard Gaussian random variables on a (sufficiently rich) probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let $(\gamma_n^{(k)})_{n\ge 1}$ be independent copies for each $k \ge 1$. Let $(H_m)_{m\ge 0}$ denote the Hermite polynomials already considered in Chapter 1. We set

$$\Psi_{\mathbf{i}} = (\mathbf{i}!)^{1/2} \prod_{j\ge 1} H_{j(\mathbf{i})}(\gamma_j).$$

The next theorem states two well-known decoupling results which were obtained in [6, 96, 120]. A general result containing both parts of the next theorem is due to Giné (see, e.g. [49, Theorem 4.2.7]).

Theorem 11.1. *Let E be a Banach space, let $m, n \ge 1$, and suppose that we are in one of the following two situations:*

1. *(symmetric case) Let $(x_{\mathbf{i}})_{|\mathbf{i}|=m} \subseteq E$ satisfy $x_{\mathbf{i}} = x_{\mathbf{i}'}$ whenever $\mathbf{i}'$ is a permutation of $\mathbf{i}$, and set*

$$F := \sum_{|\mathbf{i}|=m, |\mathbf{i}|_\infty \le n} (\mathbf{i}!/m!)^{1/2} \Psi_{\mathbf{i}} x_{\mathbf{i}}.$$

2. *(tetrahedral case) Let $(x_{\mathbf{i}})_{|\mathbf{i}|=m} \subseteq E$ satisfy $x_{\mathbf{i}} = 0$ whenever $j(\mathbf{i}) > 1$ for some $j \ge 1$, and set*

$$F := \sum_{|\mathbf{i}|=m, |\mathbf{i}|_\infty \le n} \gamma_{i_1} \cdot \ldots \cdot \gamma_{i_m} x_{\mathbf{i}}.$$

In both cases we put

$$\widetilde{F} := \sum_{|\mathbf{i}|=m, |\mathbf{i}|_\infty \le n} \gamma_{i_1}^{(1)} \cdot \ldots \cdot \gamma_{i_m}^{(m)} x_{\mathbf{i}}.$$

Then there exists a constant $C_m \ge 1$ depending only on m, such that for all $t > 0$ we have

$$\frac{1}{C_m} \mathbb{P}(\|\widetilde{F}\|_E > C_m t) \le \mathbb{P}(\|F\|_E > t) \le C_m \mathbb{P}(\|\widetilde{F}\|_E > \frac{t}{C_m}).$$

Consequently, for $1 \le p < \infty$ we have

$$\|F\|_{L^p(\Omega;E)} \eqsim_{p,m} \|\widetilde{F}\|_{L^p(\Omega;E)}.$$

Remark 11.2. The requirement that $|\mathbf{i}|_\infty \le n$ is chosen for convenience, to ensure that we are dealing with finite sums exclusively. Note however that the constants in all of our estimates do not depend on n.

Spaces of γ-radonifying operators

The class of radonifying operators which has been considered in Part I will play an important role in this chapter. We refer to Section 5.2 for the definition and the basic properties. An important role is played by spaces of iterated radonifying operators $\gamma^m(H,E)$. We define these spaces inductively by

$$\gamma^1(H,E) := \gamma(H,E), \quad \gamma^{m+1}(H,E) := \gamma(H,\gamma^m(H,E)), \quad m \geq 1.$$

To improve readability we will write $T(h,h')$ instead of $(Th)(h')$ if $T \in \gamma^2(H,E)$. Furthermore we will write $(h \otimes h') \otimes x$ to denote the operator $h \otimes (h' \otimes x) \in \gamma^2(H,E)$. Similar remarks apply when $m > 2$. For future use we record that for operators of the form

$$T = \sum_{|\mathbf{i}|=m, |\mathbf{i}|_\infty \leq n} (u_{i_1} \otimes \cdots \otimes u_{i_m}) \otimes x_{\mathbf{i}}, \quad x_{\mathbf{i}} \in E, \tag{11.1}$$

the norm in $\gamma^m(H,E)$ is given by

$$\|T\|^2_{\gamma^m(H,E)} = \mathbb{E}\Big\| \sum_{|\mathbf{i}|=m, |\mathbf{i}|_\infty \leq n} \gamma^{(1)}_{i_1} \cdot \ldots \cdot \gamma^{(m)}_{i_m} x_{\mathbf{i}} \Big\|^2_E, \tag{11.2}$$

where we use the multi-index notation from Section 11.1.

If K is a Hilbert space then $\gamma^m(H,K)$ is canonically isometric to the Hilbert space tensor product $H^{\widehat{\otimes} m}\widehat{\otimes} K$. It has been shown in [90] (see also [134]) that $\gamma^m(H,E)$ is isomorphic to $\gamma(H^{\widehat{\otimes} m}, E)$ for all $m \geq 1$ if and only if the Banach space E has Pisier's property (α) [144].

We have already considered in Proposition 5.13 the pairing

$$[T,S]_\gamma := \mathrm{tr}\,(T^*S), \qquad T \in \gamma(H,E),\ S \in \gamma(H,E^*), \tag{11.3}$$

which allows us to identify $\gamma(H,E^*)$ with a weak*-dense subspace of the dual space $\gamma(H,E)^*$. The next result from [147] (see also [90]) shows that if E is K-convex, the inclusion $\gamma(H,E^*) \hookrightarrow \gamma(H,E)^*$ is actually an isomorphism.

Proposition 11.3. *If E is K-convex, then* (11.3) *establishes an isomorphism of Banach spaces*

$$\gamma(H,E^*) \simeq (\gamma(H,E))^*.$$

We will return to K-convexity and its relevance for vector-valued Malliavin calculus in Remark 11.7 below.

Let us now consider the important special case that $H = L^2(M,\mu)$ for some σ-finite measure space (M,μ). A strongly measurable function $\phi : M^m \to E$ is said to be weakly-L^2 if $\langle \phi, x^* \rangle \in L^2(M^m)$ for all $x^* \in E^*$. We say that such a function represents an operator $T_\phi \in \gamma^m(L^2(M),E)$ if for all $f_1,\ldots,f_m \in L^2(M)$ and for all $x^* \in E^*$ we have

$$\begin{aligned}\langle T_\phi(f_1,\ldots,f_m), x^* \rangle = \int_{M^m} & f_1(t_1) \cdot \ldots \cdot f_m(t_m) \cdot \\ & \cdot \langle \phi(t_1,\ldots,t_m), x^* \rangle \, d\mu^{\otimes m}(t_1,\ldots,t_m).\end{aligned}$$

We will not always notationally distinguish between a function ϕ and the operator $T_\phi \in \gamma(L^2(M),E)$ that it represents. The subspace of operators which can be represented by a function is dense in $\gamma^m(L^2(M),E)$.

UMD Banach spaces

Let us collect some well-known facts concerning UMD Banach spaces. Let $1 < p < \infty$. A Banach space E is called a *UMD(p)-space* if there exists a constant $\beta_{p,E}$ such that for every finite L^p-martingale difference sequence $(d_j)_{j=1}^n$ with values in E and for every $\{-1,1\}$-valued sequence $(\varepsilon_j)_{j=1}^n$ we have

$$\Big(\mathbb{E}\Big\|\sum_{j=1}^n \varepsilon_j d_j\Big\|^p\Big)^{\frac{1}{p}} \le \beta_{p,E}\Big(\mathbb{E}\Big\|\sum_{j=1}^n d_j\Big\|^p\Big)^{\frac{1}{p}}.$$

It can be shown that if E is a UMD(p) space for some $1 < p < \infty$, then it is a UMD(p)-space for all $1 < p < \infty$, and henceforth a space with this property will simply be called a *UMD space*.

Examples of UMD spaces are all Hilbert spaces and the spaces $L^p(S)$ for $1 < p < \infty$ and σ-finite measure spaces (S, Σ, μ). If E is a UMD space, then $L^p(S; E)$ is a UMD space for $1 < p < \infty$. For an overview of the theory of UMD spaces and its applications in vector-valued stochastic analysis and harmonic analysis we recommend Burkholder's review article [21].

11.2 Wiener-Itô chaos in Banach spaces

In this section we will prove a Banach space analogue of the classical Wiener-Itô isometry. First we fix some notation, most of which has already been used in Chapter 1.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a (sufficiently rich) probability space, let H be a real separable Hilbert space and let $W : H \to L^2(\Omega)$ be an *isonormal Gaussian process* on H, i.e., W is a bounded linear operator from H to $L^2(\Omega)$ such that the random variables $W(h)$ are centred Gaussian and satisfy

$$\mathbb{E}(W(h_1)W(h_2)) = [h_1, h_2]_H, \quad h_1, h_2 \in H.$$

We assume that $\mathcal{F}$ is the σ-field generated by $\{W(h) : h \in H\}$. We fix an orthonormal basis $(u_j)_{j\ge1}$ of H, and consider the Gaussian sequence defined by $\gamma_j := W(u_j)$ for $j \ge 1$. For $m \ge 0$ we consider the m-th Wiener-Itô chaos

$$H^{(m)} = \overline{\mathrm{lin}}\{H_m(W(h)) : \|h\| = 1\},$$

where the closure is taken in $L^2(\Omega)$. Furthermore, let $H^{\circledS m}$ be the m-fold symmetric tensor power which is defined to be the range of the orthogonal projection $P_{\circledS} \in \mathcal{L}(H^{\widehat{\otimes} m})$ given by

$$P_{\circledS}(h_1 \otimes \cdots \otimes h_m) = \frac{1}{m!}\sum_{\pi\in S_m} h_{\pi(1)} \otimes \cdots \otimes h_{\pi(m)}, \qquad h_1, \ldots, h_m \in H,$$

where S_m is the group of permutations of $\{1, \ldots, m\}$.

Theorem 1.14 asserts that the following orthogonal decomposition holds:

$$L^2(\Omega, \mathcal{F}, \mathbb{P}) = \bigoplus_{m \geq 0} H^{(m)}.$$

Moreover, the mapping Φ_m defined by

$$\Phi_m : P_{\circledS}(u_{i_1} \otimes \cdots \otimes u_{i_m}) \mapsto (\mathbf{i}!/m!)^{1/2} \Psi_{\mathbf{i}}, \tag{11.4}$$

extends to an isometry from $H^{\circledS m}$ onto $H^{(m)}$. Recall that $\Psi_{\mathbf{i}}$ is the generalised Hermite polynomial defined in Section 11.1, to which we refer for notation.

Let us consider the vector-valued Gaussian chaos

$$H^{(m)}(E) := \overline{\mathrm{lin}}\{f \otimes x : f \in H^{(m)}, x \in E\},$$

where the closure is taken in $L^2(\Omega; E)$. The following well-known result is a consequence of the decoupling result in Theorem 11.1(1) and the Kahane-Khintchine inequalities. Extensive information on this topic can be found in the monographs [49, 97].

Proposition 11.4. *Let E be a Banach space, let $m \geq 1$, and let $1 \leq p, q < \infty$. For all $F \in H^{(m)}(E)$ we have*

$$\|F\|_{L^p(\Omega;E)} \eqsim_{m,p,q} \|F\|_{L^q(\Omega;E)}.$$

Our next goal is the construction of the spaces $\gamma^{\circledS m}(H, E)$, which will be the Banach space substitutes for the symmetric Hilbert space tensor powers $H^{\circledS m}$. We refer to Section 11.1 for the definition of the space $\gamma^m(H, E)$. For $T \in \gamma^m(H, E)$ we define its symmetrisation $P_{\circledS} T \in \gamma^m(H, E)$ by

$$(P_{\circledS} T)(h_1, \ldots, h_m) := \frac{1}{m!} \sum_{\pi \in S_m} T(h_{\pi(1)}, \ldots, h_{\pi(m)}), \quad h_1, \ldots, h_m \in H,$$

and we will say that $T \in \gamma^m(H, E)$ is symmetric if $P_{\circledS} T = T$. The mapping $P_{\circledS}$ is easily seen to be a projection in $\mathcal{L}(\gamma^m(H, E))$ and we define $\gamma^{\circledS m}(H, E)$ to be its range.

We remark that if K is a Hilbert space, then $\gamma^{\circledS m}(H, K)$ is isometrically isomorphic to the space $H^{\circledS m} \widehat{\otimes} K$, where $\widehat{\otimes}$ denotes the Hilbert space tensor product.

Now we are ready to state a Banach space-valued extension of the canonical isometry (11.4).

Theorem 11.5. *Let E be a Banach space, let $1 \leq p < \infty$, and let $m \geq 1$. The mapping*

$$(\Phi_m \otimes I) : P_{\circledS}(h_{i_1} \otimes \cdots \otimes h_{i_m}) \otimes x \mapsto (\mathbf{i}!/m!)^{1/2} \Psi_{\mathbf{i}} \otimes x,$$

extends to a bounded operator $(\Phi_m \otimes I) : \gamma^{\circledS m}(H, E) \to L^p(\Omega; E)$, which maps $\gamma^{\circledS m}(H, E)$ onto $H^{(m)}(E)$. Moreover, we have equivalence of norms

$$\|(\Phi_m \otimes I)T\|_{L^p(\Omega;E)} \eqsim_{m,p} \|T\|_{\gamma^m(H,E)}, \quad T \in \gamma^{\circledS m}(H, E).$$

Proof. Let T be a symmetric operator of the form (11.1) and observe that

$$T = \sum_{|\mathbf{i}|=m, |\mathbf{i}|_\infty \le n} P_{\circledS}(u_{i_1} \otimes \cdots \otimes u_{i_m}) \otimes x_{\mathbf{i}}.$$

Using (11.2), the decoupling result from Theorem 11.1(1) and the Kahane-Khintchine inequalities we obtain

$$\begin{aligned} \mathbb{E}\|(\Phi_m \otimes I)T\|_E^p &= \mathbb{E}\Big\| \sum_{|\mathbf{i}|=m, |\mathbf{i}|_\infty \le n} (\mathbf{i}!/m!)^{1/2} \Psi_{\mathbf{i}} x_{\mathbf{i}} \Big\|_E^p \\ &\eqsim_{m,p} \mathbb{E}\Big\| \sum_{|\mathbf{i}|=m, |\mathbf{i}|_\infty \le n} \gamma_{i_1}^{(1)} \cdot \ldots \cdot \gamma_{i_m}^{(m)} x_{\mathbf{i}} \Big\|_E^p \eqsim_{m,p} \|T\|_{\gamma^m(H,E)}^p. \end{aligned}$$

In view of Proposition 11.4 it is clear that $\Phi_m \otimes I$ maps $\gamma^{\circledS m}(H,E)$ into $H^{(m)}(E)$. To show that its range is $H^{(m)}(E)$, we observe that $\Phi_m \otimes I(h^{\otimes m} \otimes x) = H_m(W(h)) \cdot x$ for all $h \in H$ with $\|h\| = 1$ and all $x \in E$. Now the result follows from the norm estimate above and the identity

$$H^{(m)}(E) = \overline{\mathrm{lin}}\{H_m(W(h)) \cdot x : \|h\| = 1, x \in E\},$$

where the closure is taken in $L^p(\Omega; E)$. $\square$

Remark 11.6. In the special case that $E = \mathbb{R}$ and $p = 2$ we recover the classical Wiener-Itô isometry (see Theorem 1.19).

Remark 11.7. Let $m \ge 1$ and let J_m be the orthogonal projection onto $H^{(m)}$. It is well known that for all $1 < p < \infty$ the restriction of J_m to $L^p(\Omega) \cap L^2(\Omega)$ extends to a bounded projection on $L^p(\Omega)$. A Banach space E is said to be K-convex if $J_1 \otimes I$ extends to a bounded operator on $L^2(\Omega; E)$. Actually, this notion is usually defined using Rademacher instead of Gaussian random variables, but this does not affect the class of Banach spaces under consideration [63]. It has been shown by Pisier [145] that in this case the operators $J_m \otimes I$ (which will be denoted by J_m below) are bounded for all $m \ge 1$ and all $1 < p < \infty$. Every UMD space is K-convex. These facts will be used in Sections 11.4 and 11.5.

11.3 Multiple Wiener-Itô integrals in Banach spaces

As in the previous section we consider a real separable Hilbert space H and an isometry $W : H \to L^2(\Omega)$ onto a closed linear subspace consisting of Gaussian random variables.

In addition we assume in this section that $H = L^2(M, \mathcal{B}, \mu)$ for some σ-finite non-atomic measure space M. We let $\mathcal{B}_0 := \{B \in \mathcal{B} : \mu(B) < \infty\}$. For $A \in \mathcal{B}_0$ we write with a slight abuse of notation $W(A) := W(\mathbf{1}_A)$. In this way

W defines an $L^2(\Omega)$-valued measure on $\mathcal{B}_0$ which is called the white noise based on μ.

Our next goal is to construct multiple stochastic integrals for Banach space-valued functions. Our construction generalises the well known multiple stochastic integral for Hilbert space-valued functions, and in another direction, the (single) stochastic integral for Banach space-valued functions which has been constructed in [135].

For fixed $m \geq 1$ we define $\mathcal{E}_m(E)$ to be the linear space of tetrahedral simple functions $F : M^m \to \mathbb{R}$ of the form

$$F = \sum_{|\mathbf{i}|=m, |\mathbf{i}|_\infty \leq n} \mathbf{1}_{A_{i_1} \times \cdots \times A_{i_m}} \cdot x_{\mathbf{i}}, \tag{11.5}$$

where the A_j's are pairwise disjoint sets in $\mathcal{B}_0$, $n \geq 1$, and the coefficients $x_{\mathbf{i}} \in E$ vanish whenever $j(\mathbf{i}) > 1$ for some $j \geq 1$. It is easy to see that such a function F represents an operator $T_F \in \gamma^m(L^2(M), E)$ in the sense described in Section 11.1, and by taking an orthonormal basis $(u_j)_{j\geq 1}$ of $L^2(M)$ with $u_j = \mu(A_j)^{-1/2}\mathbf{1}_{A_j}$ for $j = 1, \ldots, n$, one can check that

$$\begin{aligned} \|T_F\|^2_{\gamma^m(L^2(M),E)} = \mathbb{E}\Big\| \sum_{|\mathbf{i}|=m, |\mathbf{i}|_\infty \leq n} & \gamma^{(1)}_{i_1} \cdot \ldots \cdot \gamma^{(m)}_{i_m} \cdot \\ & \cdot \mu(A_1)^{1/2} \cdot \ldots \cdot \mu(A_n)^{1/2} \cdot x_{\mathbf{i}} \Big\|^2_E . \end{aligned} \tag{11.6}$$

We recall that $(\gamma^{(k)}_j)_{j\geq 1}$ are independent Gaussian sequences for $k \geq 1$.

Lemma 11.8. *The collection of operators represented by functions in $\mathcal{E}_m(E)$ is dense in $\gamma^m(L^2(M), E)$ for all $m \geq 1$.*

Proof. This follows by reasoning as in the proof of the corresponding scalar-valued result [138, p.10], taking into account that the measure space M is non-atomic. □

Suppose that $T_F \in \gamma^m(L^2(M), E)$ is represented by a strongly measurable weakly-L^2 function F. Then T_F belongs to $\gamma^{\circledS m}(L^2(M), E)$ if and only if F agrees $\mu^{\otimes m}$-almost everywhere with its symmetrisation $\widetilde{F}$ defined by

$$\widetilde{F}(t_1, \ldots, t_m) := \frac{1}{m!} \sum_{\pi \in S_m} F(t_{\pi(1)}, \ldots, t_{\pi(m)}).$$

For $F \in \mathcal{E}_m(E)$ of the form (11.5) we define the multiple Wiener-Itô integral $I_m(F) \in L^2(\Omega; E)$ by

$$I_m(F) = \sum_{|\mathbf{i}|=m, |\mathbf{i}|_\infty \leq n} W(A_{i_1}) \cdot \ldots \cdot W(A_{i_m}) \cdot x_{\mathbf{i}}. \tag{11.7}$$

One easily checks that this definition does not depend on the representation of F as an element of $\mathcal{E}_m(E)$. Moreover, I_m is linear and $I_m(F) = I_m(\widetilde{F})$. The next theorem may be considered as a generalisation of the classical Wiener-Itô-isometry for multiple stochastic integrals to the Banach space setting.

Theorem 11.9. *Let $m \geq 1$ and $1 \leq p < \infty$. The operator $I_m : \mathcal{E}_m(E) \to L^p(\Omega; E)$ extends uniquely to a bounded operator*

$$I_m : \gamma^m(L^2(M), E) \to L^p(\Omega; E)$$

which maps $\gamma^m(L^2(M), E)$ onto $H^{(m)}(E)$. Moreover, for $F \in \gamma^m(L^2(M), E)$ we have:

(i) $I_m F = I_m \widetilde{F}$;
(ii) $\|I_m F\|_{L^p(\Omega;E)} \eqsim_{m,p} \|\widetilde{F}\|_{\gamma^m(L^2(M),E)} \leq \|F\|_{\gamma^m(L^2(M),E)}$.

Proof. First we show that for all $F \in \mathcal{E}_m(E)$ the following equivalence of norms holds:

$$\|I_m F\|_{L^p(\Omega;E)} \eqsim_{m,p} \|\widetilde{F}\|_{\gamma^m(L^2(M),E)}.$$

For that purpose we take $F \in \mathcal{E}_m(E)$ of the form (11.5). Since $I_m(F) = I_m(\widetilde{F})$ we may assume that F is symmetric, hence $x_{(i_{\pi(1)},\ldots,i_{\pi(m)})} = x_{(i_1,\ldots,i_m)}$ for all permutations $\pi \in S_m$. Let $(u_j)_{j\geq 1}$ be an orthonormal basis of $L^2(M)$ with $u_j = \mu(A_j)^{-1/2}\mathbf{1}_{A_j}$ for $j = 1, \ldots, n$, and let $(\gamma_j)_{j\geq 1}$ be the Gaussian sequence $\gamma_j = W(u_j)$ for $j \geq 1$. Using the decoupling inequalities from Theorem 11.1(2), (11.6), and the Kahane-Khintchine inequalities we obtain

$$\begin{aligned}
\|I_m F\|^p_{L^p(\Omega;E)} &= \mathbb{E}\Big\| \sum_{|\mathbf{i}|=m, |\mathbf{i}|_\infty \leq n} W(A_{i_1}) \cdot \ldots \cdot W(A_{i_m}) \cdot x_{\mathbf{i}} \Big\|^p_E \\
&= \mathbb{E}\Big\| \sum_{|\mathbf{i}|=m, |\mathbf{i}|_\infty \leq n} \gamma_{i_1} \cdot \ldots \cdot \gamma_{i_m} \cdot \mu(A_1)^{1/2} \cdot \ldots \cdot \mu(A_n)^{1/2} \cdot x_{\mathbf{i}} \Big\|^p_E \\
&\eqsim_{m,p} \mathbb{E}\Big\| \sum_{\substack{|\mathbf{i}|=m \\ |\mathbf{i}|_\infty \leq n}} \gamma^{(1)}_{i_1} \cdot \ldots \cdot \gamma^{(m)}_{i_m} \cdot \mu(A_1)^{1/2} \cdot \ldots \cdot \mu(A_n)^{1/2} \cdot x_{\mathbf{i}} \Big\|^p_E \\
&\eqsim_{m,p} \|F\|^p_{\gamma^m(L^2(M),E)}.
\end{aligned}$$

Now the first claim follows from Lemma 11.8. To prove that $I_m T \in H^{(m)}(E)$ for all $T \in \gamma^m(L^2(M), E)$ we first let $T = T_F$ for some tetrahedral function F of the form (11.5). It follows from (11.7) and the fact that

$$W(A_{j_1}) \cdot \ldots \cdot W(A_{j_m}) \in H^{(m)}$$

whenever all j_k's are different, that $I_m T \in H^{(m)}(E)$. Since I_m is continuous the same holds for general $T \in \gamma^m(L^2(M), E)$ by Lemma 11.8. To show that the mapping $I_m : \gamma^m(L^2(M), E) \to H^{(m)}(E)$ is surjective we proceed as in Theorem 11.5. The other statements are clear in view of Lemma 11.8. $\square$

11.4 The Malliavin derivative

In this section we study a vector-valued analogue of the Malliavin derivative. Consider a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a real separable Hilbert space H, and an isonormal Gaussian process $W : H \to L^2(\Omega)$. As before we assume that $\mathcal{F}$ is the σ-algebra generated by W.

Let us introduce some notation. For $n \geq 1$ we denote by $C^\infty_{pol}(\mathbb{R}^n)$ the vector space of all C^∞-functions $f : \mathbb{R}^n \to \mathbb{R}$ such that f and its partial derivatives of all orders have polynomial growth, i.e. for every multi-index α there exist positive constants C_α, p_α such that

$$|\partial_\alpha f(x)| \leq C_\alpha (1 + |x|)^{p_\alpha}.$$

Let $\mathscr{S}$ be the collection of all random variables $f : \Omega \to \mathbb{R}$ of the form

$$f = \varphi(W(h_1), \ldots, W(h_n)) \tag{11.8}$$

for some $\varphi \in C^\infty_{pol}(\mathbb{R}^n)$, $h_1, \ldots, h_n \in H$ and $n \geq 1$.

For a real Banach space E we consider the dense subspace $\mathscr{S}(E)$ of $L^p(\Omega; E)$, $1 \leq p < \infty$, consisting of all functions $F : \Omega \to E$ of the form

$$F = \sum_{i=1}^{n} f_i \cdot x_i,$$

where $f_i \in \mathscr{S}$ and $x_i \in E$, $i = 1, \ldots n$. Occasionally it will be convenient to work with the space $\mathcal{P}(E)$, which is defined similarly, except that the functions φ are required to be polynomials.

For a function $F = f \cdot x \in \mathscr{S}(E)$ with f of the form (11.8) we define its Malliavin derivative DF by

$$DF = \sum_{j=1}^{n} \partial_j \varphi(W(h_1), \ldots, W(h_n)) h_j \otimes x. \tag{11.9}$$

This definition extends to $\mathscr{S}(E)$ by linearity. For $F \in \mathscr{S}(E)$ the Malliavin derivative DF is a random variable which takes values in the algebraic tensor product $H \otimes E$, which we endow with the norm $\| \cdot \|_{\gamma(H,E)}$ (cf. Section 11.1).

The following result is the simplest case of the integration by parts formula. We omit the proof, which is the same as in the scalar-valued case [138, Lemma 1.2.1].

Lemma 11.10. *If $F \in \mathscr{S}(E)$, then $\mathbb{E}(DF(h)) = \mathbb{E}(W(h)F)$ for all $h \in H$.*

A straightforward computation shows that the following product rule holds:

$$D\langle F, G \rangle = \langle DF, G \rangle + \langle F, DG \rangle, \quad F \in \mathscr{S}(E), G \in \mathscr{S}(E^*).$$

Here $\langle \cdot, \cdot \rangle$ denotes the duality between E and E^*. Combining this with Lemma 11.10 we obtain the following integration by parts formula:

$$\mathbb{E}\langle DF(h), G\rangle = \mathbb{E}(W(h)\langle F, G\rangle) - \mathbb{E}\langle F, DG(h)\rangle, \quad F \in \mathscr{S}(E), G \in \mathscr{S}(E^*). \tag{11.10}$$

This identity is the main ingredient in the proof of the following result.

Proposition 11.11. *The Malliavin derivative D is closable as an operator from $L^p(\Omega; E)$ into $L^p(\Omega; \gamma(H, E))$ for all $1 \le p < \infty$.*

Proof. Let (F_n) be a sequence in $\mathscr{S}(\Omega) \otimes E$ be such that $F_n \to 0$ in $L^p(\Omega; E)$ and $DF_n \to X$ in $L^p(\Omega; \gamma(H, E))$ as $n \to \infty$. We must prove that $X = 0$.

Fix $h \in H$ and define

$$V_h := \{G \in \mathscr{S}(\Omega) \otimes E^* : \ W(h)G \in \mathscr{S}(\Omega) \otimes E^*\}.$$

We claim that V_h is weak*-dense in $(L^p(\Omega; E))^*$. Let $\frac{1}{p} + \frac{1}{q} = 1$. To prove this it suffices to note that the subspace $\{G \in \mathscr{S}(\Omega) : W(h)G \in \mathscr{S}(\Omega)\}$ is weak*-dense in $L^q(\Omega)$ and that $L^q(\Omega) \otimes E^*$ is weak*-dense in $(L^p(\Omega; E))^*$.

Fix $G \in V_h$. Using (11.10) and the fact that the mapping $Y \mapsto \mathbb{E}\langle Y(h), G\rangle$ defines a bounded linear functional on $L^p(\Omega; \gamma(H, E))$ we obtain

$$\mathbb{E}\langle X(h), G\rangle = \lim_{n\to\infty} \mathbb{E}\langle DF_n(h), G\rangle = \lim_{n\to\infty} \mathbb{E}(W(h)\langle F_n, G\rangle) - \mathbb{E}\langle F_n, DG(h)\rangle.$$

Since $W(h)G$ and $DG(h)$ are bounded it follows that this limit equals zero. Since V_h is weak*-dense in $(L^p(\Omega; E))^*$, we obtain that $X(h)$ vanishes almost surely. Now we choose an orthonormal basis $(h_j)_{j\ge 1}$ of H. It follows that almost surely we have $X(h_j) = 0$ for all $j \ge 1$. Hence, $X = 0$ almost surely. □

With a slight abuse of notation we will denote the closure of D again by D. Its domain in $L^p(\Omega; E)$ will be denoted by $\mathbb{D}^{1,p}(\Omega; E)$, which is a Banach space endowed with the norm

$$\|F\|_{\mathbb{D}^{1,p}(\Omega;E)} := \big(\|F\|^p_{L^p(\Omega;E)} + \|DF\|^p_{L^p(\Omega;\gamma(H,E))}\big)^{1/p}.$$

Furthermore we will write $\mathbb{D}^{1,p}(\Omega) := \mathbb{D}^{1,p}(\Omega; \mathbb{R})$.

Derivatives of higher order are defined inductively. For $n \ge 1$ we define

$$\begin{aligned} \mathbb{D}^{n+1,p}(\Omega; E) :=& \{F \in \mathbb{D}^{n,p}(\Omega; E) : D^n F \in \mathbb{D}^{1,p}(\Omega; \gamma^n(H, E))\}, \\ D^{n+1}F :=& D(D^n F), \quad F \in \mathbb{D}^{n+1,p}(\Omega; E). \end{aligned}$$

It follows from Proposition 11.11 that D^n is a closed and densely defined operator from $\mathbb{D}^{n-1,p}(\Omega; E)$ into $L^p(\Omega; \gamma^n(H, E))$. Its domain is denoted by $\mathbb{D}^{n,p}(\Omega; E)$ which is a Banach space endowed with the norm

$$\|F\|_{n,p} := \|F\|_{\mathbb{D}^{n,p}(\Omega;E)} := \Big(\|F\|^p_{L^p(\Omega;E)} + \sum_{k=1}^n \|D^k F\|^p_{L^p(\Omega;\gamma^k(H,E))}\Big)^{1/p}.$$

The main result in this section describes the behaviour of the Malliavin derivative on the E-valued Wiener-Itô chaoses. It extends [138, Proposition 1.2.2] to Banach spaces (and to $1 \le p < \infty$, but this is well-known in the scalar case).

Theorem 11.12. *Let E be a Banach space, let $1 \le p < \infty$ and let $m \ge 1$. Then we have $H^{(m)}(E) \subseteq \mathbb{D}^{1,p}(\Omega; E)$ and $D(H^{(m)}(E)) \subseteq H^{(m-1)}(\gamma(H,E))$. Moreover, the following equivalence of norms holds:*

$$\|DF\|_{L^p(\Omega;\gamma(H,E))} \eqsim_{p,m} \|F\|_{L^p(\Omega;E)}, \quad F \in H^{(m)}(E).$$

Proof. Let $(u_j)_{j\ge1}$ be an orthonormal basis of H and put $\gamma_j := W(u_j)$. Let $(\gamma_j^{(k)})_{j\ge1}$ and $(\widetilde{\gamma}_j)_{j\ge1}$ be independent copies of $(\gamma_j)_{j\ge1}$. For $\mathbf{i} = (i_1, \dots, i_m)$ and $k \ge 1$ we will write $(\mathbf{i}, k) = (i_1, \dots, i_m, k)$.

First we take $F \in H^{(m)}(E)$ of the form

$$F = \sum_{|\mathbf{i}|=m, |\mathbf{i}|_\infty \le n} \tfrac{\mathbf{i}!}{m!^{1/2}} \prod_{j=1}^n H_{j(\mathbf{i})}(\gamma_j)\, x_{\mathbf{i}}.$$

Clearly we may assume without loss of generality that the coefficients $x_{\mathbf{i}}$ are symmetric, i.e. $x_{\mathbf{i}} = x_{\mathbf{i}'}$ whenever $\mathbf{i}'$ is a permutation of $\mathbf{i}$.

It follows from Theorem 11.1(1) that

$$\begin{aligned} \mathbb{E}\|F\|_E^p &= \mathbb{E}\Big\| \sum_{|\mathbf{i}|=m, |\mathbf{i}|_\infty \le n} \tfrac{\mathbf{i}!}{m!^{1/2}} \prod_{j=1}^n H_{j(\mathbf{i})}(\gamma_j) x_{\mathbf{i}} \Big\|_E^p \\ &\eqsim_{m,p} \mathbb{E}\Big\| \sum_{|\mathbf{i}|=m} \gamma_{i_1}^{(1)} \cdot \ldots \cdot \gamma_{i_m}^{(m)} x_{\mathbf{i}} \Big\|_E^p. \end{aligned} \tag{11.11}$$

On the other hand, by a change of variables to modify the range of summation from $\{|\mathbf{i}| = m\}$ to $\{|\mathbf{i}| = m-1\}$, and rearranging terms, we obtain with the convention that $H_{-1} = 0$,

$$\begin{aligned} DF &= \sum_{|\mathbf{i}|=m, |\mathbf{i}|_\infty \le n} \tfrac{\mathbf{i}!}{m!^{1/2}} \sum_{k=1}^n \prod_{j\ne k} H_{j(\mathbf{i})}(\gamma_j) H_{k(\mathbf{i})-1}(\gamma_k) \cdot u_k \otimes x_{\mathbf{i}}, \\ &= \sum_{k=1}^n u_k \otimes \Big(m^{1/2} \sum_{|\mathbf{i}|=m-1, |\mathbf{i}|_\infty \le n} \tfrac{\mathbf{i}!}{(m-1)!} \prod_{j=1}^n H_{j(\mathbf{i})}(\gamma_j) x_{(\mathbf{i},k)} \Big) \\ &= \sum_{k=1}^n u_k \otimes \Big(\tfrac{m^{1/2}}{(m-1)!^{1/2}} \sum_{|\mathbf{i}|=m-1, |\mathbf{i}|_\infty \le n} \tfrac{\mathbf{i}!^{1/2}}{(m-1)!^{1/2}} \Psi_{\mathbf{i}} x_{(\mathbf{i},k)} \Big). \end{aligned}$$

Using the Kahane-Khintchine inequalities and Theorem 11.1(1) once more, we find

$$
\begin{aligned}
\mathbb{E}\|DF\|^p_{\gamma(H,E)} &\eqsim_p \mathbb{E}\widetilde{\mathbb{E}}\Big\|\sum_{k=1}^n \widetilde{\gamma}_k DF(u_k)\Big\|_E^p \\
&= \mathbb{E}\widetilde{\mathbb{E}}\Big\|\tfrac{m^{1/2}}{(m-1)!^{1/2}}\sum_{k=1}^n \widetilde{\gamma}_k \sum_{|\mathbf{i}|=m-1,|\mathbf{i}|_\infty\le n} \tfrac{\mathbf{i}!^{1/2}}{(m-1)!^{1/2}}\Psi_{\mathbf{i}} x_{(\mathbf{i},k)}\Big\|_E^p \\
&= \widetilde{\mathbb{E}}\mathbb{E}\Big\|\tfrac{m^{1/2}}{(m-1)!^{1/2}}\sum_{|\mathbf{i}|=m-1,|\mathbf{i}|_\infty\le n} \tfrac{\mathbf{i}!^{1/2}}{(m-1)!^{1/2}}\Psi_{\mathbf{i}}\Big(\sum_{k=1}^n \widetilde{\gamma}_k x_{(\mathbf{i},k)}\Big)\Big\|_E^p \\
&\eqsim_{m,p} \widetilde{\mathbb{E}}\mathbb{E}\Big\|\sum_{|\mathbf{i}|=m-1,|\mathbf{i}|_\infty\le n}\sum_{k=1}^n \gamma^{(1)}_{i_1}\cdot\ldots\cdot\gamma^{(m-1)}_{i_{m-1}}\widetilde{\gamma}_k x_{(\mathbf{i},k)}\Big\|_E^p.
\end{aligned}
\tag{11.12}
$$

Comparing (11.11) and (11.12) yields the norm estimate. The theorem follows by the closedness of D and the fact that functions F of the form considered above are dense in $H^{(m)}(E)$. □

Remark 11.13. In the special case where E is a UMD Banach space the result above is known. Indeed, it follows from Meyer's inequalities (Theorem 11.16 below) that

$$\|DF\|_{L^p(\Omega;\gamma(H,E))} \eqsim_{p,E} m^{1/2}\|F\|_{L^p(\Omega;E)}, \quad F\in H^{(m)}(E).$$

This formula gives an explicit dependence on m, but in contrast with Theorem 5.3 the constants depend on (the Hilbert transform constants of) E. We return to this observation in Section 11.5.

11.5 Meyer's inequalities and their consequences

Let $(P(t))_{t\ge 0}\subseteq \mathcal{L}(L^2(\Omega))$ be the Ornstein-Uhlenbeck semigroup defined by

$$P(t) := \sum_{m\ge 0} e^{-mt}J_m. \tag{11.13}$$

As is well known, this semigroup extends to a C_0-semigroup of positive contractions on $L^p(\Omega)$ for all $1\le p<\infty$. We refer the reader to [138] for proofs of these and other elementary properties.

Let E be an arbitrary Banach space. By positivity of P, $(P(t)\otimes I)_{t\ge 0}$ extends to a C_0-semigroup of contractions on the Lebesgue-Bochner spaces $L^p(\Omega;E)$ for $1\le p<\infty$ which will be denoted by $(P_E(t))_{t\ge 0}$. The domain in $L^p(\Omega;E)$ of its infinitesimal generator L_E is denoted $\mathsf{D}_p(L_E)$. The subordinated semigroup $(Q_E(t))_{t\ge 0}$ is defined by

$$Q_E(t)f := \int_0^\infty P_E(s)f\, d\nu_t(s), \tag{11.14}$$

where the probability measure ν_t is given by

$$d\nu_t(s) = \frac{t}{2\sqrt{\pi s^3}} e^{-t^2/4s}\, ds, \quad t > 0. \tag{11.15}$$

The generator of $(Q_E(t))_{t\geq 0}$ will be denoted by C_E. As is well known we have

$$C_E = -(-L_E)^{1/2}.$$

Often, when there is no danger of confusion, we will omit the subscripts E.

The next lemma is a vector-valued analogue of the representation of L as a generator associated with a Dirichlet form. We omit the proof which follows from the scalar-valued analogue in a straightforward way.

Lemma 11.14. *Let E be a UMD space. For all $F \in \mathcal{P}(E)$ and $G \in \mathbb{D}^{1,p}(\Omega; E^*)$ we have*

$$\mathbb{E}\langle L_E F, G\rangle = -\mathbb{E}[DF, DG]_\gamma.$$

In the following Lemma we collect some useful commutation relations, which follow easily from the corresponding scalar-valued results.

Lemma 11.15. *Let E be a Banach space and let $1 \leq p < \infty$.*

(i) For $F \in \mathbb{D}^{1,p}(\Omega; E)$ we have $P_E(t)F$, $Q_E(t)F \in \mathbb{D}^{1,p}(\Omega; E)$ and

$$DP_E(t)f = e^{-t}P_{\gamma(H,E)}DF, \qquad DQ_E(t)f = Q^{(1)}_{\gamma(H,E)}DF,$$

where $Q^{(1)}_{\gamma(H,E)}$ is the semigroup generated by $-(I - L_{\gamma(H,E)})^{1/2}$.

(ii) For $F \in \mathcal{P}(E)$ we have $L_E F$, $C_E F \in \mathbb{D}^{1,p}(\Omega; E)$ and

$$DL_E F = -(I - L_{\gamma(H,E)})DF, \qquad DC_E F = -(I - L_{\gamma(H,E)})^{1/2} DF.$$

Pisier proved in [146] that Meyer's inequalities extend to UMD spaces. Using γ-norms his result result can be formulated as follows.

Theorem 11.16 (Meyer's inequalities). *Let E be a UMD space and let $1 < p < \infty$. Then $\mathsf{D}_p(C_E) = \mathbb{D}^{1,p}(\Omega; E)$ and for all $f \in \mathbb{D}^{1,p}(\Omega; E)$ the following two-sided estimate holds:*

$$\|C_E f\|_{L^p(\Omega;E)} \eqsim_{p,E} \|Df\|_{L^p(\Omega;\gamma(H,E))}. \tag{11.16}$$

In Theorem 11.21 we shall state an extension of this result.

The following lemma is the crucial ingredient in the proof of Meyer's multiplier Theorem. The proof in the scalar case in [138, Lemma 1.4.1] does not extend to the vector-valued setting, since it depends heavily on the Hilbert space structure of $L^2(\Omega)$. We give a simple proof in the case that E is a UMD space, which is based on Meyer's inequalities. Recall that J_m denotes the chaos projection considered in Remark 11.7.

Lemma 11.17. *Let $1 < p < \infty$ and let E be a UMD space. For each $N \geq 1$ and $t > 0$ we have*

$$\|P(t)(I - J_0 - J_1 - \ldots - J_{N-1})\|_{\mathcal{L}(L^p(\Omega;E))} \lesssim_{E,p,N} e^{-Nt}.$$

Proof. For $F \in \mathcal{P}(E)$ we set

$$RF = D\sum_{m=1}^{\infty} m^{-1/2} J_m F, \qquad S\Big(D\sum_{m=0}^{\infty} J_m F\Big) := \sum_{m=1}^{\infty} m^{1/2} J_m F.$$

Note that the sums consists of finitely many terms since $F \in \mathcal{P}(E)$. Both operators are well-defined and L^p-bounded by Theorem 11.16. Using the fact that

$$S^N R^N F = \sum_{m=N}^{\infty} J_m F,$$

we obtain by Lemma 11.15 and Theorem 11.16,

$$\begin{aligned}
&\|P(t)(I - J_0 - J_1 - \ldots - J_{N-1})F\|_{L^p(\Omega;E)} \\
&\quad = \Big\| \sum_{m=N}^{\infty} e^{-mt} J_m F \Big\|_{L^p(\Omega;E)} = \|S^N R^N P(t) F\|_{L^p(\Omega;E)} \\
&\quad = \|S^N e^{-Nt} P(t) R^N F\|_{L^p(\Omega;E)} \leq e^{-Nt} \|S\|^N \|R\|^N \|F\|_{L^p(\Omega;E)}.
\end{aligned}$$

$\square$

Using this lemma, the remainder of the proof of Meyer's multiplier Theorem [128] in the scalar case as given in [138, Theorem 1.4.2] extends verbatim to the vector-valued setting. It is even possible to allow operator-valued multipliers.

Theorem 11.18 (Meyer's Multiplier Theorem). *Let $1 < p < \infty$, let E be a UMD space, and let $(a_k)_{k=0}^{\infty} \subseteq \mathcal{L}(L^p(\Omega;E))$ be a sequence of bounded linear operators such that $\sum_{k=0}^{\infty} \|a_k\|_{\mathcal{L}(L^p(\Omega;E))} N^{-k} < \infty$ for some $N \geq 1$. If $(\phi(n))_{n\geq 0} \subseteq \mathcal{L}(L^p(\Omega;E))$ is a sequence of operators satisfying $\phi(n) := \sum_{k=0}^{\infty} a_k n^{-k}$ for $n \geq N$, then the operator T_ϕ defined by*

$$T_\phi F := \sum_{n=0}^{\infty} \phi(n) J_n F, \quad F \in \mathcal{P}(E)$$

extends to a bounded operator on $L^p(\Omega;E)$.

As a first application of the multiplier theorem we determine the spectrum of L. We start with a simple but useful lemma.

Lemma 11.19. *Let E be a K-convex Banach space, let $1 < p < \infty$, and let $F \in L^p(\Omega;E)$ such that $J_m F = 0$ for all $m \geq 0$. Then $F = 0$ in $L^p(\Omega;E)$.*

Proof. For $G \in \mathcal{P}(E^*)$ we have

$$\mathbb{E}\langle F, G\rangle = \mathbb{E}\langle F, \sum_{m\geq 0} J_m G\rangle = \mathbb{E}\langle \sum_{m\geq 0} J_m F, G\rangle = 0.$$

This implies the result, since $\mathcal{P}(E^*)$ is dense in $L^q(\Omega; E^*)$, hence weak*-dense in $L^p(\Omega; E)^*$.

□

Proposition 11.20. *Let $1 < p < \infty$ and let E be a UMD space. Then*

$$\sigma(-L) = \{0, 1, 2, \dots\}.$$

Moreover, every integer $m \geq 0$ is an eigenvalue of $-L$ and $\ker(m + L) = H^{(m)}(E)$.

Proof. To prove that $\{0, 1, 2, \dots\} \subseteq \sigma(-L)$ we take an integer $m \geq 0$ and a non-zero $F \in H^{(m)}(E)$. Since $P(t)F = e^{-mt}F$ it follows that $F \in \mathsf{D}_p(L)$ and $(m + L)F = 0$, hence $m \in \sigma(-L)$ and $\ker(m + L) \supseteq H^{(m)}(E)$.

To show the converse inclusion for the spectrum, take $\lambda \in \mathbb{C} \setminus \{0, 1, 2, \dots\}$. To prove that $\lambda + L$ is injective, take $F \in \ker(\lambda + L)$. Since J_m is bounded for $m \geq 0$ by Remark 11.7 (UMD spaces are K-convex), it follows that $J_m LF = LJ_m F = -mJ_m F$. This implies that $(\lambda - m)J_m F = J_m(\lambda + L)F = 0$, hence $J_m F = 0$ for all $m \geq 0$, so that $F = 0$ by Lemma 11.19.

To prove surjectivity, we conclude from the Multiplier Theorem 11.18 that

$$R_\lambda := \sum_{m=0}^{\infty} \frac{1}{\lambda - m} J_m$$

extends to a bounded operator on $L^p(\Omega; E)$. Using the fact that L is closed, we infer that $(\lambda + L)R_\lambda = I$, hence $\lambda + L$ is surjective.

It remains to show that $\ker(m + L) \subseteq H^{(m)}(E)$ for all $m \geq 0$. Take $F \in \ker(m + L)$. Since

$$(m - k)J_k F = (m + L)J_k F = J_k(m + L)F = 0$$

for all integers $k \geq 0$, we have $J_k F = 0$ for all $k \neq m$. This implies that $J_k(F - J_m F) = 0$, hence $F = J_m F \in H^{(m)}(E)$ by Lemma 11.19.

□

Next we give the general form of Meyer's inequalities in the language of γ-radonifying norms. This result is stated in a slightly different setting in [112, Theorem 1.17], but the proof given there contains a gap. More precisely, the last formula for the function ψ defined in [112, p.300] should be replaced by $\psi(t) = \frac{1}{2}e^{-t/2}(I_0(\frac{t}{2}) + I_1(\frac{t}{2}))$. This function however is not contained in $L^1(0, \infty)$; but this is needed to conclude the proof.

The proof given below uses Lemma 11.17, which is based on the first order Meyer inequalities from Theorem 11.16. This allows us to adapt the argument in the scalar case from [138, Theorem 1.5.1].

Theorem 11.21 (Meyer's inequalities, general case). *Let E be a UMD space, let $1 < p < \infty$ and let $n \geq 1$. Then $\mathsf{D}_p(C^n) = \mathbb{D}^{n,p}(\Omega; E)$, and for all $F \in \mathbb{D}^{n,p}(\Omega; E)$ we have*

$$\begin{aligned} \|D^n F\|_{L^p(\Omega;\gamma^n(H,E))} &\lesssim_{p,E,n} \|C^n F\|_{L^p(\Omega;E)} \\ &\lesssim_{p,E,n} \|F\|_{L^p(\Omega;E)} + \|D^n F\|_{L^p(\Omega;\gamma^n(H,E))}. \end{aligned} \tag{11.17}$$

Proof. The proof proceeds by induction. The case $n = 1$ has been treated in Theorem 11.16. Suppose that (11.17) holds for some $n \geq 1$. Using Lemma 11.15 and the fact that the operator $C^n(I-L)^{-n/2} = (-L)^{n/2}(I-L)^{-n/2}$ is bounded on $L^p(\Omega; E)$ we obtain by the induction hypothesis

$$\begin{aligned} \mathbb{E}\|D^{n+1}F\|^p_{\gamma^{n+1}(H,E)} &\lesssim_{p,E,n} \mathbb{E}\|C^n DF\|^p_{\gamma(H,E)} \lesssim_{p,E,n} \mathbb{E}\|(I-L)^{n/2}DF\|^p_{\gamma(H,E)} \\ &= \mathbb{E}\|DC^n F\|^p_{\gamma(H,E)} \eqsim_{p,E} \mathbb{E}\|C^{n+1}F\|^p_E. \end{aligned}$$

To prove the second inequality, we note that according to Remark 11.7,

$$\|C^n(J_0 + \ldots + J_{n-1})F\|_p \lesssim_{p,E,n} \|F\|_p, \quad F \in L^p(\Omega; E).$$

Therefore it suffices to show by induction that

$$\|C^n F\|_{L^p(\Omega;E)} \lesssim_{p,E,n} \|D^n F\|_{L^p(\Omega;\gamma^n(H,E))}$$

for all $F \in \mathcal{P}(E)$ with $J_0F = \ldots = J_{n-1}F = 0$.

Let us assume that this statement holds for some $n \geq 1$ and take $F \in \mathcal{P}(E)$ satisfying $J_0F = \ldots = J_nF = 0$. It follows from Lemma 11.17 that $(P(t))_{t\geq 0}$ restricts to a C_0-semigroup $(P_n(t))_{t\geq 0}$ on

$$X_{n,p}(E) := \overline{\bigoplus_{m\geq n} H^{(m)}(E)}^{L^p(\Omega;E)},$$

satisfying the growth bound $\|P_n(t)\|_{\mathcal{L}(X_{n,p}(E))} \lesssim_{E,p,n} e^{-nt}$ for some constant K depending on n. Consequently (see e.g., [7, Proposition 3.8.2]), we have

$$\|(\alpha - L)^{1/2}F\|_p \eqsim_{p,E} \|(\beta - L)^{1/2}F\|_p, \quad F \in X_{n,p}(E),$$

for all $\alpha, \beta > -n$, and in particular is $(I-L)^{1/2}C^{-1}$ bounded on $X_{n,p}(E)$. Using Lemma 11.15 and the fact that $C^n DF \in X_{n,p}(\gamma(H,E))$, it follows that

$$\begin{aligned} \mathbb{E}\|C^{n+1}F\|^p_E &\eqsim_{p,E} \mathbb{E}\|DC^nF\|^p_{\gamma(H,E)} = \mathbb{E}\|(I-L)^{n/2}DF\|^p_{\gamma(H,E)} \\ &\lesssim_{p,E,n} \|(I-L)^{n/2}C^{-n}\|^p_{\mathcal{L}(X_{n,p}(\gamma(H,E)))}\mathbb{E}\|C^nDF\|^p_{\gamma(H,E)} \\ &\eqsim_{p,E,n} \mathbb{E}\|D^{n+1}F\|^p_{\gamma^{n+1}(H,E)}. \end{aligned}$$

□

As an application of Meyer's inequalities we will show that $\gamma(H,E)$-valued Malliavin differentiable random variables are contained in the domain of the divergence operator δ. First we give the precise definition of δ.

Fix an exponent $1 < p < \infty$ and let $\frac{1}{p} + \frac{1}{q} = 1$. For the moment let D denote the Malliavin derivative on $L^q(\Omega; E^*)$, which is a densely defined closed operator with domain $\mathbb{D}^{1,q}(\Omega; E^*)$ and taking values in $L^q(\Omega; \gamma(H, E^*))$. We let the domain $\mathsf{D}_p(\delta)$ consist of all $u \in L^p(\Omega; \gamma(H,E))$ for which there exists an $F_u \in L^p(\Omega; E)$ such that

$$\mathbb{E}[u, DG]_\gamma = \mathbb{E}\langle F_u, G\rangle \quad \text{for all } G \in \mathbb{D}^{1,q}(\Omega; E^*).$$

The function F_u, if it exists, is uniquely determined. We set

$$\delta(u) := F_u, \quad X \in \mathsf{D}_p(\delta). \tag{11.18}$$

In other words, δ is the part of the adjoint operator D^* in $L^p(\Omega; \gamma(H,E))$ which maps into $L^p(\Omega; E)$. Here we identify $L^p(\Omega; \gamma(H,E))$ and $L^p(\Omega; E)$ in a natural way with subspaces of $(L^q(\Omega; \gamma(H, E^*)))^*$ and $(L^q(\Omega; E^*))^*$ respectively.

The divergence operator δ is easily seen to be closed and densely defined. The proof of the following result is a variation of the proof of the scalar-valued result in [138, Proposition 1.5.4].

Proposition 11.22. *Let $1 < p < \infty$ and let E be a UMD space. The operator δ is bounded from $\mathbb{D}^{1,p}(\Omega; \gamma(H,E))$ into $L^p(\Omega; E)$.*

Proof. Let $u \in \mathbb{D}^{1,p}(\Omega; \gamma(H,E))$ and $G \in \mathcal{P}(E^*)$. Using Theorem 11.12 we find that $\|DJ_1G\|_p \eqsim_p \|J_1 G\|_p$, and therefore

$$\begin{aligned} |\mathbb{E}[u, D(J_0 + J_1)G]_\gamma| &\le \|u\|_{L^p(\Omega;\gamma(H,E))} \|D(J_0+J_1)G\|_{L^q(\Omega;\gamma(H,E^*))} \\ &\lesssim_{p,E} \|u\|_{L^p(\Omega;\gamma(H,E))} \|G\|_{L^q(\Omega;E^*)}. \end{aligned} \tag{11.19}$$

Now we assume that $J_0 G = J_1 G = 0$. By the Multiplier Theorem 11.18 the operator

$$T := \sum_{m=2}^{\infty} \frac{m}{m-1} J_m$$

is bounded on $L^p(\Omega; \gamma(H,E))$. By Lemma 11.17 the operator L^{-1} is well defined on $X_{1,p}(E)$, where we use the notation from the proof of Theorem 11.21. This justifies the use of L^{-1} in the following computation. Using Lemma 11.14 and Theorem 11.21 we obtain

$$\begin{aligned} |\mathbb{E}[u, DG]_\gamma| = |\mathbb{E}[u, LL^{-1}DG]_\gamma| &= |\mathbb{E}[Du, DL^{-1}DG]_\gamma| \\ &\le \|Du\|_{L^p(\Omega;\gamma^2(H,E))} \|DL^{-1}DG\|_{L^q(\Omega;\gamma^2(H,E^*))} \\ &= \|Du\|_{L^p(\Omega;\gamma^2(H,E))} \|D^2L^{-1}TG\|_{L^q(\Omega;\gamma^2(H,E^*))} \\ &\lesssim_{p,E} \|Du\|_{L^p(\Omega;\gamma^2(H,E))} \|G\|_{L^q(\Omega;E^*)}. \end{aligned} \tag{11.20}$$

Combining (11.19) and (11.20) we conclude that for all $G \in \mathcal{P}(E^*)$ we have

$$|\mathbb{E}[u, DG]_\gamma| \lesssim_{p,E} \|u\|_{\mathbb{D}^{1,p}(\Omega;E)} \|G\|_{L^q(\Omega;E^*)}.$$

It follows that there exists an $F_u \in (L^q(\Omega; E^*))^*$ such that $\mathbb{E}[u, DG]_\gamma = \mathbb{E}\langle F_u, G\rangle$. Since E is a UMD space, we conclude that $F_u \in L^p(\Omega; E)$ and we obtain the desired result. □

For $1 \le p < \infty$ we define the vector space of exact E-valued processes as

$$L^p_e(\Omega; \gamma(H, E)) = \{DF : F \in \mathbb{D}^{1,p}(\Omega; E)\}.$$

The next result is concerned with the representation of random variables as divergences of exact processes.

Proposition 11.23. *Let E be a UMD space, let $1 < p < \infty$, and let $F \in L^p(\Omega; E)$. Then $U := DL^{-1}(F - \mathbb{E}(F))$ is the unique element in $L^p_e(\Omega; \gamma(H, E))$ satisfying*

$$F = \mathbb{E}(F) + \delta(U).$$

Proof. By an easy computation we see that

$$F = \mathbb{E}(F) + \delta D(L^{-1}(F - \mathbb{E}(F))) \tag{11.21}$$

for all $F \in \mathcal{P}(E)$. It follows from Lemma 11.17 (or Proposition 11.20) that L^{-1} is well-defined and bounded on $\{G \in L^p(\Omega; E) : \mathbb{E}(G) = 0\}$. Meyer's inequalities imply that D is bounded from $\mathsf{D}_p(L)$ into $\mathbb{D}^{1,p}(\Omega; \gamma(H, E))$, and by Proposition 11.22 we have that δ is bounded from $\mathbb{D}^{1,p}(\Omega; \gamma(H, E))$ into $L^p(\Omega; E)$. Using these facts and an approximation argument with elements from $\mathcal{P}(E)$ we conclude that the right hand side of (11.21) is well-defined for all $F \in L^p(\Omega; E)$, and the identity remains valid.

To prove uniqueness, suppose that $F = \mathbb{E}(F) + \delta(DF')$ for some $F' \in \mathbb{D}^{1,p}(\Omega; E)$ with $DF' \in \mathsf{D}_p(\delta)$, and put $G := F' - L^{-1}(F - \mathbb{E}(F))$. Then $\delta DG = 0$, hence $\langle G, LP\rangle = 0$ for all polynomials $P \in \mathcal{P}(E^*)$. In particular, for all $m \ge 1$ and all $P \in \mathcal{P}(E^*) \cap H^{(m)}(E^*)$ one has $\langle G, mP\rangle = 0$, and since $\mathcal{P}(E^*) \cap H^{(m)}(E^*)$ is dense in $H^{(m)}(E^*)$, we have $\langle J_m G, \widetilde{F}\rangle = \langle G, J_m \widetilde{F}\rangle = 0$ for all $\widetilde{F} \in L^q(\Omega; E^*)$. It follows that $J_m G = 0$ for all $m \ge 1$, which implies $J_m(G - J_0 G) = 0$ for all $m \ge 0$. We conclude that $G = J_0 G$ by Lemma 11.19, hence $F' = L^{-1}(F - \mathbb{E}F) + x$ for some $x \in E$. We conclude that $DF' = DL^{-1}(F - \mathbb{E}F)$, which is the desired identity. □

We conclude the chapter with an application of the vector-valued Malliavin calculus developed in this work. We give a new proof of Theorem 11.1(1) under the additional assumption that E is a UMD space, which is based on Meyer's inequalities. This approach seems to be new even in the scalar-valued case.

Theorem 11.24. *Let E be a UMD space, let $1 < p < \infty$, and define F and $\widetilde{F}$ as in Theorem 11.1(1). Then we have*

$$\|F\|_p \eqsim_{p,m,E} \|\widetilde{F}\|_p.$$

Proof. We argue as in the proof of Theorem 11.12. By (11.11) we have

$$\mathbb{E}\|F\|_E^p = \mathbb{E}\Big\| \sum_{|\mathbf{i}|=m, |\mathbf{i}|_\infty \le n} \tfrac{\mathbf{i}!^{1/2}}{m!^{1/2}} \Psi_{\mathbf{i}} x_{\mathbf{i}} \Big\|_E^p$$

and according to (11.12),

$$\mathbb{E}\|DF\|_{\gamma(H,E)}^p \eqsim_p \widetilde{\mathbb{E}}\mathbb{E}\Big\| \tfrac{m^{1/2}}{(m-1)!^{1/2}} \sum_{|\mathbf{i}|=m-1, |\mathbf{i}|_\infty \le n} \tfrac{\mathbf{i}!^{1/2}}{(m-1)!^{1/2}} \Psi_{\mathbf{i}} \Big(\sum_{k=1}^n \widetilde{\gamma}_k x_{(\mathbf{i},k)} \Big) \Big\|_E^p.$$

Noting that $CF = m^{1/2} F$, Meyer's inequalities imply that

$$\begin{aligned} &\mathbb{E}\Big\| \sum_{|\mathbf{i}|=m, |\mathbf{i}|_\infty \le n} \tfrac{\mathbf{i}!^{1/2}}{m!^{1/2}} \Psi_{\mathbf{i}} x_{\mathbf{i}} \Big\|_E^p \\ &\quad \eqsim_{p,m,E} \widetilde{\mathbb{E}}\mathbb{E}\Big\| \sum_{|\mathbf{i}|=m-1, |\mathbf{i}|_\infty \le n} \tfrac{\mathbf{i}!^{1/2}}{(m-1)!^{1/2}} \Psi_{\mathbf{i}} \Big(\sum_{k=1}^n \widetilde{\gamma}_k x_{(\mathbf{i},k)} \Big) \Big\|_E^p. \end{aligned}$$

The desired result is obtained by repeating this procedure $m-1$ times. $\square$

12

The Clark-Ocone Formula

Let $\mathscr{H}$ be a separable real Hilbert space, let $T > 0$, and let $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ be the augmented filtration generated by an $\mathscr{H}$-cylindrical Brownian motion $(W_{\mathscr{H}}(t))_{t\in[0,T]}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. In this chapter we will prove that if E is a UMD Banach space, $1 \le p < \infty$, and $F \in \mathbb{D}^{1,p}(\Omega; E)$ is $\mathcal{F}_T$-measurable, then

$$F = \mathbb{E}(F) + \int_0^T P_{\mathbb{F}}(DF)\, dW_{\mathscr{H}},$$

where D is the Malliavin derivative of F and $P_{\mathbb{F}}$ is the projection onto the $\mathbb{F}$-adapted elements in a suitable Banach space of L^p-stochastically integrable $\mathcal{L}(\mathscr{H}, E)$-valued processes.

12.1 The Skorokhod integral

As in Chapter 11, $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space, H is a separable real Hilbert space, and $W : H \to L^2(\Omega)$ is an *isonormal Gaussian process.* We assume that $\mathcal{F}$ is the σ-field generated by $\{W(h) : h \in H\}$.

The Malliavin derivative acting on possibly vector-valued functions will be denoted by D. We collect some versions of the product rule which will be useful below.

Proposition 12.1. *Let $1 \le p, q, r < \infty$ such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$.*

(i) For all $F \in \mathbb{D}^{1,p}(\Omega; E)$ and $G \in \mathbb{D}^{1,q}(\Omega; E^*)$ we have $\langle F, G\rangle \in \mathbb{D}^{1,r}(\Omega)$ and

$$D\langle F, G\rangle = \langle DF, G\rangle + \langle F, DG\rangle.$$

(ii) For all $F \in \mathbb{D}^{1,p}(\Omega)$ and $G \in \mathbb{D}^{1,q}(\Omega; E)$ we have $FG \in \mathbb{D}^{1,r}(\Omega; E)$ and

$$D(FG) = F\, DG + DF \otimes G.$$

(iii) *For all $F \in \mathbb{D}^{1,p}(\Omega;E)$ and $G \in \mathbb{D}^{1,q}(\Omega;E^*)$ and $h \in H$ we have $\langle DF(h), G\rangle \in L^r(\Omega)$ and*

$$\mathbb{E}\langle DF(h), G\rangle = \mathbb{E}(W(h)\langle F, G\rangle) - \mathbb{E}\langle F, DG(h)\rangle.$$

We continue with a useful Lemma concerning the divergence operator, which has already introduced in (11.18).

Lemma 12.2. *We have $\mathscr{S}(\Omega)\otimes\gamma(H,E) \subseteq \mathsf{D}_p(\delta)$ and*

$$\delta(f\otimes R) = \sum_{j\geq 1} W(h_j)f\otimes Rh_j - R(Df), \qquad f\in\mathscr{S}(\Omega),\ R\in\gamma(H,E).$$

Here $(h_j)_{j\geq 1}$ denotes an arbitrary orthonormal basis of H.

Proof. For $f\in\mathscr{S}(\Omega)$, $R\in\gamma(H,E)$, and $G\in\mathscr{S}(\Omega)\otimes E^*$ we obtain, using the integration by parts formula (11.10) (or Proposition 12.1(iii)),

$$\begin{aligned}
\mathbb{E}\langle f\otimes R, DG\rangle &= \sum_{j\geq 1}\mathbb{E}\langle f\otimes Rh_j, DG(h_j)\rangle \\
&= \sum_{j\geq 1}\mathbb{E}(W(h_j)\langle f\otimes Rh_j, G\rangle) - \mathbb{E}\langle [Df,h_j]_H\otimes Rh_j, G\rangle \\
&= \mathbb{E}\Big\langle \sum_{j\geq 1} W(h_j)f\otimes Rh_j - \sum_{j\geq 1}[Df,h_j]_H\otimes Rh_j, G\Big\rangle \\
&= \mathbb{E}\Big\langle \sum_{j\geq 1} W(h_j)f\otimes Rh_j - R(Df), G\Big\rangle.
\end{aligned}$$

The sum $\sum_{j\geq 1} W(h_j)f\otimes Rh_j$ converges in $L^p(\Omega;E)$. This follows from the Kahane-Khintchine inequalities and the fact that $(W(h_j))_{j\geq 1}$ is a sequence of independent standard Gaussian variables; note that the function f is bounded. $\square$

We shall now assume that $H = L^2(0,T;\mathscr{H})$, where T is a fixed positive real number and $\mathscr{H}$ is a separable real Hilbert space. We will show that if the Banach space E is a UMD space, the divergence operator δ is an extension of the stochastic integral for adapted $\mathcal{L}(\mathscr{H},E)$-valued processes constructed recently in [134]. Let us start with a summary of its construction.

Let $W_{\mathscr{H}} = (W_{\mathscr{H}}(t))_{t\in[0,T]}$ be an $\mathscr{H}$-cylindrical Brownian motion on $(\Omega,\mathcal{F},\mathbb{P})$, i.e., for each $t\in[0,T]$, $W_{\mathscr{H}}(t)$ is a bounded linear operator from $\mathscr{H}$ into $L^2(\Omega)$ having the following properties:

(1) $W_{\mathscr{H}}h := (W_{\mathscr{H}}(t)h)_{t\in[0,T]}$ is a real-valued Brownian motion for all $h\in\mathscr{H}$;
(2) $\mathbb{E}(W_{\mathscr{H}}(s)g\cdot W_{\mathscr{H}}(t)h) = (s\wedge t)[g,h]$ for all $s,t\in[0,T]$ and $g,h\in\mathscr{H}$.

We shall assume that $W_{\mathscr{H}}$ is adapted to a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ satisfying the usual conditions, i.e., $W_{\mathscr{H}}h$ is adapted to $\mathbb{F}$ each $h\in\mathscr{H}$. The Itô isometry defines an isonormal process $W : L^2(0,T;\mathscr{H})\to L^2(\Omega)$ by

$$W(\phi) := \int_0^T \phi \, dW_{\mathscr{H}}, \qquad \phi \in L^2(0,T;\mathscr{H}).$$

Following [134] we say that a process $X : (0,T) \times \Omega \to \gamma(\mathscr{H}, E)$ is an elementary adapted process with respect to the filtration $\mathbb{F}$ if it is of the form

$$X(t,\omega) = \sum_{i=1}^{m} \sum_{j=1}^{n} \mathbf{1}_{(t_{i-1},t_i]}(t) 1_{A_{ij}}(\omega) \sum_{k=1}^{l} h_k \otimes x_{ijk}, \tag{12.1}$$

where $0 \le t_0 < \cdots < t_n \le T$, the sets $A_{ij} \in \mathcal{F}_{t_{i-1}}$ are disjoint for each j, and $h_k, \dots, h_k \in \mathscr{H}$ are orthonormal. The stochastic integral with respect to $W_{\mathscr{H}}$ of such a process is defined by

$$I(X) := \int_0^T X \, dW_{\mathscr{H}} := \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{l} 1_{A_{ij}} (W_{\mathscr{H}}(t_i) h_k - W_{\mathscr{H}}(t_{i-1}) h_k) \otimes x_{ijk},$$

Elementary adapted processes define elements of $L^p(\Omega; \gamma(L^2(0,T;\mathscr{H}), E))$ in a natural way. The closure of these elements in $L^p(\Omega; \gamma(L^2(0,T;\mathscr{H}), E))$ is denoted by $L^p_{\mathbb{F}}(\Omega; \gamma(L^2(0,T;\mathscr{H}), E))$.

Proposition 12.3 ([134, Theorem 3.5]). *Let E be a UMD space and let $1 < p < \infty$. The stochastic integral uniquely extends to a bounded operator*

$$I : L^p_{\mathbb{F}}(\Omega; \gamma(L^2(0,T;\mathscr{H}), E)) \to L^p(\Omega; E).$$

Moreover, for all $X \in L^p_{\mathbb{F}}(\Omega; \gamma(L^2(0,T;\mathscr{H}), E))$ we have the two-sided estimate

$$\|I(X)\|_{L^p(\Omega;E)} \eqsim \|X\|_{L^p(\Omega;\gamma(L^2(0,T;\mathscr{H}),E))},$$

with constants only depending on p and E.

A consequence of this result is the following lemma, which will be useful in the proof of Theorem 12.12.

Lemma 12.4. *Let E be a UMD space and let $1 < p, q < \infty$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$. For all $X \in L^p_{\mathbb{F}}(\Omega; \gamma(L^2(0,T;\mathscr{H}), E))$ and $Y \in L^q_{\mathbb{F}}(\Omega; \gamma(L^2(0,T;\mathscr{H}), E^*))$ we have*

$$\mathbb{E}\langle I(X), I(Y) \rangle = \mathbb{E}\langle X, Y \rangle.$$

Proof. When X and Y are elementary adapted the result follows by direct computation. The general case follows from Proposition 12.3 applied to E and E^*, noting that E^* is a UMD space as well. $\square$

In the next approximation result we identify $L^2(0,t;\mathscr{H})$ with a closed subspace of $L^2(0,T;\mathscr{H})$. The simple proof is left to the reader.

Lemma 12.5. *Let $1 \le p < \infty$, let $0 < t \le T$, and let $(\psi_n)_{n\ge1}$ be an orthonormal basis of $L^2(0,t;\mathscr{H})$. The linear span of the functions*

$$f(W(\psi_1),\dots,W(\psi_n)) \otimes (h \otimes x),$$

with $f \in \mathscr{S}(\Omega)$, $h \in H$, $x \in E$, is dense in $L^p(\Omega,\mathcal{F}_t;\gamma(\mathscr{H},E))$.

The next result shows that the divergence operator δ is an extension of the stochastic integral I. This means that δ is a vector-valued Skorokhod integral.

Theorem 12.6. *Let E be a UMD space and let $1 < p < \infty$ be fixed. The space $L^p_{\mathbb{F}}(\Omega;\gamma(L^2(0,T;\mathscr{H}),E))$ is contained in $\mathsf{D}_p(\delta)$ and*

$$\delta(X) = I(X) \quad \text{for all} \quad X \in L^p_{\mathbb{F}}(\Omega;\gamma(L^2(0,T;\mathscr{H}),E)).$$

Proof. Fix $0 < t \le T$, let $(h_k)_{k\ge1}$ be an orthonormal basis of $\mathscr{H}$, and put $X := 1_A \sum_{k=1}^n h_k \otimes x_k$ with $A \in \mathcal{F}_t$ and $x_k \in E$ for $k = 1,\dots,n$. Let $(\psi_j)_{j\ge1}$ be an orthonormal basis of $L^2(0,t;\mathscr{H})$. By Lemma 12.5 we can approximate X in $L^p(\Omega,\mathcal{F}_t;\gamma(\mathscr{H},E))$ with a sequence $(X_l)_{l\ge1}$ in $\mathscr{S}(\Omega,\gamma(\mathscr{H},E))$ of the form

$$X_l := \sum_{m=1}^{M_l} f_{lm}(W(\psi_1),\dots,W(\psi_n)) \otimes (h_m \otimes x_{lm})$$

with $x_{lm} \in E$.

Now let $0 < t < u \le T$. From $\psi_m \perp \mathbf{1}_{(t,u]} \otimes h$ in $L^2(0,T;\mathscr{H})$ it follows that $DX_l(\mathbf{1}_{(t,u]} \otimes h) = 0$ for all $h \in \mathscr{H}$. By Lemma 12.2,

$$\mathbf{1}_{(t,u]} \otimes X_l = \sum_{m=1}^{M_l} f_{lm}(W(\psi_1),\dots,W(\psi_n)) \otimes ((\mathbf{1}_{(t,u]} \otimes h_m) \otimes x_{lm})$$

belongs to $\mathsf{D}_p(\delta)$ and

$$\begin{aligned}\delta(\mathbf{1}_{(t,u]} \otimes X_l) &= \sum_{m=1}^{M_l} W(\mathbf{1}_{(t,u]} \otimes h_m) f_{lm}(W(\psi_1),\dots,W(\psi_n)) \otimes x_{lm} \\ &= I(\mathbf{1}_{(t,u]} \otimes X_l).\end{aligned}$$

Noting that $\mathbf{1}_{(t,u]} \otimes X_l \to \mathbf{1}_{(t,u]} \otimes X$ in $L^p(\Omega;\gamma(L^2(0,T;\mathscr{H}),E))$ as $l \to \infty$, the closedness of δ implies that $\mathbf{1}_{(t,u]} \otimes X \in \mathsf{D}_p(\delta)$ and

$$\delta(\mathbf{1}_{(t,u]} \otimes X_l) = I(\mathbf{1}_{(t,u]} \otimes X_l).$$

By linearity, it follows that the elementary adapted processes of the form (12.1) with $t_0 > 0$ are contained in $\mathsf{D}_p(\delta)$ and that I and δ coincide for such processes.

To show that this equality extends to all $X \in L^p_{\mathbb{F}}(\Omega;\gamma(L^2(0,T;\mathscr{H}),E))$ we take a sequence X_n of elementary adapted processes of the above form converging to X. Since I is a bounded operator from $L^p_{\mathbb{F}}(\Omega;\gamma(L^2(0,T;\mathscr{H}),E))$ into $L^p(\Omega;E)$, it follows that $\delta(X_n) = I(X_n) \to I(X)$ as $n \to \infty$. The fact that δ is closed implies that $X \in \mathsf{D}_p(\delta)$ and $\delta(X) = I(X)$. □

12.2 A Clark-Ocone formula

Our next aim is to prove that the space $L^p_{\mathbb{F}}(\Omega;\gamma(L^2(0,T;\mathscr{H}),E))$, which has been introduced in the previous section, is a complemented subspace of $L^p(\Omega;\gamma(L^2(0,T;\mathscr{H}),E))$. For this purpose we need a number of auxiliary results. We refer to Section 5.1 for the definition of the notion of γ-boundedness and the notation $\gamma(\mathscr{T})$.

Proposition 12.7. *Let $\mathscr{T}$ be a γ-bounded subset of $\mathcal{L}(E,F)$ and let H be a separable real Hilbert space. For each $T\in\mathscr{T}$ let $\widetilde{T}\in\mathcal{L}(\gamma(H,E),\gamma(H,F))$ be defined by $\widetilde{T}R := T\circ R$. The collection $\widetilde{\mathscr{T}} = \{\widetilde{T}: \ T\in\mathscr{T}\}$ is γ-bounded, with $\gamma(\widetilde{\mathscr{T}}) = \gamma(\mathscr{T})$.*

Proof. Let $(\gamma_j)_{j\ge 1}$ and $(\widetilde{\gamma}_j)_{j\ge 1}$ be two sequences of independent standard Gaussian random variables, on probability spaces $(\Omega,\mathcal{F},\mathbb{P})$ and $(\widetilde{\Omega},\widetilde{\mathcal{F}},\widetilde{\mathbb{P}})$ respectively. By the Fubini theorem,

$$\begin{aligned}
\mathbb{E}\Big\|\sum_{j=1}^n \gamma_j\widetilde{T}_jR_j\Big\|^2_{\gamma(H,F)} &= \mathbb{E}\widetilde{\mathbb{E}}\Big\|\sum_{i=1}^\infty \widetilde{\gamma}_i\sum_{j=1}^n \gamma_jT_jR_jh_i\Big\|^2_F\\
&= \widetilde{\mathbb{E}}\mathbb{E}\Big\|\sum_{j=1}^n \gamma_jT_j\sum_{i=1}^\infty \widetilde{\gamma}_iR_jh_i\Big\|^2_F\\
&\le \gamma^2(\mathscr{T})\widetilde{\mathbb{E}}\mathbb{E}\Big\|\sum_{j=1}^n \gamma_j\sum_{i=1}^\infty \widetilde{\gamma}_iR_jh_i\Big\|^2_E\\
&= \gamma^2(\mathscr{T})\mathbb{E}\widetilde{\mathbb{E}}\Big\|\sum_{i=1}^\infty \widetilde{\gamma}_i\sum_{j=1}^n \gamma_jR_jh_i\Big\|^2_E\\
&= \gamma^2(\mathscr{T})\mathbb{E}\Big\|\sum_{j=1}^n \gamma_jR_j\Big\|^2_{\gamma(H,E)}.
\end{aligned}$$

This proves the inequality $\gamma(\widetilde{\mathscr{T}})\le\gamma(\mathscr{T})$. The reverse inequality holds trivially. $\square$

The next proposition is a result by Bourgain [17], known as the vector-valued Stein inequality. We refer to [36, Proposition 3.8] for a detailed proof.

Proposition 12.8. *Let E be a UMD space and let $(\mathcal{F}_t)_{t\in[0,T]}$ be a filtration on $(\Omega,\mathcal{F},\mathbb{P})$. For all $1<p<\infty$ the conditional expectations $\{\mathbb{E}(\cdot|\mathcal{F}_t): \ t\in[0,T]\}$ define a γ-bounded set in $\mathcal{L}(L^p(\Omega;E))$.*

We will use a multiplier result due to Kalton and Weis [90], which has already been stated in Proposition 5.16. This time we need a slightly more general version, which is stated below for the convenience of the reader. In its formulation we make the observation that every step function $f:(0,T)\to\gamma(\mathscr{H},E)$ defines an element $R_g\in\gamma(L^2(0,T;\mathscr{H}),E)$ by the formula

$$R_f \phi := \int_0^T f(t)\phi(t)\,dt.$$

Since R_f determines f uniquely almost everywhere, in what follows we shall always identify R_f and f.

Proposition 12.9. *Let E and F be real Banach spaces and let $M : (0,T) \to \mathcal{L}(E,F)$ have γ-bounded range $\{M(t) : t \in (0,T)\} =: \mathscr{M}$. Assume that for all $x \in E$, $t \mapsto M(t)x$ is strongly measurable. Then the mapping $M : f \mapsto [t \mapsto M(t)f(t)]$ extends to a bounded operator from $\gamma(L^2(0,T;\mathscr{H}),E)$ to $\gamma(L^2(0,T;\mathscr{H}),F)$ of norm $\|M\| \le \gamma(\mathscr{M})$.*

Here we identified $M(t) \in \mathcal{L}(E,F)$ with $\widetilde{M(t)} \in \mathcal{L}(\gamma(\mathscr{H},E),\gamma(\mathscr{H},F))$ as in Proposition 12.7.

The next result is taken from [134].

Proposition 12.10. *Let H be a separable real Hilbert space and let $1 \le p < \infty$. Then $f \mapsto [h \mapsto f(\cdot)h]$ defines an isomorphism of Banach spaces*

$$L^p(\Omega;\gamma(H,E)) \simeq \gamma(H, L^p(\Omega;E)).$$

After these preparations we are ready to state the result announced above. We fix a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ and define, for step functions $f : (0,T) \to \gamma(\mathscr{H}, L^p(\Omega;E))$,

$$(P_{\mathbb{F}} f)(t) := \mathbb{E}(f(t)|\mathcal{F}_t), \tag{12.2}$$

where $\mathbb{E}(\cdot|\mathcal{F}_t)$ is considered as a bounded operator acting on $\gamma(\mathscr{H}, L^p(\Omega;E))$ as in Proposition 12.7.

Lemma 12.11. *Let E be a UMD space, and let $1 < p,q < \infty$ satisfy $\frac{1}{p}+\frac{1}{q} = 1$.*

(i) $P_{\mathbb{F}}$ extends to a bounded operator on $\gamma(L^2(0,T;\mathscr{H}), L^p(\Omega;E))$.
(ii) As a bounded operator on $L^p(\Omega;\gamma(L^2(0,T;\mathscr{H}),E))$, $P_{\mathbb{F}}$ is a projection onto the subspace $L^p_{\mathbb{F}}(\Omega;\gamma(L^2(0,T;\mathscr{H}),E))$.
(iii) For $X \in L^p(\Omega;\gamma(L^2(0,T;\mathscr{H}),E))$ and $Y \in L^q(\Omega;\gamma(L^2(0,T;\mathscr{H}),E^*))$ we have

$$\mathbb{E}\langle X, P_{\mathbb{F}} Y\rangle = \mathbb{E}\langle P_{\mathbb{F}} X, Y\rangle.$$

(iv) For all $X \in L^p(\Omega;\gamma(L^2(0,T;\mathscr{H}),E))$ we have $\mathbb{E}P_{\mathbb{F}}X = \mathbb{E}X$.

Proof. (i): From Propositions 12.7 and 12.8 we infer that the collection of conditional expectations $\{\mathbb{E}(\cdot|\mathcal{F}_t) : t \in [0,T]\}$ is γ-bounded in $\mathcal{L}(\gamma(\mathscr{H}, L^p(\Omega;E)))$. The boundedness of $P_{\mathbb{F}}$ then follows from Proposition 12.9. For step functions $f : (0,T) \to \gamma(\mathscr{H}, L^p(\Omega;E))$ it is clear from (12.2) that $P_{\mathbb{F}}^2 f = P_{\mathbb{F}} f$, which means that $P_{\mathbb{F}}$ is a projection.

(ii): By the identification of Proposition 12.10, $P_{\mathbb{F}}$ acts as a bounded projection in the space $L^p(\Omega;\gamma(L^2(0,T;\mathscr{H}),E))$. For elementary adapted processes $X \in L^p(\Omega;\gamma(L^2(0,T;\mathscr{H}),E))$ we have $P_{\mathbb{F}}X = X$, which implies that the range of $P_{\mathbb{F}}$ contains $L^p_{\mathbb{F}}(\Omega;\gamma(L^2(0,T;\mathscr{H}),E))$. To prove the converse inclusion we fix a step function $X:(0,T)\to\gamma(\mathscr{H},L^p(\Omega;E))$ and observe that $P_{\mathbb{F}}X$ is adapted in the sense that $(P_{\mathbb{F}}X)(t)$ is strongly $\mathcal{F}_t$-measurable for every $t\in[0,T]$. As is shown in [134, Proposition 2.12], this implies that $P_{\mathbb{F}}X\in L^p_{\mathbb{F}}(\Omega;\gamma(L^2(0,T;\mathscr{H}),E))$. By density it follows that the range of $P_{\mathbb{F}}$ is contained in $L^p_{\mathbb{F}}(\Omega;\gamma(L^2(0,T;\mathscr{H}),E))$.

(iii): Keeping in mind the identification of Proposition 12.10, for step functions with values in the finite rank operators from $\mathscr{H}$ to E this follows from (12.2) by elementary computation. The result then follows from a density argument.

(iv): Identifying a step function $f:(0,T)\to\gamma(\mathscr{H},L^p(\Omega;E))$ with the associated operator in $\gamma(L^2(0,T;\mathscr{H}),L^p(\Omega;E))$ and viewing $\mathbb{E}$ as a bounded operator from $\gamma(L^2(0,T;\mathscr{H}),L^p(\Omega;E))$ to $\gamma(L^2(0,T;\mathscr{H}),E)$, by (12.2) we have

$$\mathbb{E}P_{\mathbb{F}}f(t)=\mathbb{E}\mathbb{E}(f(t)|\mathcal{F}_t)=\mathbb{E}f(t).$$

Thus $\mathbb{E}P_{\mathbb{F}}f=\mathbb{E}f$ for all step functions $f:(0,T)\to\gamma(\mathscr{H},L^p(\Omega;E))$, and hence for all $f\in\gamma(L^2(0,T;\mathscr{H}),L^p(\Omega;E))$ by density. The result now follows by an application of Proposition 12.10. □

Now let $\mathbb{F}=(\mathcal{F}_t)_{t\in[0,T]}$ be the augmented filtration generated by $W_{\mathscr{H}}$. It has been proved in [134, Theorem 4.7] that if E is a UMD space and $1<p<\infty$, and if $F\in L^p(\Omega;E)$ is $\mathcal{F}_T$-measurable, then there exists a unique $X\in L^p_{\mathbb{F}}(\Omega;\gamma(L^2(0,T;\mathscr{H}),E))$ such that

$$F=\mathbb{E}(F)+I(X).$$

The following two results give an explicit expression for X. They extend the classical Clark-Ocone formula and its Hilbert space extension to UMD spaces.

Theorem 12.12 (Clark-Ocone representation, first L^p-version). *Let E be a UMD space and let $1<p<\infty$. If $F\in\mathbb{D}^{1,p}(\Omega;E)$ is $\mathcal{F}_T$-measurable, then*

$$F=\mathbb{E}(F)+I(P_{\mathbb{F}}(DF)).$$

Moreover, $P_{\mathbb{F}}(DF)$ is the unique $Y\in L^p_{\mathbb{F}}(\Omega;\gamma(L^2(0,T;\mathscr{H}),E))$ satisfying $F=\mathbb{E}(F)+I(Y)$.

Proof. We may assume that $\mathbb{E}(F)=0$.

Let $X\in L^p_{\mathbb{F}}(\Omega;\gamma(L^2(0,T;\mathscr{H}),E))$ be such that $F=I(X)=\delta(X)$. Let $\frac{1}{p}+\frac{1}{q}=1$, and let $Y\in L^q(\Omega;\gamma(L^2(0,T;\mathscr{H}),E^*))$ be arbitrary. By Lemma 12.11, Theorem 12.6, and Lemma 12.4 we obtain

$$\begin{aligned}\mathbb{E}\langle P_{\mathbb{F}}(DF),Y\rangle&=\mathbb{E}\langle DF,P_{\mathbb{F}}Y\rangle=\mathbb{E}\langle F,\delta(P_{\mathbb{F}}Y)\rangle\\&=\mathbb{E}\langle\delta(X),\delta(P_{\mathbb{F}}Y)\rangle=\mathbb{E}\langle I(X),I(P_{\mathbb{F}}Y)\rangle\\&=\mathbb{E}\langle X,P_{\mathbb{F}}Y\rangle=\mathbb{E}\langle P_{\mathbb{F}}X,Y\rangle=\mathbb{E}\langle X,Y\rangle.\end{aligned}$$

Since this holds for all $Y \in L^q(\Omega; \gamma(L^2(0,T;\mathscr{H}), E^*))$, it follows that $X = P_{\mathbb{F}}(DF)$. The uniqueness of $P_{\mathbb{F}}(DF)$ follows from the injectivity of I as a bounded linear operator from $L^p_{\mathbb{F}}(\Omega; \gamma(L^2(0,T;\mathscr{H}), E))$ to $L^p(\Omega, \mathcal{F}_T)$. □

With a little extra effort we can prove a bit more:

Theorem 12.13 (Clark-Ocone representation, second L^p-version). *Let E be a UMD space and let $1 < p < \infty$. The operator $P_{\mathbb{F}} \circ D$ has a unique extension to a bounded operator from $L^p(\Omega, \mathcal{F}_T; E)$ to $L^p_{\mathbb{F}}(\Omega; \gamma(L^2(0,T;\mathscr{H}), E))$, and for all $F \in L^p(\Omega, \mathcal{F}_T; E)$ we have the representation*

$$F = \mathbb{E}(F) + I((P_{\mathbb{F}} \circ D)F).$$

Moreover, $(P_{\mathbb{F}} \circ D)F$ is the unique $Y \in L^p_{\mathbb{F}}(\Omega; \gamma(L^2(0,T;\mathscr{H}), E))$ satisfying $F = \mathbb{E}(F) + I(Y)$.

Proof. It follows from Theorem 12.12 that $F \mapsto I((P_{\mathbb{F}} \circ D)F)$ extends uniquely to a bounded operator on $L^p(\Omega, \mathcal{F}_T; E)$, since it equals $F \mapsto F - \mathbb{E}(F)$ on the dense subspace $\mathbb{D}^{1,p}(\Omega, \mathcal{F}_T; E)$. The proof is finished by recalling that I is an isomorphism from $L^p_{\mathbb{F}}(\Omega; \gamma(L^2(0,T;\mathscr{H}), E))$ onto its range in $L^p(\Omega, \mathcal{F}_T)$. □

12.3 Extension to L^1

We continue with an extension of Theorem 12.13 to random variables in the space $L^1(\Omega, \mathcal{F}_T; E)$. As before, $\mathbb{F} = (\mathcal{F}_t)_{t \in [0,T]}$ is the augmented filtration generated by the $\mathscr{H}$-cylindrical Brownian motion $W_{\mathscr{H}}$.

We denote by $L^0(\Omega; F)$ the vector space of all strongly measurable random variables with values in the Banach space F, identifying random variables that are equal almost surely. Endowed with the metric

$$d(X, Y) = \mathbb{E}(\|X - Y\| \wedge 1),$$

$L^0(\Omega; F)$ is a complete metric space, and we have $\lim_{n\to\infty} X_n = X$ in $L^0(\Omega; F)$ if and only if $\lim_{n\to\infty} X_n = X$ in measure in F.

We let $L^0(\Omega; \gamma(L^2(0,T;\mathscr{H}), E))$ denote the closure of the elementary adapted processes in $L^0_{\mathbb{F}}(\Omega; \gamma(L^2(0,T;\mathscr{H}), E))$. By the results of [134], the stochastic integral I has a unique extension to a linear homeomorphism from $L^0_{\mathbb{F}}(\Omega; \gamma(L^2(0,T;\mathscr{H}), E))$ onto its image in $L^0(\Omega, \mathcal{F}_T; E)$.

Theorem 12.14 (Clark-Ocone representation, L^1-version). *Let E be a UMD space. The operator $P_{\mathbb{F}} \circ D$ has a unique extension to a continuous linear operator from $L^1(\Omega, \mathcal{F}_T; E)$ to $L^0_{\mathbb{F}}(\Omega; \gamma(L^2(0,T;\mathscr{H}), E))$, and for all $F \in L^1(\Omega, \mathcal{F}_T; E)$ we have the representation*

$$F = \mathbb{E}(F) + I((P_{\mathbb{F}} \circ D)F).$$

Moreover, $(P_{\mathbb{F}} \circ D)F$ is the unique element $Y \in L^0_{\mathbb{F}}(\Omega; \gamma(L^2(0,T;\mathscr{H}), E))$ satisfying $F = \mathbb{E}(F) + I(Y)$.

Proof. We shall use the process $\xi_X : (0,T) \times \Omega \to \gamma(L^2(0,T;\mathscr{H}), E)$ associated with a strongly measurable random variable $X : \Omega \to \gamma(L^2(0,T;\mathscr{H}), E)$, defined by

$$(\xi_X(t,\omega))f := (X(\omega))(\mathbf{1}_{[0,t]}f), \qquad f \in L^2(0,T;\mathscr{H}).$$

Some properties of this process have been studied in [134, Section 4].

Let $(F_n)_{n\geq 1}$ be a sequence of $\mathcal{F}_T$-measurable random variables in $\mathscr{S}(\Omega)\otimes E$ which is Cauchy in $L^1(\Omega, \mathcal{F}_T; E)$. By [134, Lemma 5.4] there exists a constant $C \geq 0$, depending only on E, such that for all $\delta > 0$ and $\varepsilon > 0$ and all $m, n \geq 1$,

$$\begin{aligned}
&\mathbb{P}\big(\|P_{\mathbb{F}}(DF_n - DF_m)\|_{\gamma(L^2(0,T;\mathscr{H}),E)} > \varepsilon\big) \\
&\qquad \leq \frac{C\delta^2}{\varepsilon^2} + \mathbb{P}\big(\sup_{t\in[0,T]} \|I(\xi_{P_{\mathbb{F}}(DF_n - DF_m)}(t))\| \geq \delta\big) \\
&\qquad \stackrel{(*)}{=} \frac{C\delta^2}{\varepsilon^2} + \mathbb{P}\big(\sup_{t\in[0,T]} \|\mathbb{E}(F_n - F_m|\mathcal{F}_t) - \mathbb{E}(F_n - F_m)\| \geq \delta\big) \\
&\qquad \stackrel{(**)}{\leq} \frac{C\delta^2}{\varepsilon^2} + \frac{1}{\delta}\mathbb{E}\|F_n - F_m - \mathbb{E}(F_n - F_m)\|.
\end{aligned}$$

In this computation, $(*)$ follows from Theorem 12.12 which gives

$$\mathbb{E}(F|\mathcal{F}_t) - \mathbb{E}(F) = \mathbb{E}\big(I(P_{\mathbb{F}}DF)\big|\mathcal{F}_t\big) = \mathbb{E}\big(I(\xi_{P_{\mathbb{F}}DF}(T))\big|\mathcal{F}_t\big) = I(\xi_{P_{\mathbb{F}}DF}(t)).$$

The estimate $(**)$ follows from Doob's maximal inequality. Since the right-hand side in the above computation can be made arbitrarily small, this proves that $(P_{\mathbb{F}}(DF_n))_{n\geq 1}$ is Cauchy in measure in $\gamma(L^2(0,T;\mathscr{H}), E)$.

For $F \in L^1(\Omega, \mathcal{F}_T; E)$ this permits us to define

$$(P_{\mathbb{F}} \circ D)F := \lim_{n\to\infty} P_{\mathbb{F}}(DF_n),$$

where $(F_n)_{n\geq 1}$ is any sequence of $\mathcal{F}_T$-measurable random variables in $\mathscr{S}(\Omega)\otimes E$ satisfying $\lim_{n\to\infty} F_n = F$ in $L^1(\Omega, \mathcal{F}_T; E)$. It is easily checked that this definition is independent of the approximation sequence. The resulting linear operator $P_{\mathbb{F}} \circ D$ has the stated properties. This time we use the fact that I is a homeomorphism from $L^0_{\mathbb{F}}(\Omega; \gamma(L^2(0,T;\mathscr{H}), E))$ onto its image in $L^0(\Omega, \mathcal{F}_T; E)$; this also gives the uniqueness of $(P_{\mathbb{F}} \circ D)F$. $\square$

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List of symbols

General mathematics

$\mathbb{N} = \{0, 1, 2, 3, \ldots\}$
$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots, \}$
$\mathbb{Q}$, rational numbers
$\mathbb{R}$, real numbers
$\overline{\lim}$, limes superior
$\underline{\lim}$, limes inferior
p', conjugate exponent, $\frac{1}{p} + \frac{1}{p'} = 1$
δ_{mn}, 1 if $m = n$ and 0 otherwise
∂_k, partial derivative
lsc, lower semicontinuous
$a \vee b = \max\{a, b\}$
$a \wedge b = \min\{a, b\}$
$a \lesssim b$, p. 6
$a \gtrsim b$, p. 6
$a \eqsim b$, p. 6

Spaces

$AC(J; X)$, p. 153
$AC^p(J; X)$, p. 153
$AC^p_{\mathrm{loc}}(J; X)$, p. 153
$\mathcal{E}(\Sigma_\theta)$, p. 115
$\mathcal{F}(\Sigma_\theta^+)$, p. 116
$H_0^\infty(\Sigma_\theta)$, p. 113
$H^\infty(\Sigma_\theta^+)$, p. 116
$\mathscr{L}(E, F)$, bounded linear operators
$\mathscr{L}(E)$, bounded linear operators
E^*, dual Banach space of E
$\mathcal{L}_2(H, G)$, Hilbert-Schmidt operators
$\gamma(H, X)$,, p. 109
$\mathcal{B}_{\mathrm{b}}(S; X)$, bounded Borel functions
$C(S; X)$, continuous functions
$C_{\mathrm{b}}(S; X)$, bounded cont. functions
$\mathcal{F}C^k_{\mathrm{b}}(E; H_0)$, p. 32
$\mathcal{F}\mathscr{P}(E; H_0)$, p. 32

Measure and probability

a.e., almost everywhere
$\mathbb{E}$, expectation
$\mathscr{B}(X)$, Borel σ-algebra
$\mathscr{P}(X)$, p. 139
$\widehat{\mu}$, Fourier transform, p. 21
$\mu \otimes \nu$, product measure
$\mu * \nu$, convolution, p. 38
$\mu_n \rightharpoonup \mu$, p. 140

Operators

Miscellaneous

Index

Summary

Analysis of Infinite Dimensional Diffusions

This thesis is concerned with analytic aspects of stochastic differential equations in infinite dimensional state spaces. Such equations provide a mathematical description of various phenomena in physics, biology, finance, and other fields of science.

Part I of this thesis contains a study of operators which arise as infinitesimal generators of transition semigroups associated with stochastic differential equations in Banach spaces. The focus is on a class of elliptic differential operators on Wiener spaces, for which a detailed analysis is presented in suitable L^p-spaces. The main results are L^p-estimates and domain characterisations for the elliptic operators and their square roots, which generalise the classical Meyer inequalities. As an application, it is shown that the boundedness of the Riesz transform is preserved under L^p-second quantisation of analytic Hilbert space contraction semigroups. The methods are analytic in spirit with a probabilistic flavour, and build on recent advances in operator theory, in particular the holomorphic functional calculus for sectorial operators and randomised boundedness of operators on Banach spaces.

The underlying framework is inspired by the theory of perturbed Hodge-Dirac operators, which provides a unified setting for various problems in harmonic analysis including the Kato square root problem. In this setting, randomised gradient bounds for transition semigroups and Littlewood-Paley-Stein inequalities for the associated generators are obtained.

By duality, the transition semigroups studied in Part I of this thesis induce a flow in the space of probability measures, representing the evolution of the law of the underlying stochastic process. In Part II of this thesis a framework is developed for the study of these evolutions as gradient flows associated with entropy functionals on the Wasserstein space over an infinite dimensional Banach space.

For this purpose a Wasserstein distance for probability measures on a Banach space is considered, where the underlying metric is induced by the reproducing kernel Hilbert space of the noise in the stochastic equation.

It is proved that a continuity equation can be associated with smooth curves of probability measures, which allows for the introduction of velocity fields associated with smooth curves and subdifferentials of functionals, in the spirit of Riemannian geometry.

For functionals satisfying appropriate displacement convexity conditions it is shown that the *metric* formulation of a gradient flow in the sense of an evolution variational inequality is equivalent to a *differential geometric* formulation in the sense of the Riemannian structure. Under natural assumptions on the reproducing kernel Hilbert spaces it is proved that entropy functionals associated with Gaussian measures are displacement convex. The corresponding gradient flows are shown to satisfy Fokker-Planck equations involving the generators considered in Part I of this thesis.

The Malliavin calculus is a differential calculus on an infinite dimensional space endowed with a Gaussian measure. The scalar-valued theory extends in a natural way to Hilbert spaces, but the straightforward Banach space-valued extension breaks down. In Part III of this thesis a Banach space-valued theory having most of the good features of the scalar theory is developed. This extension relies on the systematic use of radonifying operators. Among the obtained results are analogues of the Wiener-Itô isometry, two-sided L^p-estimates for multiple stochastic integrals, and boundedness of the Malliavin derivative on each vector-valued Wiener-Itô chaos. Some results require geometric assumptions on the Banach space under consideration, such as the Clark-Ocone representation formula for random variables, which holds if the Banach space has the so-called UMD property.

Samenvatting

Analyse van Oneindig-Dimensionale Diffusies

Dit proefschrift is gewijd aan analytische aspecten van stochastische differentiaalvergelijkingen in oneindig-dimensionale toestandsruimten. Zulke vergelijkingen geven een wiskundige beschrijving van diverse verschijnselen in de natuurkunde, biologie, econometrie, en andere delen van de wetenschap.

In deel I van dit proefschrift worden operatoren bestudeerd die optreden als generatoren van overgangshalfgroepen behorende bij stochastische differentiaalvergelijkingen in Banachruimten. De nadruk ligt op een klasse van elliptische differentiaaloperatoren op Wienerruimten, waarvoor een gedetailleerde analyse in zekere L^p-ruimten wordt gepresenteerd. De hoofdresultaten zijn L^p-afschattingen en domeinkarakteriseringen voor de elliptische operatoren en hun wortels, die de klassieke Meyer-ongelijkheden generaliseren. Als toepassing wordt bewezen dat de begrensdheid van de Riesz-transformatie behouden blijft onder L^p-tweede quantisatie van analytische contractiehalfgroepen op Hilbertruimten. De gebruikte methoden zijn analytisch met een probabilistische component, en borduren voort op recente ontwikkelingen in de operatorentheorie, in het bijzonder de holomorfe functionaalcalculus voor sectoriële operatoren en gerandomiseerde begrensdheid van operatoren op Banachruimten.

De onderliggende structuur maakt gebruik van de theorie van verstoorde Hodge-Dirac-operatoren. Deze theorie verschaft een algemeen kader voor verschillende problemen in de harmonische analyse, waaronder het wortelprobleem van Kato. In deze context worden gerandomiseerde gradient-afschattingen voor overgangshalfgroepen en Littlewood-Paley-Stein-ongelijkheden voor de bijbehorende generatoren bewezen.

De overgangshalfgroepen die bestudeerd zijn in Deel I geven aanleiding tot een stroming in de ruimte van kansmaten, die de evolutie van de verdeling van het onderliggende stochastische proces beschrijft. In Deel II van dit proefschrift wordt een kader ontwikkeld waarbinnen deze evoluties beschreven kunnen worden als gradient-stromingen behorende bij entropie-functionalen op de Wassersteinruimte over een oneindig-dimensionale Banachruimte.

In dit kader wordt een Wassersteinafstand voor kansmaten op een Banachruimte beschouwd, waarbij de onderliggende metriek bepaald wordt door de reproducerende kern Hilbertruimte van de ruis in de stochastische vergelijking. Er wordt bewezen dat er een continuïteitsvergelijking kan worden toegevoegd aan een pad van kansmaten, die het mogelijk maakt snelheidsvelden voor gladde paden en subdifferentialen voor functionalen in te voeren.

Onder geschikte convexiteits-aannamen wordt bewezen dat de *metrische* formulering van een gradient-stroming in de zin van een evolutie-variationele ongelijkheid equivalent is aan een *differentiaalgeometrische* formulering in de zin van de Riemannse structuur. Onder natuurlijke voorwaarden op de reproducerende kern Hilbertruimten wordt bewezen dat entropie-functionalen voor een Gaussmaat verplaatsings-convex zijn. Er wordt aangetoond dat de bijbehorende gradient-stromingen voldoen aan Fokker-Planck-vergelijkingen, waarin de generatoren die in Deel I van dit proefschrift beschouwd worden een rol spelen.

De Malliavincalculus is een differentiaalrekening op een oneindig-dimensionale ruimte voorzien van een Gaussmaat. De scalaire theorie generaliseert op natuurlijke wijze naar Hilbertruimten, maar er bestaat geen voor de hand liggende uitbreiding naar Banachruimten. In Deel III van dit proefschrift wordt een Banachruimte-waardige theorie ontwikkeld, die de meeste goede eigenschappen van de scalaire theorie behoudt. In deze uitbreiding wordt systematisch gebruik gemaakt van radonificerende operatoren. Onder de verkregen resultaten zijn een analogon van de Wiener-Itô-isometrie, tweezijdige L^p-afschattingen voor stochastische integralen en begrensdheid van de Malliavin-afgeleide op elke vectorwaardige Wiener-Itô-chaos. Voor sommige resultaten zijn geometrische aannamen op de Banachruimte vereist. Een voorbeeld is de Clark-Ocone formule die wordt bewezen voor Banachruimten met de zogenaamde UMD-eigenschap.

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Delft, March 2009 *Jan Maas*

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Curriculum Vitae

Jan Maas was born in Leidschendam on 6th April 1982. In 2000 he completed his secondary education at the Aloysius College in The Hague. Between 2000 and 2005, he pursued the study of Applied Mathematics at Delft University of Technology, where he obtained his M.Sc. degree "cum laude" with a specialisation in functional analysis. In September 2005 he began his Ph.D. research under the supervision of prof. dr. J.M.A.M. van Neerven at Delft University of Technology. Part of this research was carried out during a half-year stay at the University of New South Wales in Sydney, Australia in 2007. In his spare time Jan Maas is an enthusiastic jazz guitarist.